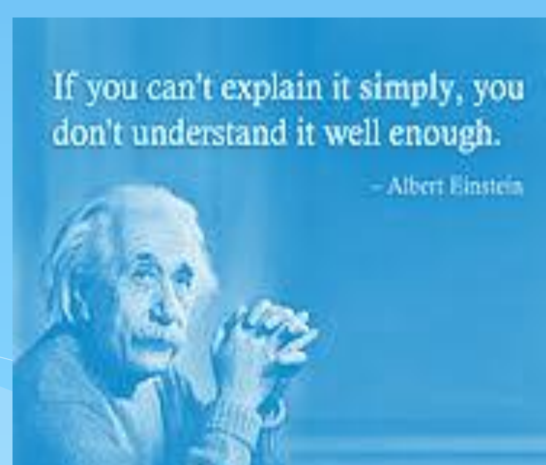
MINISTRY OF EDUCATION AND SCIENCE OF UKRAINE
STATE HIGHER EDUCATIONAL INSTITUTION
«NATIONAL MINING UNIVERSITY»
DEPARTMENT OF STRUCTURAL THEORETICAL AND APPLIED MECHANICS

DOLGOV A.M.

STRENGTH OF MATERIALS

TEXTBOOK

ILLUSTRATIVE MATERIAL



Dnepropetrovsk - 2015

Introduction

The three fundamental areas of engineering mechanics are statics, dynamics, and strength of materials.

Statics and dynamics are devoted primarily to the study of the external effects upon rigid bodies—that is, bodies for which the change in shape (deformation) can be neglected.

In contrast, strength of materials deals with the effects and deformations that are caused by the applied loads.

A machine part or structure must be strong enough to carry the applied load without breaking and, at the same time, the deformations must not be excessive.

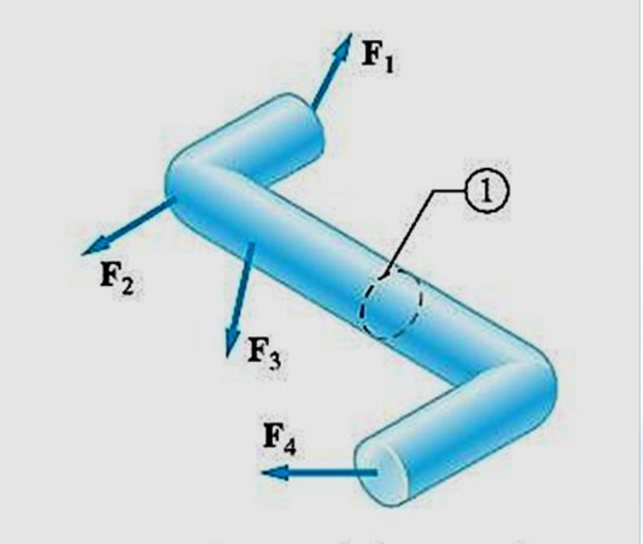
Equilibrium analysis will determine the force P, but not the strength or the rigidity of the bar.

In strength of materials, the statics solution is extended to include an analysis of the forces acting inside the bar to be certain that the bar will neither break nor deform

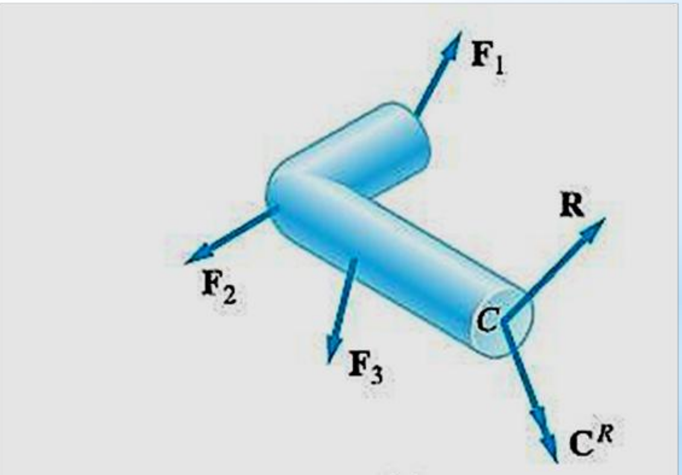
Analysis of Internal Forces

Method of sections

This method introduces an imaginary cutting plane that isolates a segment of the structure. The cutting plane must include the cross section of the member of interest. The axial force acting in the member can then be found from the FBD of the isolated segment because it now appears as an external force on the FBD.

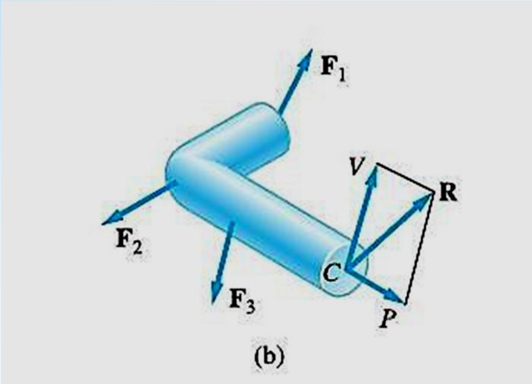


External forces acting on a body.

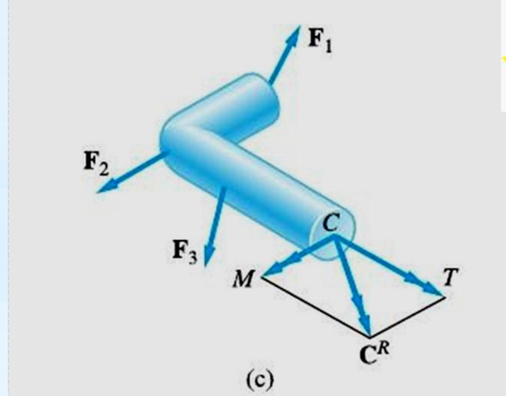


(a)

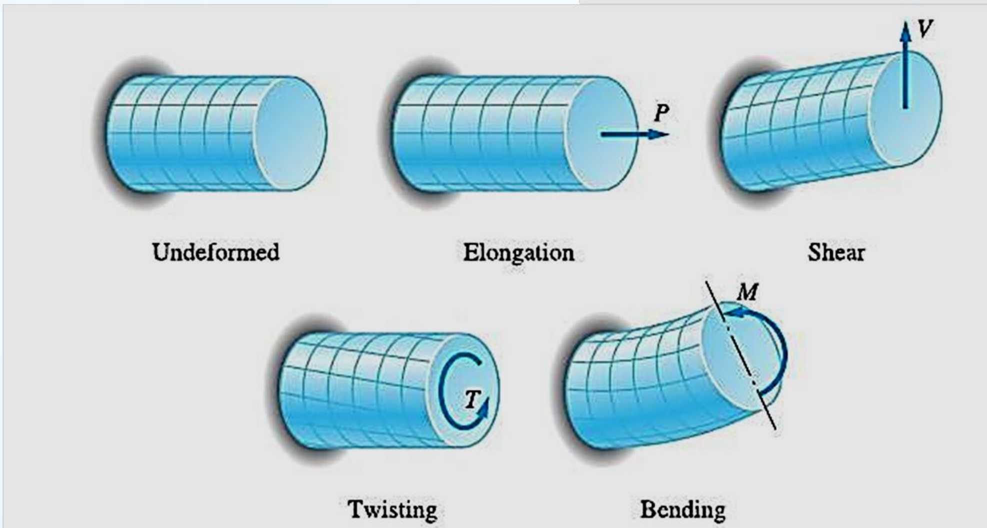
Free-body diagram for determining the internal force system acting on section ①.



Resolving the internal force **R** into the axial force P and the shear force V.



Resolving the internal couple $\mathbf{C}^R$ into the torque T and the bending moment M.



P:

Deformations produced by the components of internal forces and couples.

P:The component of the resultant force that is perpendicular to the cross section, tending to elongate or shorten the bar, is called the normal force.

V: The component of the resultant force lying in the plane of the cross section, tending to shear (slide) one segment of the bar relative to the other segment, is called the shear force.

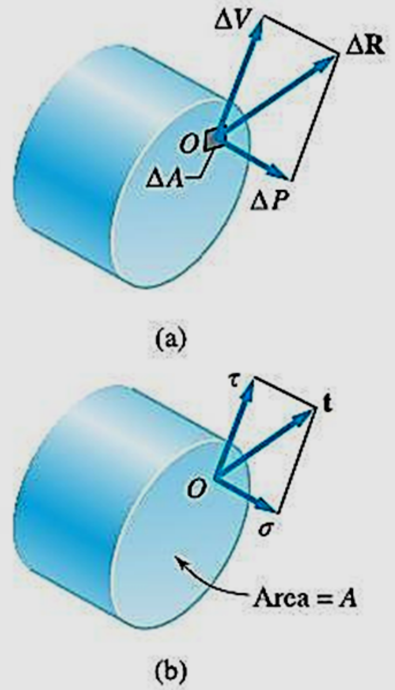
T: The component of the resultant couple that tends to twist (rotate) the bar is called the twisting moment or torque.

M: The component of the resultant couple that tends to bend the bar is called the bending moment.

STRESS

The stress vector acting on the cross section at point O is defined as

$$\mathbf{t} = \lim_{\Delta A \to 0} \frac{\Delta \mathbf{R}}{\Delta A}$$



Normal and shear stresses acting on the cross section at point O

$$\sigma = \lim_{\Delta A \to 0} \frac{\Delta P}{\Delta A} = \frac{dP}{dA} \qquad \tau = \lim_{\Delta A \to 0} \frac{\Delta V}{\Delta A} = \frac{dV}{dA}$$

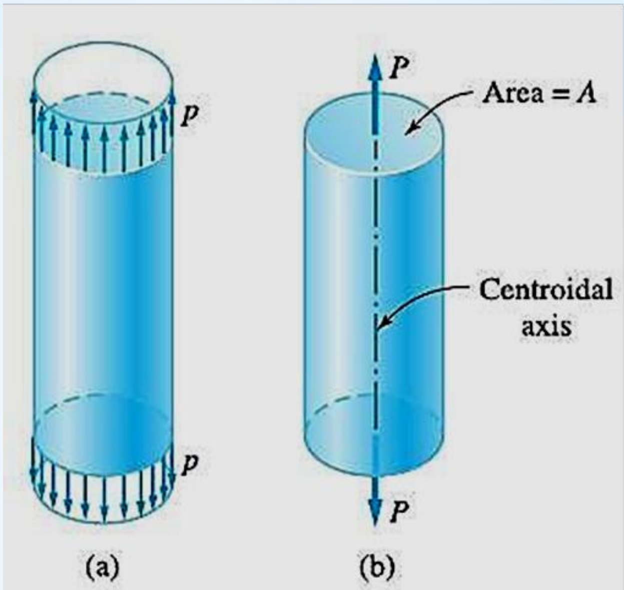
The commonly used sign convention for axial forces is to define tensile forces as positive and compressive forces as negative. This convention is carried over to normal stresses: Tensile stresses are considered to be positive, compressive stresses negative.

If the stresses are uniformly distributed

$$\sigma = \frac{P}{A} \qquad \tau = \frac{V}{A}$$

Axially Loaded Bars

When the loading is uniform, its resultant passes through the centroid of the loaded area.

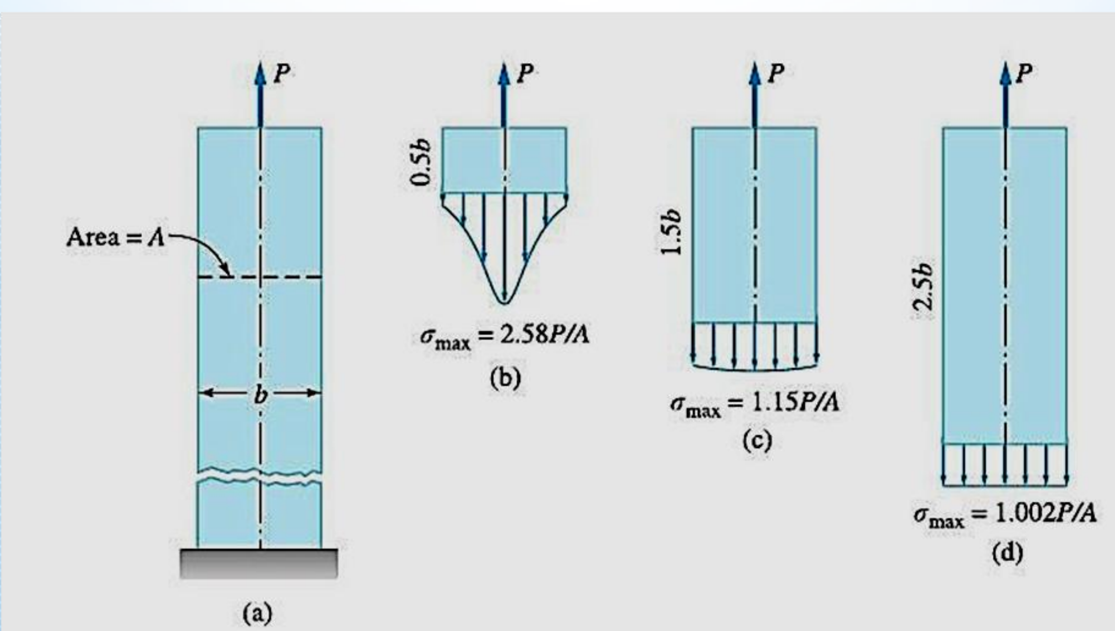


A bar loaded axially by (a) uniformly distributed load of intensity p; and (b) a statically equivalent centroidal force $P = pA$.

