TM-9 part 2

Determine velocity v_1 and acceleration a_1 of body 1 of the given mechanical system (fig. 1) when translation of body 1 is equal to $x_1 = 1$ m. It is known that masses of bodies are $m_1 = 10$ kg, $m_2 = 20$ kg, $m_3 = 30$ kg, respectively. External radius of body 2 is $R_2 = 1$ m, internal radius of body 2 is $r_2 = 0.5$ m, and a radius of body 3 is $R_3 = 0.75$ m. Radii of inertia of bodies 2 and 3 are $i_2 = i_3 = 0.75$ m. Coefficient of friction of body 1 is $f = 0.1$, moment of force $M_2 = 10$ N·m, force $F = 50$ N, and angle $\alpha = 45°$. The mechanical system is in equilibrium at the moment of time $t_0 = 0$, friction of strings on pulleys can be neglected.

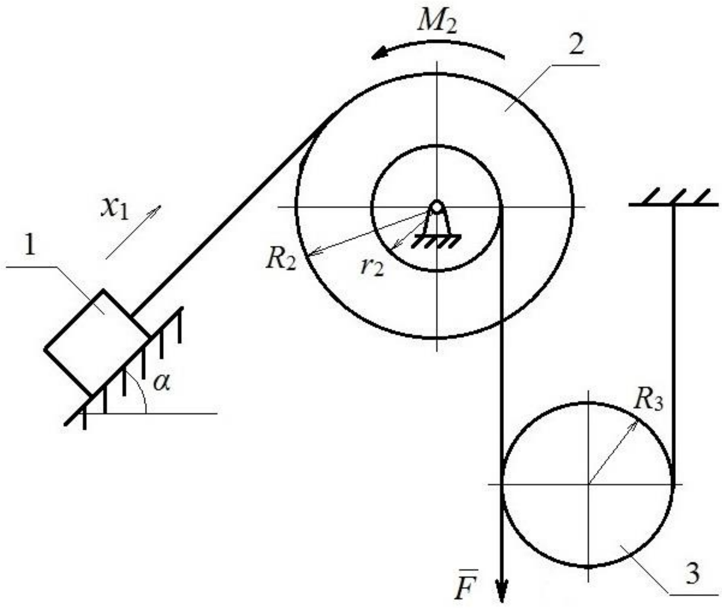

Fig. 1 Initial scheme

Solution

Given:
$x_1 = 1$ m
$m_1 = 10$ kg
$m_2 = 20$ kg
$m_3 = 30$ kg
$R_2 = 1$ m
$r_2 = 0.5$ m
$R_3 = 0.75$ m
$i_2 = i_3 = 0.75$ m
$f = 0.1$
$M_2 = 10$ N·m
$F = 50$ N
$\alpha = 45°$
Find:
v_1, a_1