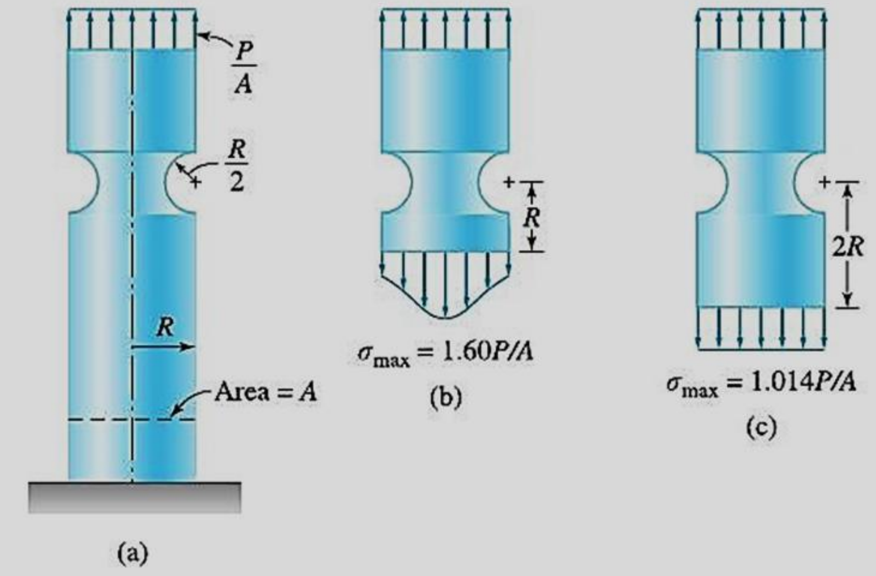
$$\boxed{\sigma = \frac{P}{A}}$$

Saint Venant's principle

The difference between the effects of two different but statically equivalent loads becomes very small at sufficiently large distances from the load.

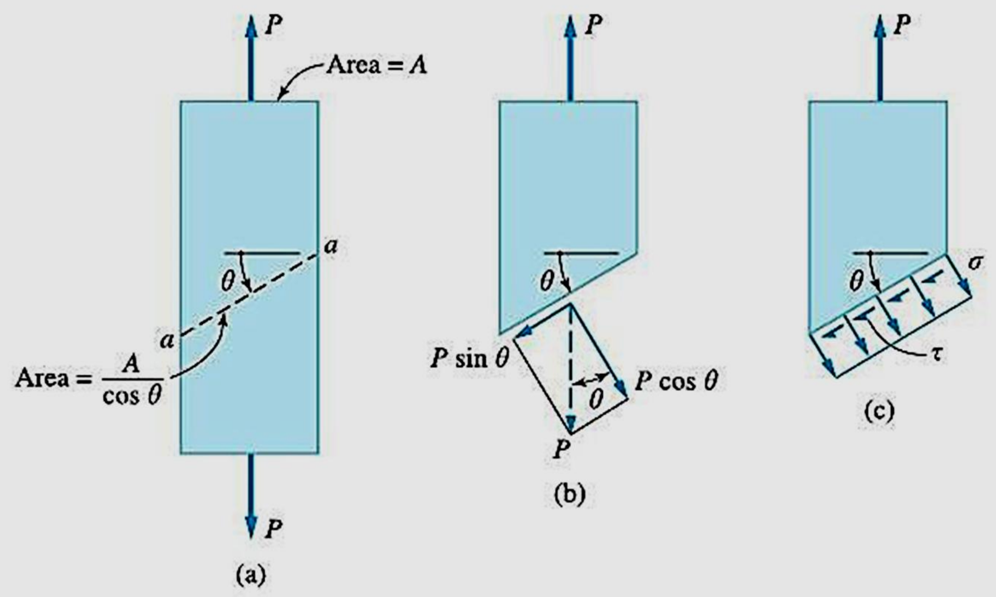


Normal stress distribution in a strip caused by a concentrated load.



Normal stress distribution in a grooved bar.

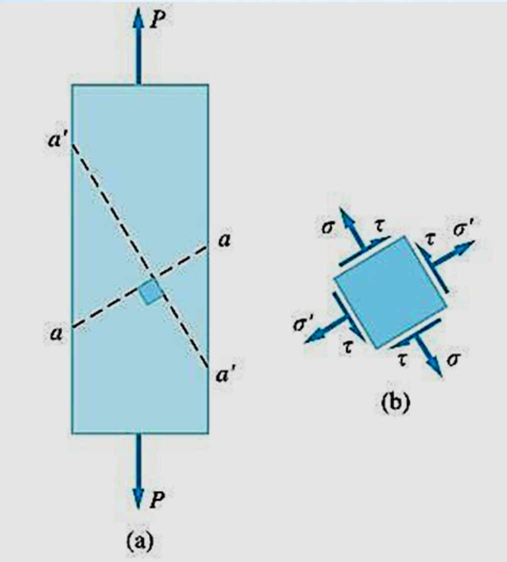
Stresses on inclined planes



Determining the stresses acting on an inclined section of a bar.

$$\boxed{\sigma = \frac{P\cos\theta}{A/\cos\theta} = \frac{P}{A}\cos^2\theta}$$

$$\boxed{\tau = \frac{P\sin\theta}{A/\cos\theta} = \frac{P}{A}\sin\theta\cos\theta = \frac{P}{2A}\sin 2\theta}$$



Stresses acting on two mutually perpendicular inclined sections of a bar.

$$\sigma' = \frac{P}{A}\sin^2\theta \qquad \tau' = -\frac{P}{2A}\sin 2\theta$$

$$\boxed{\tau' = -\tau}$$

The shear stresses that act on complementary planes have the same magnitude but opposite sense.

Procedure for stress analysis

Equilibrium Analysis

- *If necessary, find the external reactions using a free-body diagram (FBD) of the entire structure.*
- *Compute the axial force P in the member using the method of sections.*

This method introduces an imaginary cutting plane that isolates a segment of the structure. The cutting plane must include the cross section of the member of interest. The axial force acting in the member can then be found from the FBD of the isolated segment.

Computation of Stress

• *After the axial force has been found by equilibrium analysis, the normal stress in the member can be obtained from $\sigma = {}^{P}/_{A}$, where A is the cross-sectional area of the member at the cutting plane.*

Design Considerations

*For purposes of design, the computed stress must be compared with the **allowable stress**, also called the **working stress**.*

To prevent failure of the member, the computed stress must be less than the working stress.

Note on the Analysis of Trusses

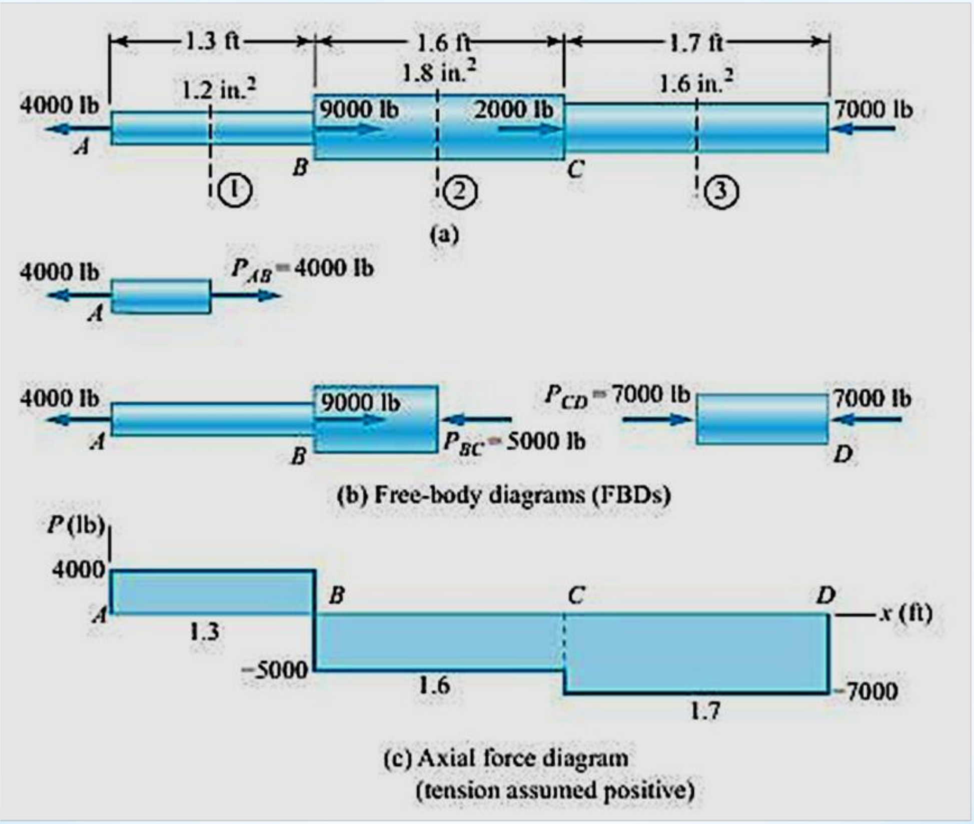
(1) weights of the members are negligible compared to the applied loads;

(2) joints behave as smooth pins;

(3) all loads are applied at the joints. Under these assumptions, each member of the truss is an axially loaded bar.

Sample Problem

The bar ABCD (a) consists of three cylindrical steel segments with different lengths and cross-sectional areas. Axial loads arc applied as shown. Calculate the normal stress in each segment.



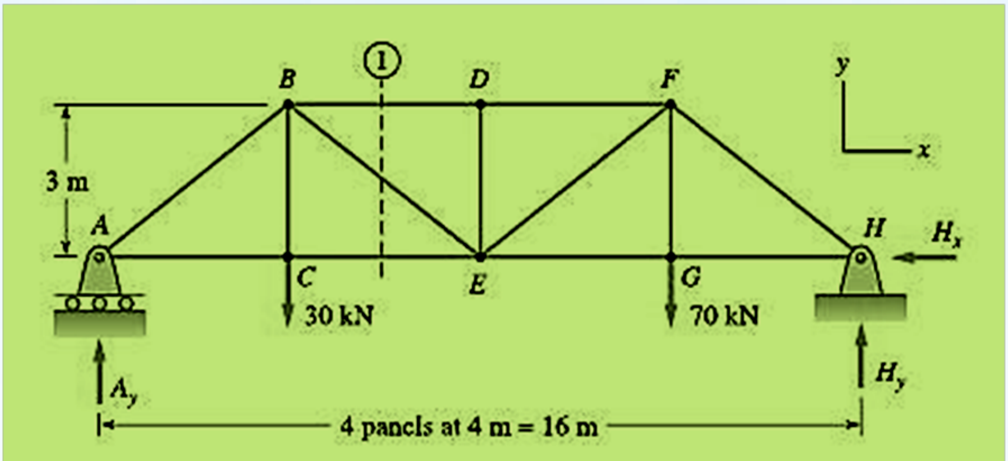
$$\sigma_{AB} = \frac{P_{AB}}{A_{AB}} = \frac{4000 \text{ lb}}{1.2 \text{ in.}^2} = 3330 \text{ psi (T)} \qquad \textit{Answer}$$

$$\sigma_{BC} = \frac{P_{BC}}{A_{BC}} = \frac{5000 \text{ lb}}{1.8 \text{ in.}^2} = 2780 \text{ psi (C)} \qquad \textit{Answer}$$

$$\sigma_{CD} = \frac{P_{CD}}{A_{CD}} = \frac{7000 \text{ lb}}{1.6 \text{ in.}^2} = 4380 \text{ psi (C)} \qquad \textit{Answer}$$

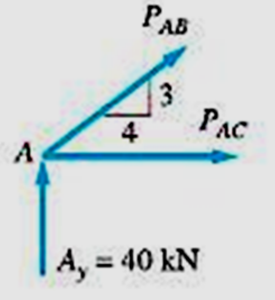
Sample Problem

For the truss shown, calculate the normal stresses in (1) member AC and (2) member BD. The cross-sectional area of each member is 900 mm^2.



(a)

Equilibrium analysis using the FBD of the entire truss in Fig. (a) gives the following values for the external reactions: $A_y = 40$ kN, $H_y = 60$ kN, and $H_x = 0$.



(b) FBD of pin A

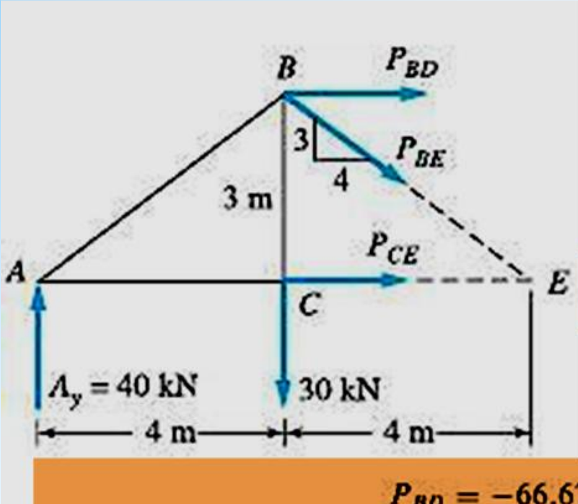
$$\sum F_y = 0 \quad +\uparrow \quad 40 + \frac{3}{5} P_{AB} = 0$$

$$\sum F_x = 0 \quad \overset{+}{\rightarrow} \quad P_{AC} + \frac{4}{5} P_{AB} = 0$$

$$P_{AC} = 53.33 \text{ kN (tension)}$$

$$\sigma_{AC} = \frac{P_{AC}}{A_{AC}} = \frac{53.33 \text{ kN}}{900 \text{ mm}^2} = \frac{53.33 \times 10^3 \text{ N}}{900 \times 10^{-6} \text{ m}^2}$$

$$= 59.3 \times 10^6 \text{ N/m}^2 = 59.3 \text{ MPa (T)} \qquad \textit{Answer}$$



$$\sum M_E = 0 \quad +\circlearrowleft \quad -40(8) + 30(4) - P_{BD}(3) = 0$$

$$P_{BD} = -66.67 \text{ kN} = 66.67 \text{ kN (C)}$$

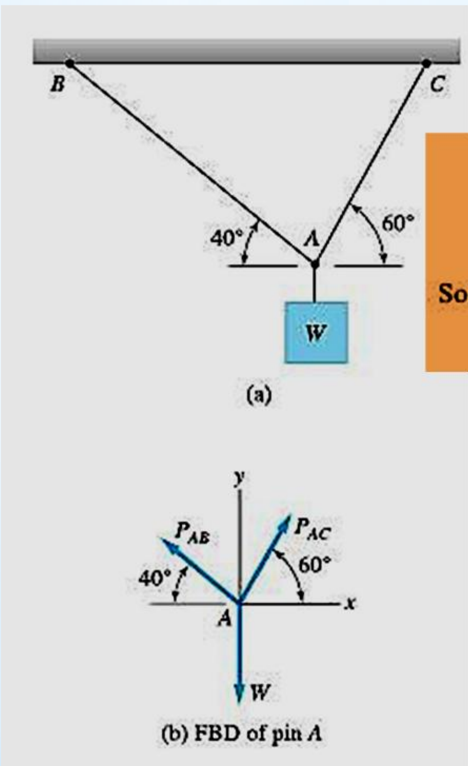
Therefore, the normal stress in member *BD* is

$$\sigma_{BD} = \frac{P_{BD}}{A_{BD}} = \frac{-66.67 \text{ kN}}{900 \text{ mm}^2} = \frac{-66.67 \times 10^3 \text{ N}}{900 \times 10^{-6} \text{ m}^2}$$

$$= -74.1 \times 10^6 \text{ N/m}^2 = 74.1 \text{ MPa (C)} \qquad \textit{Answer}$$

Sample Problem

Figure (a) shows a two-member truss supporting a block of weight W. The cross-sectional areas of the members arc 800 mm2 for AB and 400 mm2 for AC. Determine the maximum safe value of W if the working stresses are 110 MPa for AB and 120 MPa for AC.



(a)

(b) FBD of pin A

$$\sum F_x = 0 \quad \overset{+}{\rightarrow} \quad P_{AC}\cos 60° - P_{AB}\cos 40° = 0$$

$$\sum F_y = 0 \quad +\uparrow \quad P_{AC}\sin 60° + P_{AB}\sin 40° - W = 0$$

Solving simultaneously, we get

$$P_{AB} = 0.5077W \qquad P_{AC} = 0.7779W$$

Design for Normal Stress in Bar AB

$$P_{AB} = (\sigma_w)_{AB}A_{AB}$$

$$0.5077W = (110 \times 10^6\ \mathrm{N/m^2})(800 \times 10^{-6}\ \mathrm{m^2})$$

$$W = 173.3 \times 10^3\ \mathrm{N} = 173.3\ \mathrm{kN}$$

Design for Normal Stress in Bar AC

$$P_{AC} = (\sigma_w)_{AC}A_{AC}$$

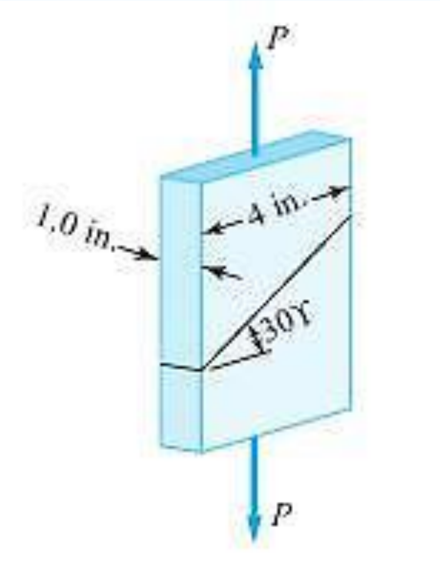
$$0.7779W = (120 \times 10^6\ \mathrm{N/m^2})(400 \times 10^{-6}\ \mathrm{m^2})$$

$$W = 61.7 \times 10^3\ \mathrm{N} = 61.7\ \mathrm{kN}$$

Choose the Correct Answer
The maximum safe value of W is the smaller of the preceding two values—namely.
W = 61.7kN ***Answer***

Sample Problem

The rectangular wood panel is formed by gluing together two boards along the 30-degree scam. Determine the largest axial force P that can be carried safely by the panel if the working stress for the wood is 1120 psi and the normal and shear stresses in the glue are limited to 700 psi and 450 psi respectively.



Design for Working Stress in Wood

$$P = \sigma_w A = 1120(4 \times 1.0) = 4480 \text{ lb}$$

Design for Normal Stress in Glue

$$\sigma = \frac{P}{A}\cos^2\theta$$

$$700 = \frac{P}{(4 \times 1.0)}\cos^2 30^\circ$$

$$P = 3730 \text{lb}$$

Design for Shear Stress in Glue

$$\sigma = \frac{P}{2A}\sin 2\theta$$

$$450 = \frac{P}{2(4 \times 1.0)}\sin 60^\circ$$

$$P = 4160 \text{lb}$$

Choose the Correct Answer

Comparing the above three solutions, we see that the largest safe axial load that can be safely applied is governed by the normal stress in the glue, its value being P = 3730 lb Answer

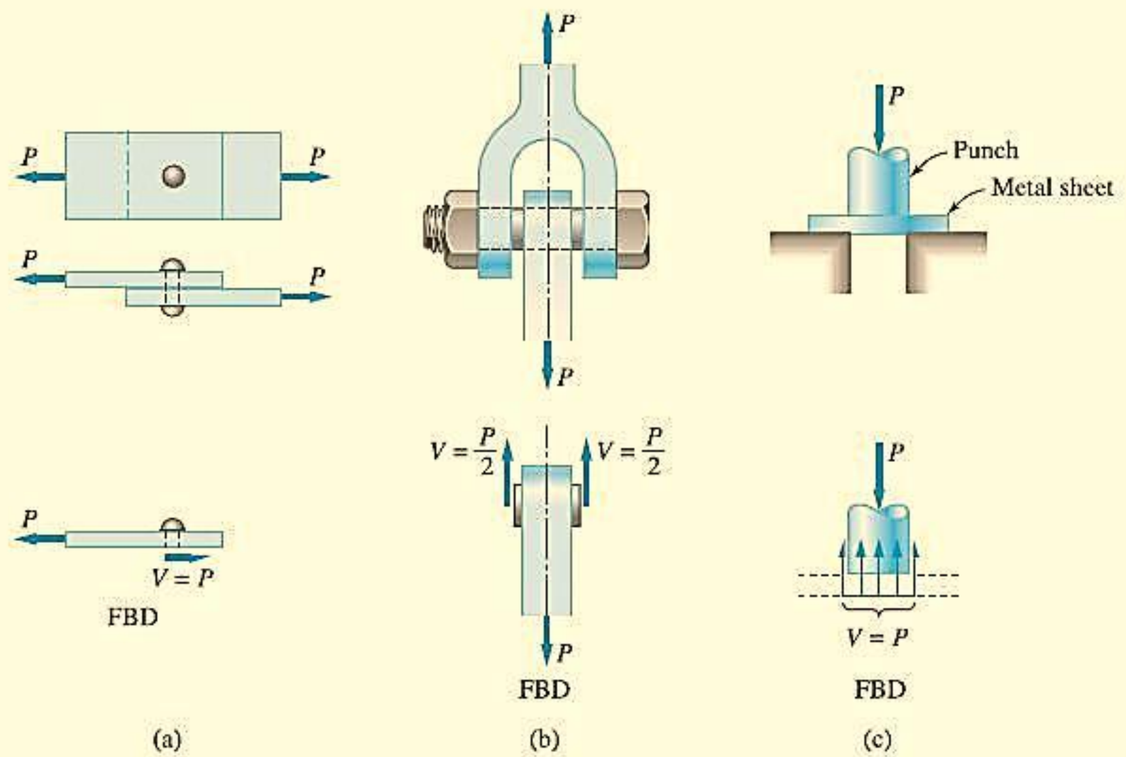
Shear Stress

By definition, normal stress acting on an interior plane is directed perpendicular to that plane.

Shear stress is tangent to the plane on which it acts.

When the shear force V is uniformly distributed over the shear area A, so that the shear stress can be computed as

$$\tau = \frac{V}{A}$$



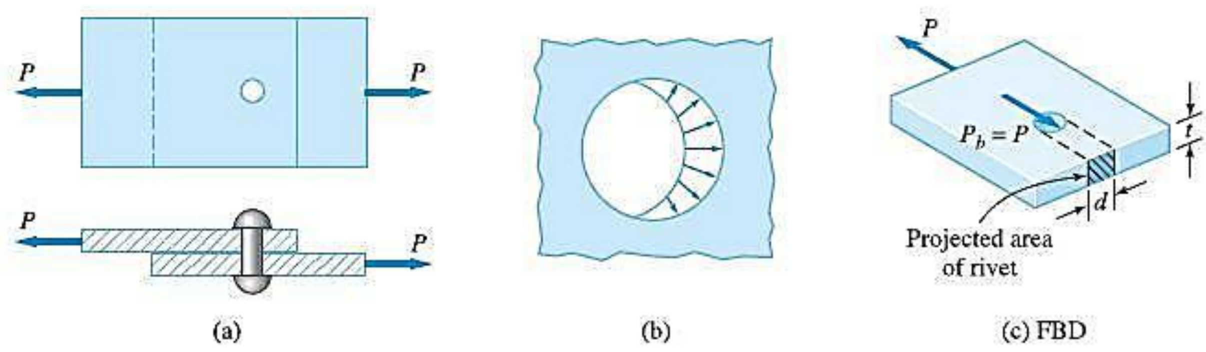
Examples of direct shear: (a) single shear in a rivet; (b) double shear in a bolt; and (c) shear in a metal sheet produced by a punch.

Bearing Stress

If two bodies are pressed against each other, compressive forces are developed on the area of contact. The pressure caused by these surface loads is called bearing stress.

We assume that the bearing stress is uniformly distributed over a reduced area. The reduced area is taken to be the projected area of the rivet.

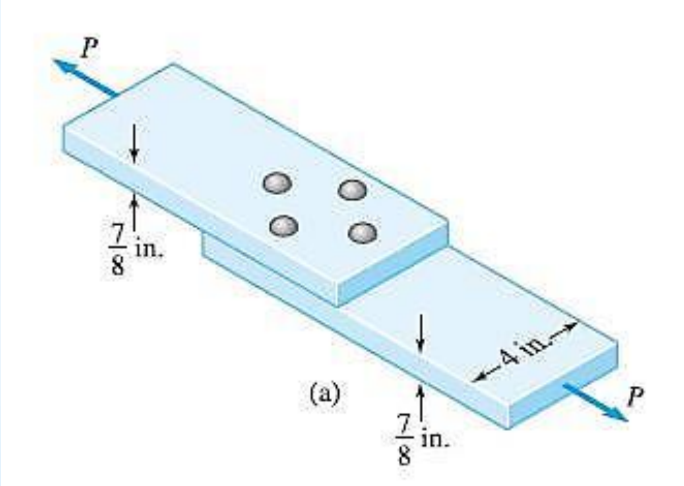
$$\boxed{\sigma_b = \frac{P_b}{A_b} = \frac{P}{td}}$$

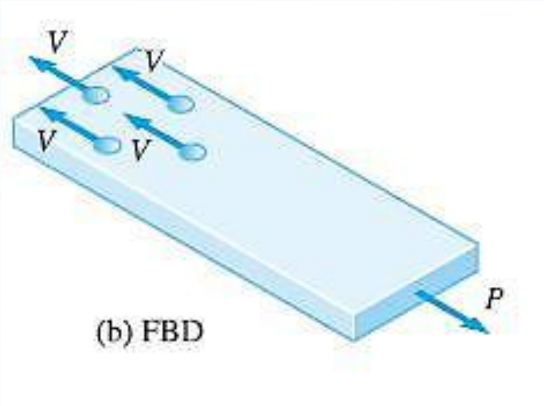


Example of bearing stress: (a) a rivet in a lap joint; (b) bearing stress is not constant; (c) bearing stress caused by the bearing force P_b is assumed to be uniform on projected area td.