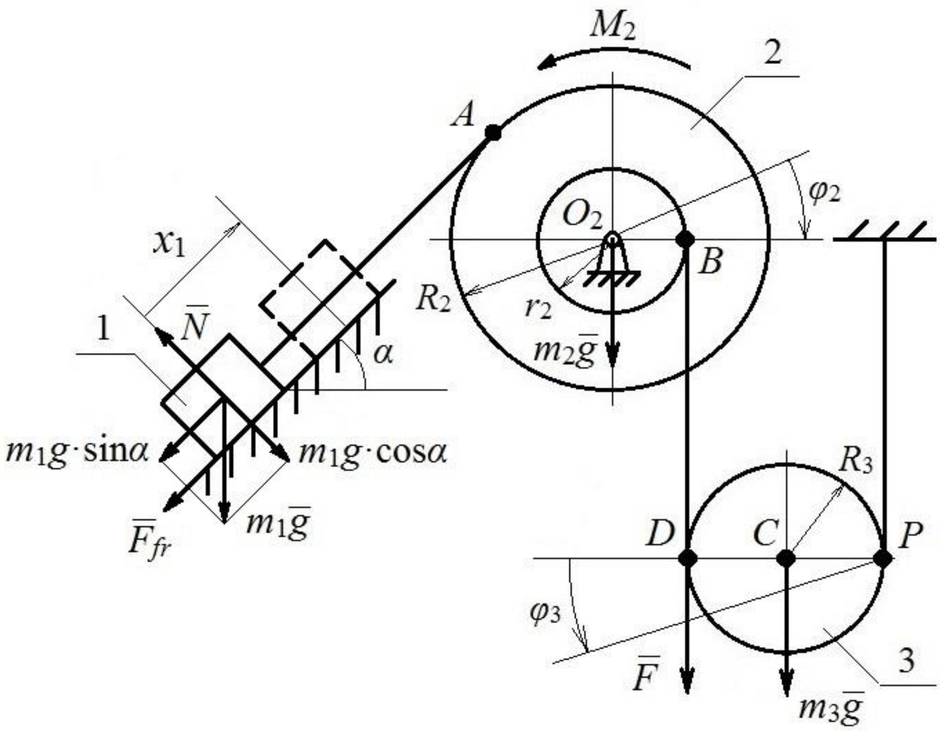

Fig. 2 Calculation scheme

Apply the theorem of the change in the kinetic energy of a system (1).

$$T - T_0 = \sum A_i^E + \sum A_i^I, \qquad (1)$$

where T and T_0 are kinetic energies of the system in initial and current positions, respectively, $\sum A_i^E$ and $\sum A_i^I$ are sums of works done by external and internal forces, which are applied to the system during the translation from the initial to current position, respectively.

For the given system, which consists of absolutely rigid bodies that are connected via inextensible strings, the sum of works done by internal forces $\sum A_i^I = 0$.

It is also given that the system is in equilibrium at the initial position, hence $T_0 = 0$. Then, formula (1) can be transformed into

$$T = \sum A_i^E. \qquad (2)$$

Construct a calculation scheme (fig. 2) in order to determine the kinetic energy of the system T and the sum of works done by external forces $\sum A_i^E$. Indicate the given system in a current position in the figure using a dashed line. Indicate additional points A, B, C, D, P, O_2 and directions of motion of bodies 2 and 3. Also, indicate all external forces that are applied to the system, such as forces of gravity of bodies m_1g, m_2g, m_3g, force of friction of body 1 on the inclined surface F_{fr}, and normal reaction of the surface N.

Construct kinematic dependencies between velocities and translations of points in the system. Express velocities and translations of points through the parameters of body 1 (x_1, v_1).

$$x_1 = \varphi_2 \cdot R_2,$$
$$\varphi_2 \cdot r_2 = \varphi_3 \cdot 2R_3,$$

from where the following is obtained

$$\varphi_2 = \frac{x_1}{R_2}, \qquad (3)$$

$$\varphi_3 = \frac{\varphi_2 \cdot r_2}{2R_3} = \frac{x_1 \cdot r_2}{2R_2 \cdot R_3}, \qquad (4)$$

$$x_C = \varphi_3 \cdot R_3 = \frac{x_1 \cdot r_2}{2R_2}, \qquad (5)$$

$$x_D = \varphi_3 \cdot 2R_3 = \frac{x_1 \cdot r_2}{R_2}. \qquad (6)$$

Respectively,

$$\omega_2 = \frac{d\varphi_2}{dt} = \frac{dx_1}{dt} \cdot \frac{1}{R_2} = \frac{v_1}{R_2}, \qquad (7)$$

$$\omega_3 = \frac{d\varphi_3}{dt} = \frac{v_1 \cdot r_2}{2R_2 \cdot R_3}, \qquad (8)$$

$$v_C = \frac{dx_C}{dt} = \frac{v_1 \cdot r_2}{2R_2}. \qquad (9)$$

Kinetic energy of the system

$$T = T_1 + T_2 + T_3. \qquad (10)$$

Then, the kinetic energies of bodies considering the dependencies (7-9)