Sample Problem

The lap joint is fastened by four rivets of 3/4-in. diameter. Find the maximum load P that can be applied if the working stresses are 14 ksi for shear in the rivet and 18 ksi for bearing in the plate. Assume that the applied load is distributed evenly among the four rivets, and neglect friction between the plates.





The equilibrium condition is V = P/4.

Design for Shear Stress in Rivets

$$V = \tau A$$

$$\frac{P}{4} = (14 \times 10^3)\left[\frac{\pi(3/4)^2}{4}\right]$$

$$P = 24\,700 \text{ lb}$$

Design for Bearing Stress in Plate

$$P_b = \sigma_b t d$$

$$\frac{P}{4} = (18 \times 10^3)(7/8)(3/4)$$

$$P = 47\,300 \text{ lb}$$

P= 24700 lb **Answer**

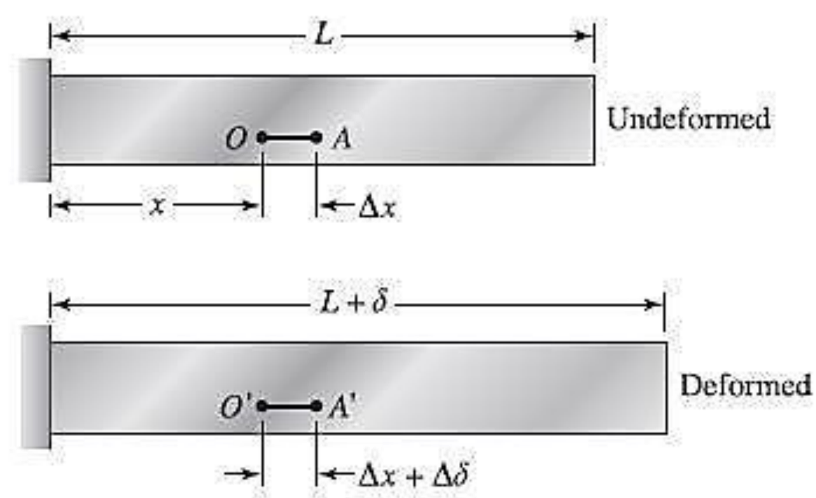
Strain

In general terms, strain is a geometric quantity that measures the deformation of a body.
There are two types of strain: normal strain, which characterizes dimensional changes, and shear strain, which describes distortion (changes in angles).

Axial Deformation. Normal (axial) strain

The normal strain is defined as the elongation per unit length

$$\epsilon = \frac{\delta}{L}$$



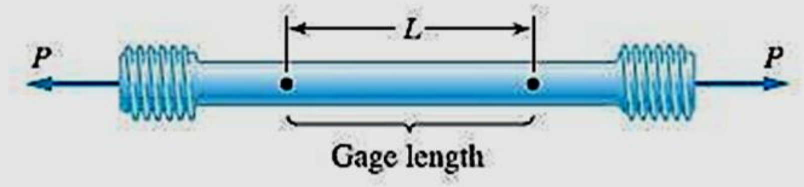
Deformation of a prismatic bar.

Strain in a point

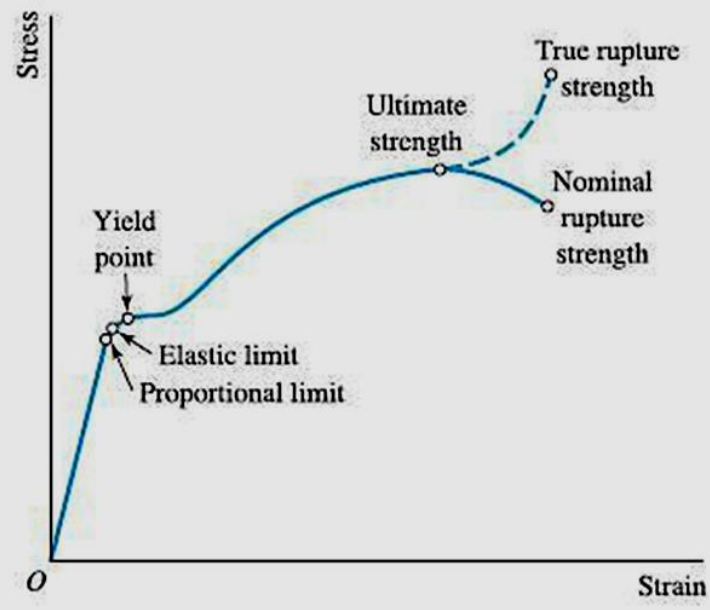
$$\epsilon = \lim_{\Delta x \to 0} \frac{\Delta\delta}{\Delta x} = \frac{d\delta}{dx}$$

$$\delta = \int_0^L d\delta = \int_0^L \epsilon\, dx$$

Stress-Strain Diagram



Specimen used in the standard tension test.



Stress-strain diagram obtained from the standard tension test on a structural steel specimen.

Proportional Limit and Hooke's Law

The stress-strain diagram is a straight line from the origin O to a point called the proportional limit. This plot is a manifestation of Hooke's law:

Stress is proportional to strain

$$\boxed{\sigma = E\epsilon}$$

where E is a material property known as the modulus of elasticity or Young's modulus.

Elastic Limit

A material is said to be elastic if, after being loaded, the material returns to its original shape when the load is removed.
The permanent deformation that remains after the removal of the load is called the permanent set.

Yield Point

The point where the stress-strain diagram becomes almost horizontal is called the yield point, and the corresponding stress is known as the yield stress or yield strength.

Ultimate Stress

The ultimate stress or ultimate strength is the highest stress on the stress-strain curve.

Rupture Stress

The rupture stress or rupture strength is the stress at which failure occurs.

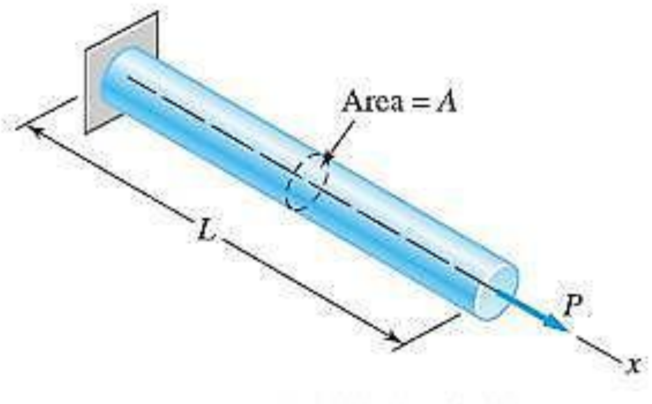
Failed tensile test specimen showing necking, or narrowing, of the cross section.

Working stress and factor of safety

The working stress also called the allowable stress is the maximum safe axial stress used in design. In most designs, the working stress should be limited to values not exceeding the proportional limit so that the stresses remain in the elastic range.
It is customary to base the working stress on either the yield stress or the ultimate stress divided by a suitable number N, called the factor of safety

$$\boxed{\sigma_w = \frac{\sigma_{yp}}{N} \quad \text{or} \quad \sigma_w = \frac{\sigma_{\text{ult}}}{N}}$$

Axially Loaded Bars



Axially loaded bar.

From the Hooke's law

$$\sigma = E\epsilon$$

But $\epsilon = \frac{\delta}{L}$ **. Then**

$$\boxed{\delta = \frac{\sigma L}{E} = \frac{PL}{EA}}$$

If the strain (or stress) in the bar is not uniform, then the axial strain varies with the x-coordinate and the elongation of the bar can be obtained by integration

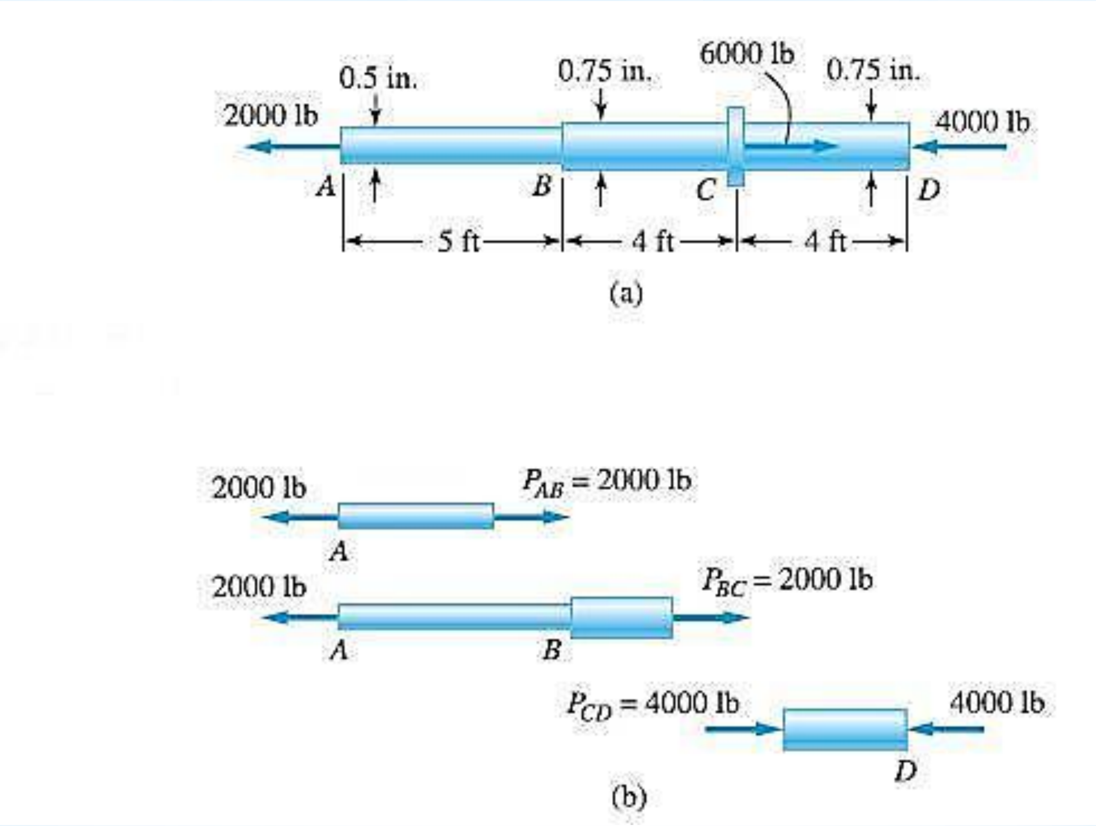
$$\boxed{\delta = \int_0^L \frac{\sigma}{E}\,dx = \int_0^L \frac{P}{EA}\,dx}$$

The magnitude of the internal force P must be found from equilibrium analysis.

Note that a positive (tensile) P results in positive δ (elongation); conversely, a negative P (compression) gives negative δ (shortening).

Sample Problem

The steel propeller shaft ABCD carries the axial loads shown in Fig. Determine the change in the length of the shaft caused by these loads. Use $E = 29 \times 10^6$psi for steel.



$$P_{AB} = P_{BC} = 2000 \text{ lb (T)} \qquad P_{CD} = 4000 \text{ lb (C)}$$

$$\delta = \sum \frac{PL}{EA} = \frac{1}{E}\left[\left(\frac{PL}{A}\right)_{AB} + \left(\frac{PL}{A}\right)_{BC} - \left(\frac{PL}{A}\right)_{CD}\right]$$

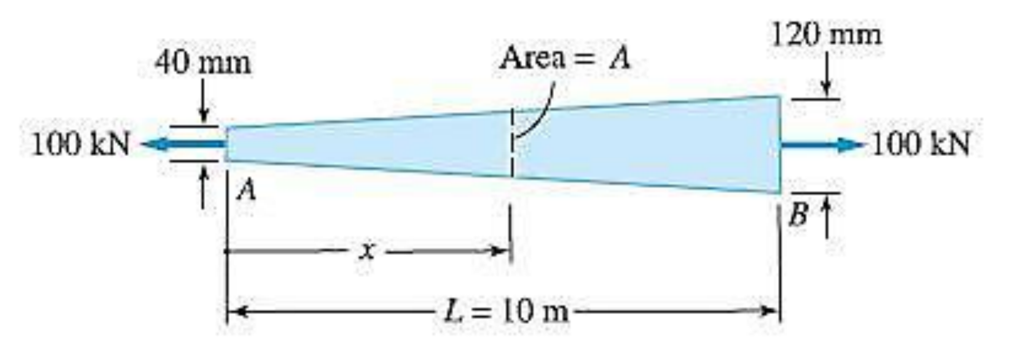
$$= \frac{1}{29 \times 10^6}\left[\frac{2000(5 \times 12)}{\pi(0.5)^2/4} + \frac{2000(4 \times 12)}{\pi(0.75)^2/4} - \frac{4000(4 \times 12)}{\pi(0.75)^2/4}\right]$$

$$= 0.013\,58 \text{ in.} \quad \text{(elongation)}$$

Answer

Sample Problem

The cross section of the 10-m-long flat steel bar AB has a constant thickness of 20 mm, but its width varies as shown in the figure. Calculate the elongation of the bar due to the 100-kN axial load. Use E = 200 GPa for steel.