$$T_1 = \frac{m_1 \cdot v_1^2}{2}, \qquad (11)$$

$$T_2 = \frac{I_2 \cdot \omega_2^2}{2} = \frac{m_2 \cdot i_2^2 \cdot \left(\frac{v_1}{R_2}\right)^2}{2}, \qquad (12)$$

$$T_3 = \frac{m_3 \cdot v_C^2}{2} + \frac{I_3 \cdot \omega_3^2}{2} = \frac{m_3 \cdot \left(\frac{v_1 \cdot r_2}{2R_2}\right)^2}{2} + \frac{\frac{m_3 \cdot i_3^2}{2} \cdot \left(\frac{v_1 \cdot r_2}{2R_2 \cdot R_3}\right)^2}{2}, \quad (13)$$

where moments of inertia of bodies 2 and 3 are as follows

$$I_2 = m_2 \cdot i_2^2 = 20 \cdot 0.75^2 = 11.25\ (\text{kg} \cdot \text{m}^2),$$

$$I_3 = \frac{m_3 \cdot i_3^2}{2} = \frac{30 \cdot 0.75^2}{2} = 8.438\ (\text{kg} \cdot \text{m}^2).$$

Substitute (11-13) in (10)

$$T = \frac{m_1 \cdot v_1^2}{2} + \frac{m_2 \cdot i_2^2 \cdot \left(\frac{v_1}{R_2}\right)^2}{2} + \frac{m_3 \cdot \left(\frac{v_1 \cdot r_2}{2R_2}\right)^2}{2} + \frac{\frac{m_3 \cdot i_3^2}{2} \cdot \left(\frac{v_1 \cdot r_2}{2R_2 \cdot R_3}\right)^2}{2}$$

and take the common multiplier $\frac{v_1^2}{2}$ outside the parenthesis

$$T = \frac{v_1^2}{2}\left[m_1 + m_2 \cdot i_2^2 \cdot \left(\frac{1}{R_2}\right)^2 + m_3 \cdot \left(\frac{r_2}{2R_2}\right)^2 + \frac{m_3 \cdot i_3^2}{2} \cdot \left(\frac{r_2}{2R_2 \cdot R_3}\right)^2\right] = \frac{v_1^2}{2} \cdot M_{red} \quad (14)$$

where the quantity in parenthesis $M_{red} = 24.06$ is reduced mass, kg.

Determine the work done by external forces

$$\sum A_i^E = A(m_1 g) + A(F_{fr}) + A(N) + A(m_2 g) + A(M_2) + A(m_3 g) + A(F). \quad (15)$$

The work done by force of gravity of body 1 is determined as a product of a magnitude of the force of gravity, translation of a point, to which it is applied (in this case translation of body 1) and a cosine of the angle between the direction of force and the direction of translation of body 1. Determine the works done by other forces and moments of forces analogically, while considering the dependencies (3-6).

$$A(m_1 g) = m_1 g \cdot x_1 \cdot \cos(m_1 g;\ x_1) = m_1 g \cdot x_1 \cdot \cos(90° + \alpha) =$$
$$= 10 \cdot 9.8 \cdot x_1 \cdot (-0.707) = -69.29 \cdot x_1\ (\text{J}).$$

Work done by the force of friction

$$A(F_{fr}) = F_{fr} \cdot x_1 \cdot \cos(F_{fr};\ x_1) = F_{fr} \cdot x_1 \cdot \cos 180° =$$
$$= 6.93 \cdot x_1 \cdot (-1) = -\ 6.93 \cdot x_1\ (\text{J}),$$

where $F_{fr} = N \cdot f = m_1 g \cdot \cos\alpha \cdot f = 10 \cdot 9.8 \cdot 0.707 \cdot 0.1 = 6.93$ (N),

and $N = m_1 g \cdot \cos\alpha$ from the equation of equilibrium on the axis, which is perpendicular to the surface.

Work done by a normal reaction of the surface

$$A(N) = N \cdot x_1 \cdot \cos(N;\ x_1) = N \cdot x_1 \cdot \cos 90° = 0.$$

The work done by force of gravity of body 2 is equal to 0 because the translation of point O_2 is equal to 0.

$$A(m_2 g) = m_2 g \cdot x_{O_2} \cdot \cos(m_2 g;\ x_{O_2}) = 0.$$

The work done by the moment of force M_2

$$A(M_2) = M_2 \cdot \varphi_2 \cdot \cos(M_2;\ \varphi_2) = M_2 \cdot \frac{x_1}{R_2} \cdot \cos 180° =$$
$$= 10 \cdot \frac{x_1}{1} \cdot (-1) = -10 \cdot x_1\ (\text{J}).$$

The work done by force of gravity of body 3

$$A(m_3 g) = m_3 g \cdot x_C \cdot \cos(m_3 g;\ x_C) = m_3 g \cdot \frac{x_1 \cdot r_2}{2R_2} \cdot \cos 0° =$$
$$= 30 \cdot 9.8 \cdot \frac{x_1 \cdot 0.5}{2 \cdot 1} \cdot 1 = 73.5 \cdot x_1\ (\text{J}).$$

The work done by force F

$$A(F) = F \cdot x_D \cdot \cos(F;\ x_D) = F \cdot \frac{x_1 \cdot r_2}{R_2} \cdot \cos 0° =$$
$$= 50 \cdot \frac{x_1 \cdot 0.5}{1} \cdot 1 = 25 \cdot x_1 \text{ (J)}.$$

Substitute works done by all forces in (15)

$$\sum A_i^E = m_1 g \cdot x_1 \cdot \cos(90° + \alpha) + F_{fr} \cdot x_1 \cdot \cos 180° +$$
$$+ M_2 \cdot \frac{x_1}{R_2} \cdot \cos 180° + m_3 g \cdot \frac{x_1 \cdot r_2}{2R_2} \cdot \cos 0° + F \cdot \frac{x_1 \cdot r_2}{R_2} \cdot \cos 0°,$$

take the common multiplier x_1 outside the parenthesis

$$\sum A_i^E = x_1 \cdot [m_1 g \cdot \cos(90° + \alpha) + F_{fr} \cdot \cos 180° +$$
$$+ M_2 \cdot \frac{1}{R_2} \cdot \cos 180° + m_3 g \cdot \frac{r_2}{2R_2} \cdot \cos 0° + F \cdot \frac{r_2}{R_2} \cdot \cos 0°] =$$
$$= x_1 \cdot [-69.29 - 6.93 - 10 + 73.5 + 25] = x_1 \cdot F_{red}, \qquad (16)$$

where $F_{red} = 12.28$ is reduced force, N.

Equate (14) and (16), which is the transformed expression (2)

$$\frac{v_1^2}{2} \cdot M_{red} = x_1 \cdot F_{red}, \qquad (17)$$

from where the following is obtained

$$v_1 = \sqrt{\frac{2 \cdot x_1 \cdot F_{red}}{M_{red}}},$$

substitute x_1 = 1 m from the initial data and obtain

$$v_1 = \sqrt{\frac{2 \cdot 1 \cdot 12.28}{24.06}} = 1.01 \text{ (m/s)}.$$

Differentiate (17) with respect to the time parameter

$$\frac{d\left(\frac{v_1^2}{2} \cdot M_{red}\right)}{dt} = \frac{d(x_1 \cdot F_{red})}{dt},$$
$$\frac{1}{2} \cdot 2v_1 \cdot a_1 \cdot M_{red} = v_1 \cdot F_{red},$$
$$a_1 = \frac{F_{red}}{M_{red}} = \frac{12.28}{24.06} = 0.51 \text{ (m/s}^2\text{)}.$$

Answer: v_1 = 1.01 m/s, a_1 = 0.51 m/s^2.