Equilibrium requires that the internal axial force P = 100 kN is constant along the entire length of the bar. However, the cross-sectional area A of the bar varies with the .x-coordinate.

The cross-sectional areas at A and B are A_A = 20 x 40 = 800 mm^2 and A_B = 20 x 120 = 2400 mm^2. Between A and B the cross-sectional area is a linear function of x:

$$A = A_A + (A_B - A_A)\frac{x}{L} = 800\ \text{mm}^2 + (1600\ \text{mm}^2)\frac{x}{L}$$

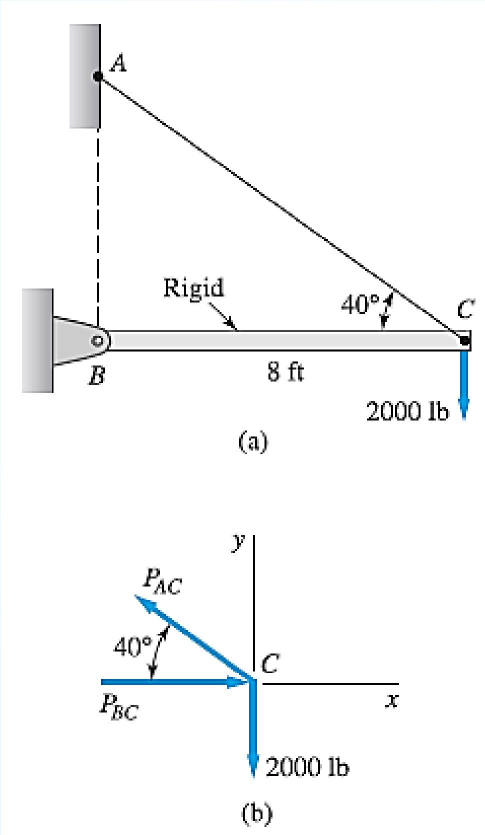
$$\delta = \int_0^L \frac{P}{EA}\,dx = \int_0^{10\ \text{m}} \frac{100\times10^3}{(200\times10^9)[(800+160x)\times10^{-6}]}\,dx$$

$$= 0.5\int_0^{10\ \text{m}} \frac{dx}{800+160x} = \frac{0.5}{160}[\ln(800+160x)]_0^{10}$$

$$= \frac{0.5}{160}\ln\frac{2400}{800} = 3.43\times10^{-3}\ \text{m} = 3.43\ \text{mm} \qquad \textit{Answer}$$

Sample Problem

The rigid bar AC is supported by the steel rod AC of cross-sectional area 0.25 in^2. Find the vertical displacement of point C caused by the 2000-lb load. Use E= 29 x 10^6 psi for steel.

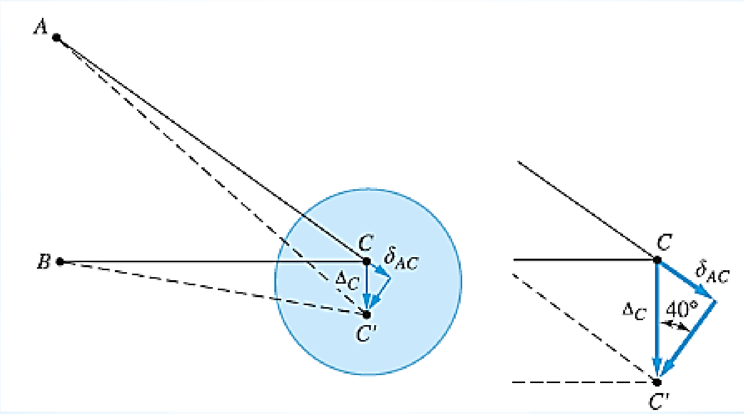


$$\Sigma F_y = 0 \quad +\uparrow \quad P_{AC}\sin 40^\circ - 2000 = 0 \qquad P_{AC} = 3111 \text{ lb}$$

$$L_{AC} = \frac{L_{BC}}{\cos 40^\circ} = \frac{8 \times 12}{\cos 40^\circ} = 125.32 \text{ in.}$$

$$\delta_{AC} = \left(\frac{PL}{EA}\right)_{AC} = \frac{3111(125.32)}{(29 \times 10^6)(0.25)} = 0.053\,78 \text{ in.} \quad \text{(elongation)}$$

The geometric relationship between δ_{AC} and the displacement Δ_C of C is illustrated in the displacement diagram.

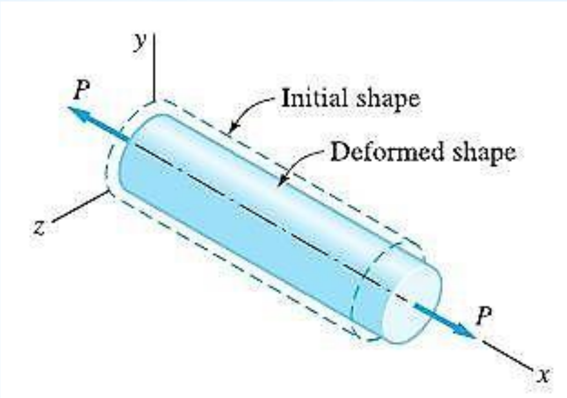


$$\Delta_C = \frac{\delta_{AC}}{\sin 40^\circ} = \frac{0.053\,78}{\sin 40^\circ} = 0.0837 \text{ in.}$$

Generalized Hooke's Law

Uniaxial loading; Poisson's ratio

Experiments show that when a bar is stretched by an axial force, there is a contraction in the transverse dimensions.



Transverse dimensions contract as the bar is stretched by an axial force P.

The transverse strain is uniform throughout the cross section and is the same in any direction in the plane of the cross section.

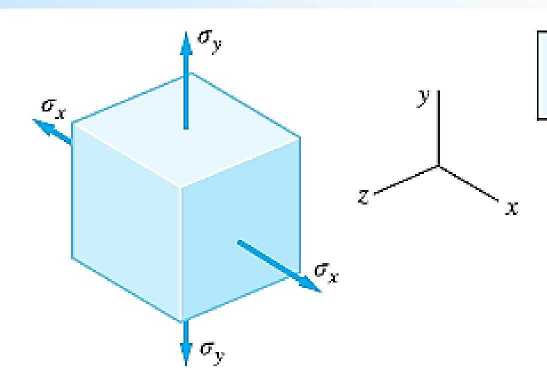
$$\epsilon_y = \epsilon_z = -\nu\epsilon_x$$

Where υ - Poisson's ratio is a dimensionless quantity that ranges between 0.25 and 0.33 for metals.

The generalized Hooke's law for uniaxial loading:

$$\epsilon_x = \frac{\sigma_x}{E} \qquad \epsilon_y = \epsilon_z = -\nu\frac{\sigma_x}{E}$$

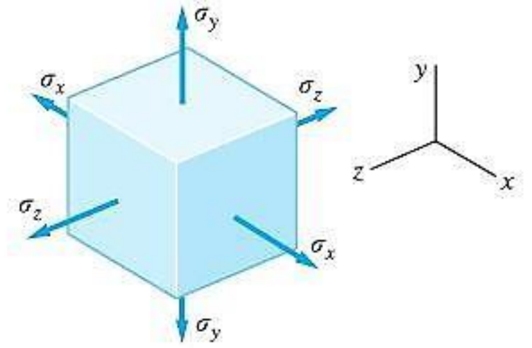
Biaxial Loading



$$\epsilon_x = \frac{1}{E}(\sigma_x - \nu\sigma_y) \quad \epsilon_y = \frac{1}{E}(\sigma_y - \nu\sigma_x) \quad \epsilon_z = -\frac{\nu}{E}(\sigma_x + \sigma_y)$$

$$\sigma_x = \frac{(\epsilon_x + \nu\epsilon_y)E}{1-\nu^2} \qquad \sigma_y = \frac{(\epsilon_y + \nu\epsilon_x)E}{1-\nu^2}$$

Triaxial Loading



Stresses acting on a material element in triaxial loading.

$$\epsilon_x = \frac{1}{E}[\sigma_x - \nu(\sigma_y + \sigma_z)]$$

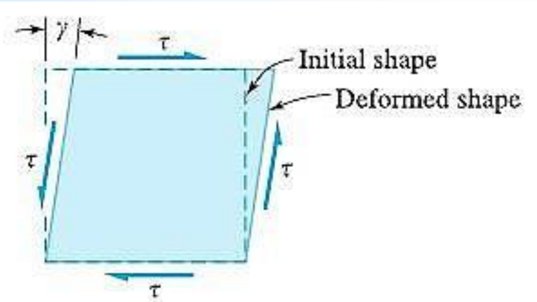
$$\epsilon_y = \frac{1}{E}[\sigma_y - \nu(\sigma_z + \sigma_x)]$$

$$\epsilon_z = \frac{1}{E}[\sigma_z - \nu(\sigma_x + \sigma_y)]$$

Shear loading

Shear stress causes the deformation shown in Fig. The lengths of the sides of the element do not change, but the element undergoes a distortion from a rectangle to a parallelogram. The shear strain, which measures the amount of distortion, is the angle γ. It can be shown that the relationship between shear stress τ and shear strain γ is linear within the elastic range, that is,

$$\boxed{\tau = G\gamma}$$



Deformation of a material element caused by shear stress.

which is Hooke's law for shear. The material constant *G* is called the shear modulus of elasticity (or simply shear modulus), or the modulus of rigidity.

$$\boxed{G = \frac{E}{2(1+\nu)}}$$

EXTENTION AND COMPRESSION

Problem

Steel rod (Young's modulus $E = 2 \cdot 10^4$ $\kappa N/cm^2$) is under the action of axial forces *P* and *2P*. Draw longitudinal forces *N* and normal stresses σ_z diagrams. Analyze strength of a rod if working stress is $[\sigma]$=16 $\kappa N/cm^2$. Determine elongation of the rod Δl.

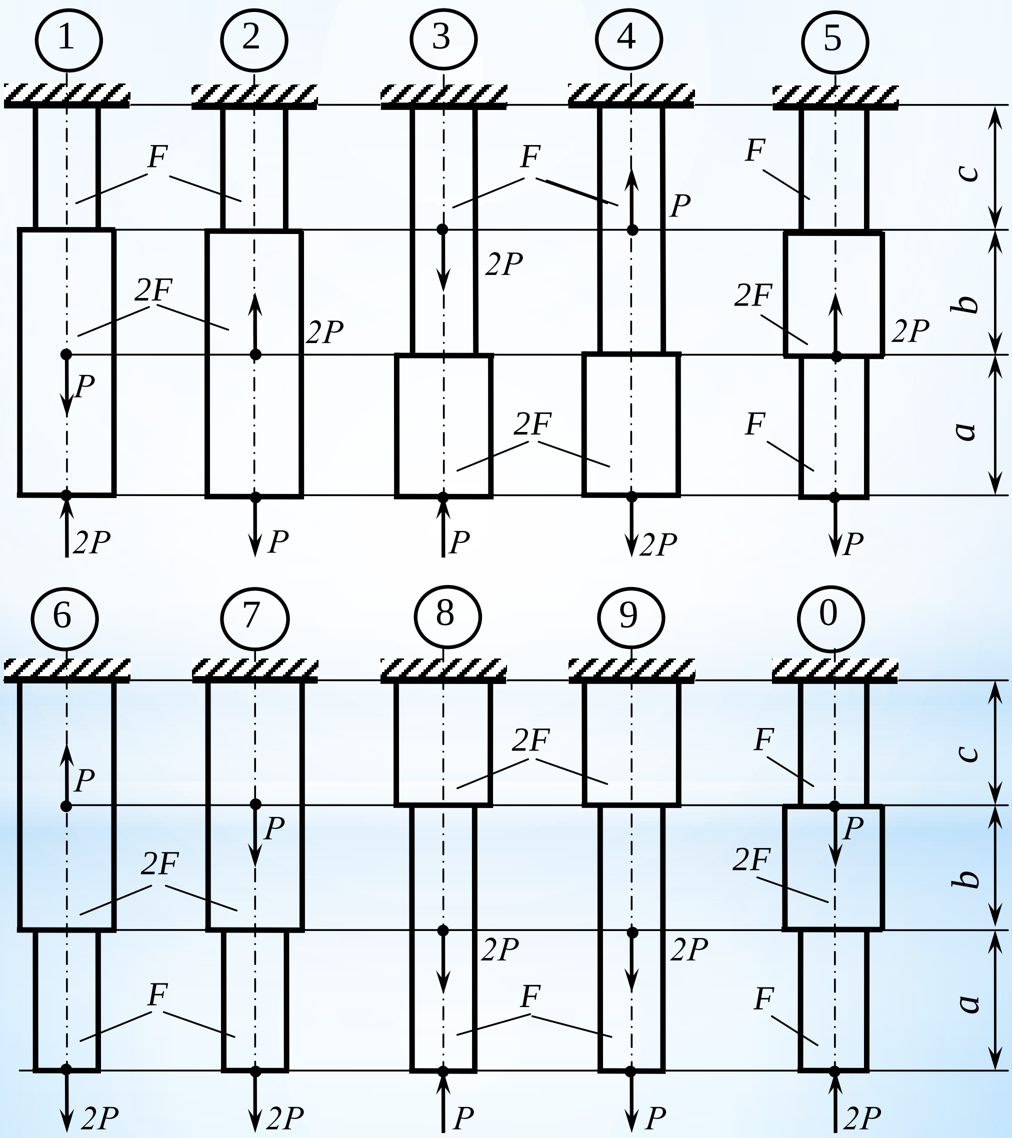


Fig.1

DATA

№	F, cm²	a, m	b, m	c, m	P, кN
1	2,0	1,2	1,4	1,6	11
2	2,2	1,4	1,6	1,4	12
3	2,4	1,8	1,6	1,2	13
4	2,6	1,6	2,0	1,0	14
5	2,8	2,0	1,8	1,2	15
6	3,0	2,2	1,6	1,4	16
7	3,2	2,4	1,4	1,6	17
8	3,4	2,6	1,2	1,8	18
9	3,6	2,8	1,0	1,4	19
10	3,8	2,4	1,6	1,2	20
11	2,2	1,6	1,4	1,2	10
12	2,4	1,6	1,8	1,0	11
13	2,6	2,0	1,8	1,0	13
14	2,8	1,8	2,0	1,4	14

Problem sample

Given: $E = 2 \cdot 10^4$ $\kappa N/cm^2$, $a = 200$ cm; $b = 150$ cm, $c = 100$ cm F=10 cm^2, $P_1 = 100$ κN и $P_2 = 300$ κN, [σ]=16 $\kappa N/cm^2$ (Fig. 2).

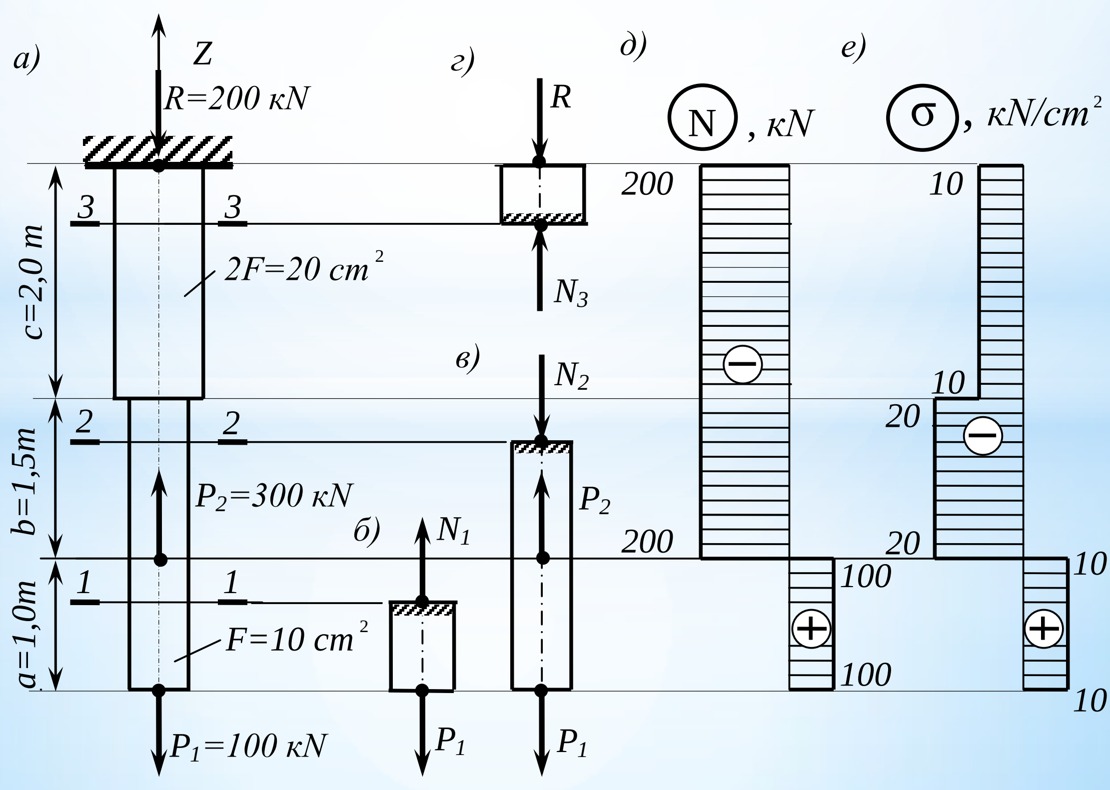


Fig 2

Solution.

1. Define reaction R in rigid clamp.

$\sum Z = 0: \quad -R + P_2 - P_1 = 0; \quad R = P_2 - P_1 = 300 - 100 = 200\ \kappa N.$

2. Draw N diagram.

$$N_1 = P_1 = 100\ \kappa N.$$

$$N_2 = P_2 - P_1 = 300 - 100 = 200\ \kappa N.$$

$$N_3 = R = 200\ \kappa N.$$

3. Draw σ_z diagram.

$$\sigma_{z_k} = N_k / F_k\ ,$$

$$\sigma_{z_1} = \frac{N_1}{F_1} = \frac{N_1}{F} = +\frac{100}{10} = +10\ \kappa N/cm^2,$$

$$\sigma_{z_2} = \frac{N_2}{F_2} = \frac{N_2}{F} = -\frac{200}{10} = -20\ \kappa N/cm^2,$$

$$\sigma_{z_3} = \frac{N_3}{F_3} = \frac{N_3}{2F} = -\frac{200}{20} = -10\ \kappa N/cm^2.$$

4. Analyze strength of the rod.

Strength condition is $\sigma_z^{\max} \le [\sigma]$. In our case

$$\sigma_z^{\max} = \left|\sigma_{z_2}\right| = 20\ \ \kappa N/cm^2 > [\sigma] = 16\ \kappa N/cm^2,$$

Then the area of the second segment is to be increased:

$$\boldsymbol{F_2 \ge \left|N_2\right| / [\sigma] = 200/16 = 12{,}5}\ \ cm^2.$$

Take on a second segment $F_2 = 12{,}5\ cm^2$.

5. Calculate elongation of the rod Δl.

$$\Delta l = \sum_k \frac{N_k l_k}{EF_k},$$

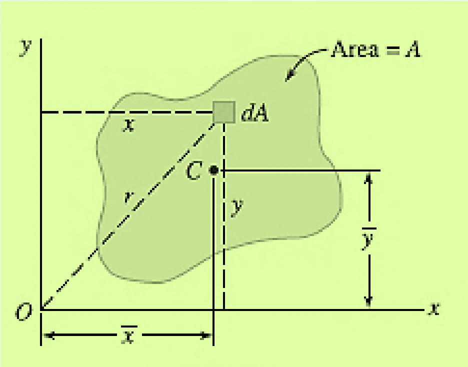
$$\Delta l = \frac{N_1 l_1}{EF_1} + \frac{N_2 l_2}{EF_2} + \frac{N_3 l_3}{EF_3} = \frac{100 \cdot 100}{2 \cdot 10^4 \cdot 10} - \frac{200 \cdot 150}{2 \cdot 10^4 \cdot 12{,}5} - \frac{200 \cdot 200}{2 \cdot 10^4 \cdot 20} = -0{,}17\ \text{см}.$$

Hence, the length of the rod decreases $1{,}7$ мм.

Properties of Plane Areas

First Moments of Area; Centroid

The first moments of a plane area *A* about the *x*- and *y*-axes are defined as



$$Q_x = \int_A y\,dA \qquad Q_y = \int_A x\,dA$$

,where *dA* is an infinitesimal element of *A* located at *(x,y),* as shown in Fig.

The centroid *C* of the area is defined as the point in the *xy*-plane that has the coordinates

$$\bar{x} = \frac{Q_y}{A} \qquad \bar{y} = \frac{Q_x}{A}$$

.

The following are useful properties of the first moments of area:

- **If the origin of the *xy*-coordinate system is the centroid of the area (in which case** $\bar{x} = \bar{y} = 0$), **then** $Q_x = Q_y = 0.$
- **Whenever the area has an axis of symmetry, the centroid of the area will lie on that axis.**

Second Moments of Area

We define the second moments of a plane area *A* with respect to the *xy*-axes by

$$I_x = \int_A y^2\,dA \qquad I_y = \int_A x^2\,dA \qquad I_{xy} = \int_A xy\,dA$$

.

The integrals *Ix* and *Iy* are commonly called the moments of inertia, whereas *Ixy* is known as the product of inertia.

We define the polar moment of inertia of an area about point *O* (strictly speaking, about an axis through *O*, perpendicular to the plane of the area) by

$$J_O = \int_A r^2 \, dA$$

where *r* is the distance from *O* to the area element *dA*.

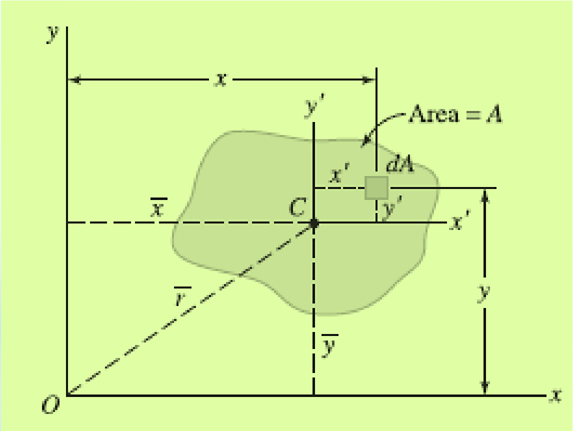
***The polar moment of inertia of an area about a point O equals the sum of the moments of inertia of the area about two perpendicular axes that intersect at O*.**

$$J_O = I_x + I_y$$

Parallel-Axis Theorems

The parallel-axis theorem for the moment of inertia of an area

$$I_x = \bar{I}_x + A\bar{y}^2$$



The parallel-axis theorem for products of inertia

$$I_{xy} = \bar{I}_{xy} + A\bar{x}\bar{y}$$

The parallel-axis theorem for the polar moment of inertia

$$J_O = \bar{J}_C + A\bar{r}^2$$

where $\bar{r} = \sqrt{\bar{x}^2 + \bar{y}^2}$ is the distance between O and C

Radii of Gyration

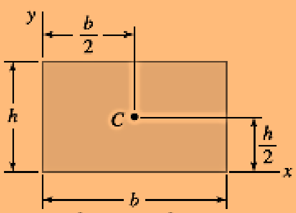
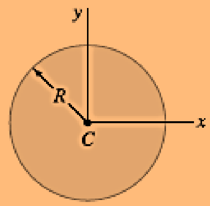
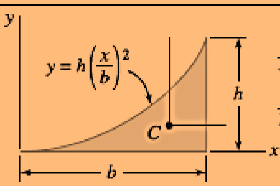
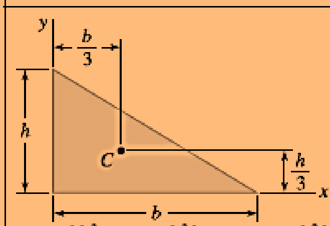
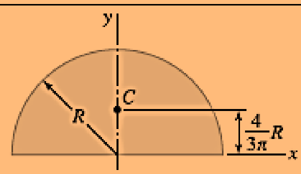
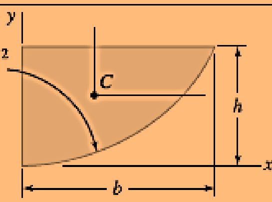
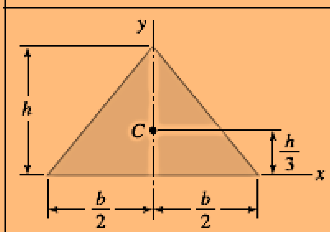
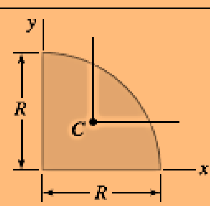
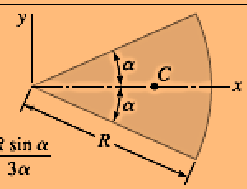
The radii of gyration about the *x*-axis, the *y*-axis, and the point *O* are defined as

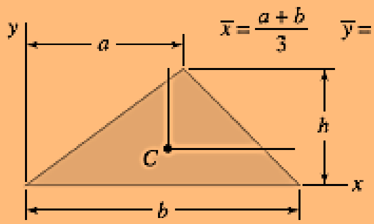
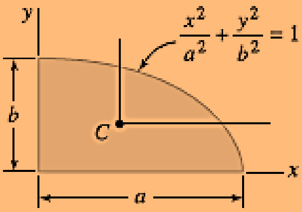
$$k_x = \sqrt{\frac{I_x}{A}} \qquad k_y = \sqrt{\frac{I_y}{A}} \qquad k_O = \sqrt{\frac{J_O}{A}}$$

The radii of gyration are related by

$$k_O^2 = k_x^2 + k_y^2$$

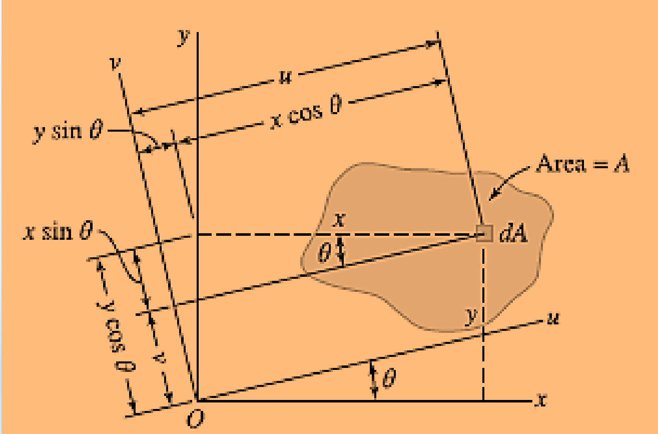
Review of Properties of Plane Areas

Rectangle	Circle	Half parabolic complement
 $\bar{I}_x = \frac{bh^3}{12}$ $\bar{I}_y = \frac{b^3h}{12}$ $\bar{I}_{xy} = 0$ $I_x = \frac{bh^3}{3}$ $I_y = \frac{b^3h}{3}$ $I_{xy} = \frac{b^2h^2}{4}$	 $I_x = I_y = \frac{\pi R^4}{4}$ $I_{xy} = 0$	 $\bar{x} = \frac{3b}{4}$ $\bar{y} = \frac{3h}{10}$ $\bar{I}_x = \frac{37bh^3}{2100}$ $I_x = \frac{bh^3}{21}$ $\bar{I}_y = \frac{b^3h}{80}$ $I_y = \frac{b^3h}{5}$ $\bar{I}_{xy} = \frac{b^2h^2}{120}$ $I_{xy} = \frac{b^2h^2}{12}$
Right triangle	**Semicircle**	**Half parabola**
 $\bar{I}_x = \frac{bh^3}{36}$ $\bar{I}_y = \frac{b^3h}{36}$ $\bar{I}_{xy} = -\frac{b^2h^2}{72}$ $I_x = \frac{bh^3}{12}$ $I_y = \frac{b^3h}{12}$ $I_{xy} = \frac{b^2h^2}{24}$	 $\bar{I}_x = 0.1098R^4$ $\bar{I}_{xy} = 0$ $I_x = I_y = \frac{\pi R^4}{8}$ $I_{xy} = 0$	$y = h\left(\frac{x}{b}\right)^2$ $\bar{x} = \frac{3b}{8}$ $\bar{y} = \frac{3h}{5}$  $\bar{I}_x = \frac{8bh^3}{175}$ $I_x = \frac{2bh^3}{7}$ $\bar{I}_y = \frac{19b^3h}{480}$ $I_y = \frac{2b^3h}{15}$ $\bar{I}_{xy} = \frac{b^2h^2}{60}$ $I_{xy} = \frac{b^2h^2}{6}$
Isosceles triangle	**Quarter circle**	**Circular sector**
 $\bar{I}_x = \frac{bh^3}{36}$ $\bar{I}_y = \frac{b^3h}{48}$ $\bar{I}_{xy} = 0$ $I_x = \frac{bh^3}{12}$ $I_{xy} = 0$	$\bar{x} = \frac{4R}{3\pi}$ $\bar{y} = \frac{4R}{3\pi}$  $\bar{I}_x = \bar{I}_y = 0.054\,88R^4$ $I_x = I_y = \frac{\pi R^4}{16}$ $\bar{I}_{xy} = -0.016\,47R^4$ $I_{xy} = \frac{\pi R^4}{8}$	 $\bar{x} = \frac{2R \sin \alpha}{3\alpha}$ $I_x = \frac{R^4}{8}(2\alpha - \sin 2\alpha)$ $I_y = \frac{R^4}{8}(2\alpha + \sin 2\alpha)$ $I_{xy} = 0$

Triangle	Quarter ellipse
	
$\bar{x} = \frac{a+b}{3}$ $\quad \bar{y} = \frac{h}{3}$	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ $\quad \bar{x} = \frac{4a}{3\pi}$ $\quad \bar{y} = \frac{4b}{3\pi}$
$\bar{I}_x = \frac{bh^3}{36}$ $\qquad I_x = \frac{bh^3}{12}$	$\bar{I}_x = 0.05488ab^3$ $\qquad I_x = \frac{\pi ab^3}{16}$
$\bar{I}_y = \frac{bh}{36}(a^2 - ab + b^2)$ $\qquad I_y = \frac{bh}{12}(a^2 + ab + b^2)$	$\bar{I}_y = 0.05488a^3b$ $\qquad I_y = \frac{\pi a^3 b}{16}$
$\bar{I}_{xy} = \frac{bh^2}{72}(2a - b)$ $\qquad I_{xy} = \frac{bh^2}{24}(2a + b)$	$\bar{I}_{xy} = -0.01647a^2b^2$ $\qquad I_{xy} = \frac{a^2b^2}{8}$

Transformation of Second Moments of Area

In general, the values of Ix, Iy, and Ixy for a plane area depend on the location of the origin of the coordinate system and the orientation of the xy-axes. In the previous section, we reviewed the effect of translating the coordinate axes (parallel-axis theorem). Here the changes caused by rotating the coordinate axes are given.



$$I_u = \frac{I_x + I_y}{2} + \frac{I_x - I_y}{2}\cos 2\theta - I_{xy}\sin 2\theta$$

$$I_v = \frac{I_x + I_y}{2} - \frac{I_x - I_y}{2}\cos 2\theta + I_{xy}\sin 2\theta$$

$$I_{uv} = \frac{I_x - I_y}{2}\sin 2\theta + I_{xy}\cos 2\theta$$

Principal Moments of Inertia and Principal Axes

The axes for which I_x is max, I_y is min and I_{xy}=0 are called principal axes. The respective moments are called principal moments of inertia.

The expression for the principal moments of inertia:

$$\left.\begin{matrix} I_1 \\ I_2 \end{matrix}\right\} = \frac{I_x + I_y}{2} \pm \sqrt{\left(\frac{I_x - I_y}{2}\right)^2 + I_{xy}^2}$$

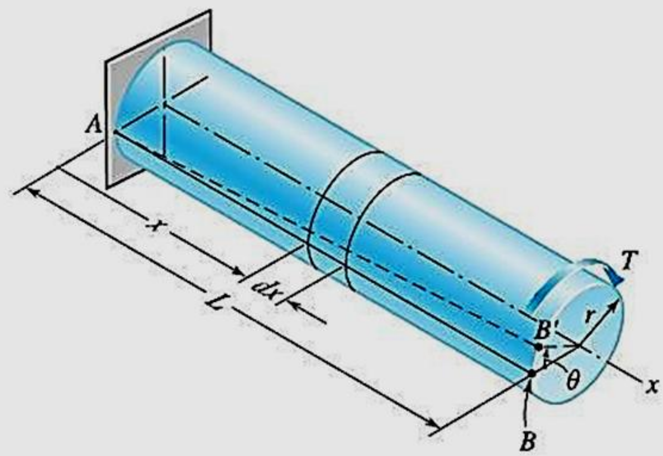
$$\tan 2\theta = -\frac{2I_{xy}}{I_x - I_y}$$

Torsion of Circular Shafts

In many engineering applications, members are required t carry torsional loads. Consider the torsion of circular shaft

Because a circular cross section is an efficient shape for resisting torsional loads, circular shafts are commonly used to transmit power in rotating machinery.

Simplifying assumptions



Deformation of a circular shaft caused by the torque T. The initially straight line AB deforms into a helix.

During the deformation, the cross sections are not distorted in any manner—they remain plane, and the radius r does not change.

The length L of the shaft remains constant.

Based on these observations, we make the following assumptions:

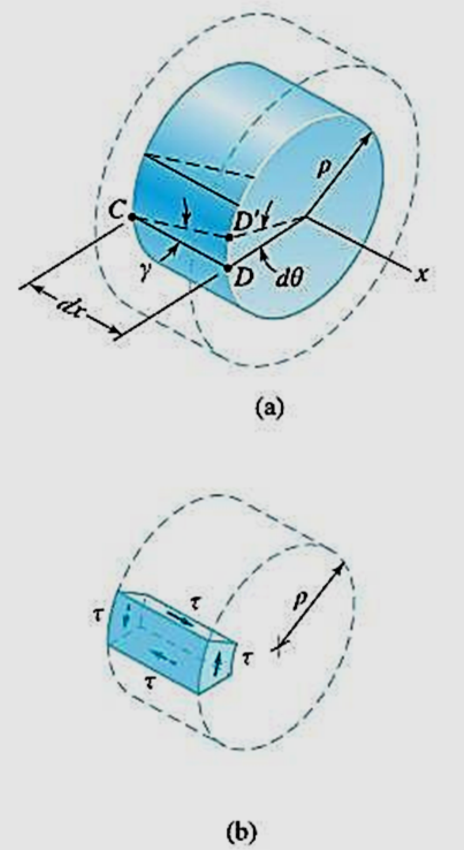
• Circular cross sections remain plane (do not warp) and perpendicular to the axis of the shaft.

• Cross sections do not deform (there is no strain in the plane of the cross section).

• The distances between cross sections do not change (the axial normal strain is zero).

Each cross section rotates as a rigid entity about the axis of the shaft.

Compatibility



(a) Shear strain of a material element caused by twisting of the shaft; (b) the corresponding shear stress.

Angle γ is a shear strain of the element

$$\overline{DD'} = \rho\, d\theta = \gamma\, dx$$

$$\gamma = \frac{d\theta}{dx}\rho$$

The quantity $\frac{d\theta}{dx}$ is the angle of twist per unit length.

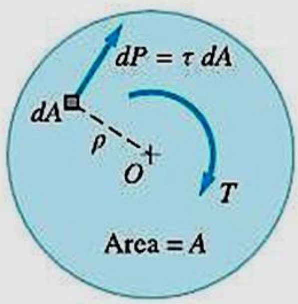
From the Hooke's law

$$\tau = G\gamma = G\frac{d\theta}{dx}\rho$$

The shear stress varies linearly with the radial distance ρ from the axis of

Equilibrium

For the shaft to be in equilibrium, the resultant of the shear stress acting on a cross section must be equal to the internal torque *T* acting on that cross section.



Calculating the resultant of the shear stress acting on the cross section. Resultant is a couple equal to the internal torque T.

$$dP = \tau\, dA = G(d\theta/dx)\rho\, dA$$

The moment (torque) of *dP* about the center *O* is

$$\rho\, dP = G(d\theta/dx)\rho^2\, dA.$$

$$\int_A \rho\, dP = T \qquad G\frac{d\theta}{dx}\int_A \rho^2\, dA = T$$

$$\int_A \rho^2\, dA = J$$ **- polar moment of inertia**

Angle of twist

$$\frac{d\theta}{dx} = \frac{T}{GJ}$$

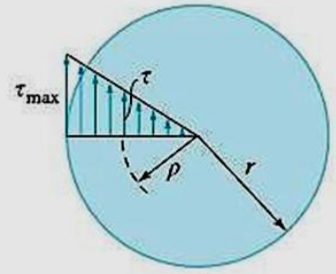
$$\boxed{\theta = \int_0^L d\theta = \int_0^L \frac{T}{GJ}\, dx}$$

T=const

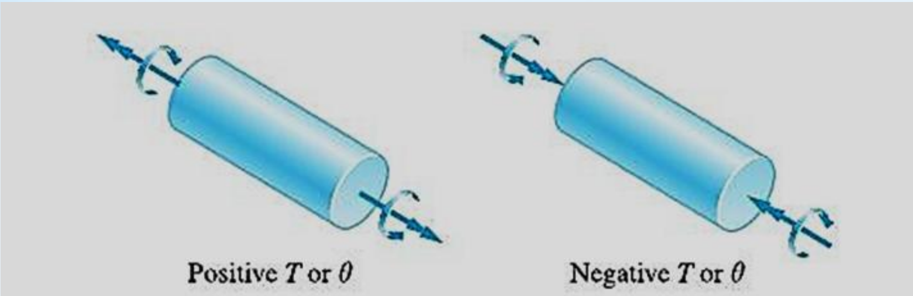
$$\boxed{\theta = \frac{TL}{GJ}}$$

Torsion formulas

$$\boxed{\tau = \frac{T\rho}{J}}$$

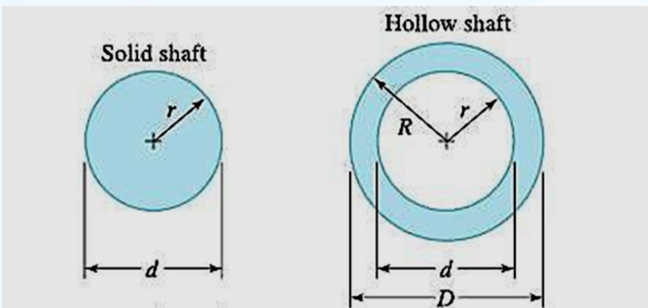


Distribution of shear stress along the radius of a circular shaft.



Sign conventions for torque T and angle of twist θ.

$$\tau_{max} = \frac{Tr}{J}$$



$$J = \frac{\pi r^4}{2} = \frac{\pi d^4}{32} \qquad J = \frac{\pi}{2}(R^4 - r^4) = \frac{\pi}{32}(D^4 - d^4)$$

Polar moments of inertia of circula

$$\text{Solid shaft:} \quad \tau_{max} = \frac{2T}{\pi r^3} = \frac{16T}{\pi d^3}$$

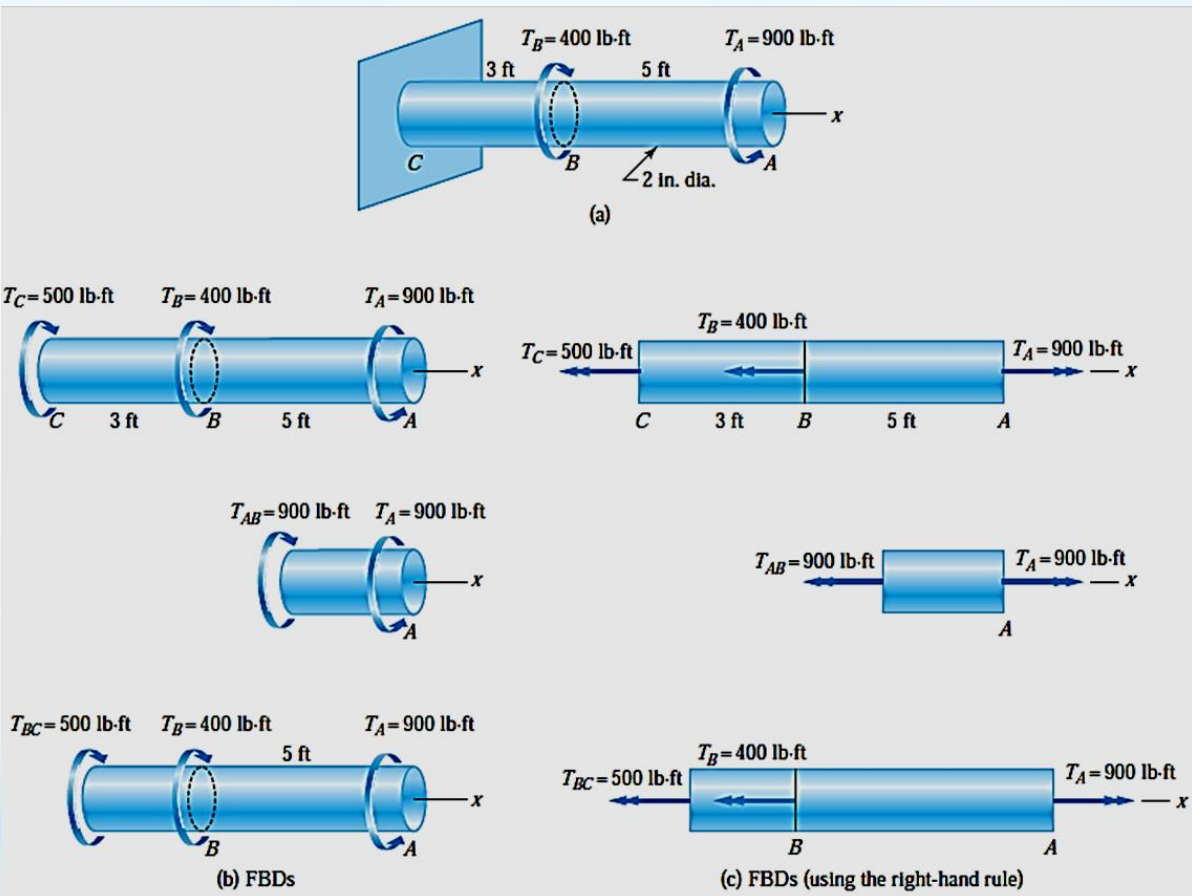
$$\text{Hollow shaft:} \quad \tau_{max} = \frac{2TR}{\pi(R^4 - r^4)} = \frac{16TD}{\pi(D^4 - d^4)}$$

Statically indeterminate problems

- Draw the required free-body diagrams and write the equations of equilibrium.
- Derive the compatibility equations from the restrictions imposed on the angles of twist.
- Use the torque-twist relationships to express the angles of twist in the compatibility equations in terms of the torques.
- Solve the equations of equilibrium and compatibility for the torques.

Sample Problem

A 2-in.-diameter solid steel cylinder is built into the support at *C* and subjected to the torques T_A and T_B. Determine the maximum shear stresses in segments *AB* and *BC* of the cylinder, and compute the angle of rotation of end *A*. Use *G* = 12 x 10^6 psi for steel.



By making use of section method determine the torque in each of the two segments of the cylinder:

$$T_{AB} = 900\ lb.ft, T_{BC} = 500\ lb.ft, T_C = 500\ lb.ft$$

$$(\tau_{\max})_{AB} = \frac{T_{AB} r}{J} = \frac{(900 \times 12)(1.0)}{1.5708} = 6880 \text{ psi}$$

$$(\tau_{\max})_{BC} = \frac{T_{BC} r}{J} = \frac{(500 \times 12)(1.0)}{1.5708} = 3820 \text{ psi}$$

The rotation of end A of the cylinder is obtained by summing the angles of twist of the two segments:

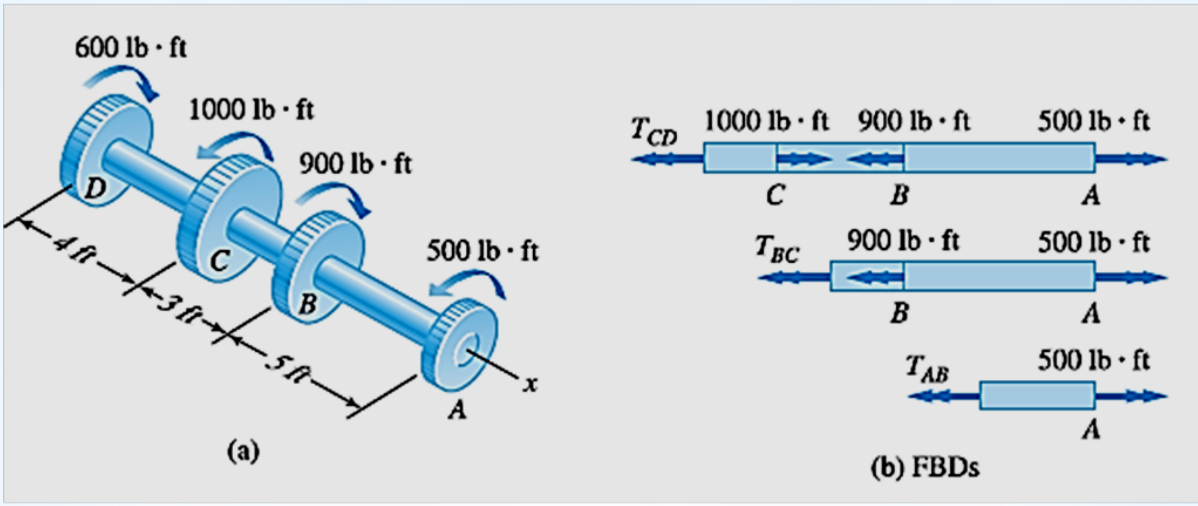
$$\theta_A = \theta_{A/B} + \theta_{B/C}$$

$$\theta_A = \frac{T_{AB} L_{AB} + T_{BC} L_{BC}}{GJ} = \frac{(900 \times 12)(5 \times 12) + (500 \times 12)(3 \times 12)}{(12 \times 10^6)(1.5708)}$$
$$= 0.045\,84 \text{rad} = 2.63^\circ$$

The positive result indicates that the rotation vector of *A* is in the positive .*x*-direction: that is, θ_A is directed counterclockwise when viewed from *A* toward *C*.

Sample Problem

The four rigid gears, loaded as shown in Fig.(a), are attached to a 2-in.-diameter steel shaft. Compute the angle of rotation of gear *A* relative to gear *D*. Use G = 12 x 10^6 psi for the shaft.



It is convenient to represent the torques as vectors (using the right-hand rule) on the FBDs in Fig. (b). We assume that the internal torques are positive according to the sign convention introduced earlier (positive torque vectors point away from the cross section). Applying the equilibrium condition to each FBD, we obtain

$$500 - 900 + 1000 - T_{CD} = 0$$

$$500 - 900 - T_{BC} = 0$$

$$500 - T_{AB} = 0$$

which yield

$$T_{AB} = 500 \text{ lb}\cdot\text{ft} \qquad T_{BC} = -400 \text{ lb}\cdot\text{ft} \qquad T_{CD} = 600 \text{ lb}\cdot\text{ft}$$

The rotation of gear *A* relative to gear *D* can be viewed as the rotation of gear *A* if gear *D* were fixed. This rotation is obtained by summing the angles of twist of the three segments:

$$\theta_{A/D} = \frac{T_{AB}L_{AB} + T_{BC}L_{BC} + T_{CD}L_{CD}}{GJ}$$

$$= \frac{(500 \times 12)(5 \times 12) - (400 \times 12)(3 \times 12) + (600 \times 12)(4 \times 12)}{[\pi(2)^4/32](12 \times 10^6)}$$

$$= 0.028\,27 \text{ rad} = 1.620^\circ \qquad \textit{Answer}$$

Sample Problem

A solid steel shaft in a rolling mill transmits 20 kW of powe at 2 Hz. Determine the smallest safe diameter of the shaft i the shear stress is not to exceed 40 MPa and the angle of twist is limited to 6° in a length of 3 m. Use G = 83 GPa.

This problem illustrates a design that must possess sufficient strength as well as rigidity.

Determine the torque:

$$T = \frac{\mathscr{P}}{2\pi f} = \frac{20 \times 10^3}{2\pi(2)} = 1591.5\ \mathrm{N \cdot m}$$

Satisfy the strength condition:

$$\tau_{\max} = \frac{16T}{\pi d^3} \qquad 40 \times 10^6 = \frac{16(1591.5)}{\pi d^3}$$

which yields $d = 58.7 \times 10^{-3}$ m $= 58.7$ mm.

Satisfy the requirement of rigidity:

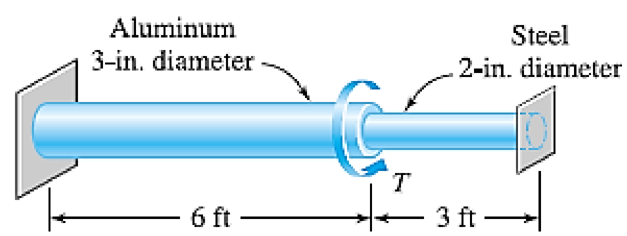
$$\theta = \frac{TL}{GJ} \qquad 6\left(\frac{\pi}{180}\right) = \frac{1591.5(3)}{(83 \times 10^9)(\pi d^4/32)}$$

from which we obtain $d = 48.6 \times 10^{-3}$ m $= 48.6$ mm.

To satisfy both strength and rigidity requirements, we must choose the larger diameter namely,

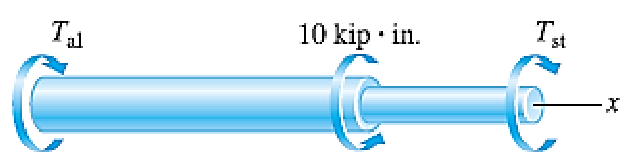
Sample Problem

The shaft consists of a 3-in.-diameter aluminum segment that is rigidly joined to a 2-in.-diameter steel segment. The ends of the shaft are attached to rigid supports. Calculate the maximum shear stress developed in each segment when the torque *T* = 10kip•in. is applied. Use *G* = 4 x 10^6 psi for aluminum and *G* = 12 x 10^6 psi for steel.



(a)

From the FBD of the entire shaft in Fig. (b), the equilibrium equation is



(b) FBD

$$\Sigma M_x = 0 \qquad (10 \times 10^3) - T_{st} - T_{al} = 0$$

Compatibility. **A second relationship between the torques is obtained by noting that the right end of the aluminum segment must rotate through the same angle as the left end of the steel segment. Then**

$$\left(\frac{TL}{GJ}\right)_{st} = \left(\frac{TL}{GJ}\right)_{al}$$

$$\frac{T_{st}(3 \times 12)}{(12 \times 10^6)\frac{\pi}{32}(2)^4} = \frac{T_{al}(6 \times 12)}{(4 \times 10^6)\frac{\pi}{32}(3)^4}$$

Whence

$$T_{al} = 4576 \text{ lb} \cdot \text{in.} \qquad T_{st} = 5424 \text{ lb} \cdot \text{in.}$$

$$(\tau_{max})_{al} = \left(\frac{16T}{\pi d^3}\right)_{al} = \frac{16(4576)}{\pi(3)^3} = 863 \text{ psi}$$

$$(\tau_{max})_{st} = \left(\frac{16T}{\pi d^3}\right)_{st} = \frac{16(5424)}{\pi(2)^3} = 3450 \text{ psi}$$

Answer

TORTION OF CIRCULAR SHAFTS

Problem

Steel circular shaft (shear module $G = 0{,}8 \cdot 10^4$ кN/cm^2) is loaded by 4 torques M_i (Fig. 1). (1) Draw torque diagram; (2) Define safe diameters of the shaft at $[\tau] = 8$ кN/cm^2; (3) Draw twist angle diagram.

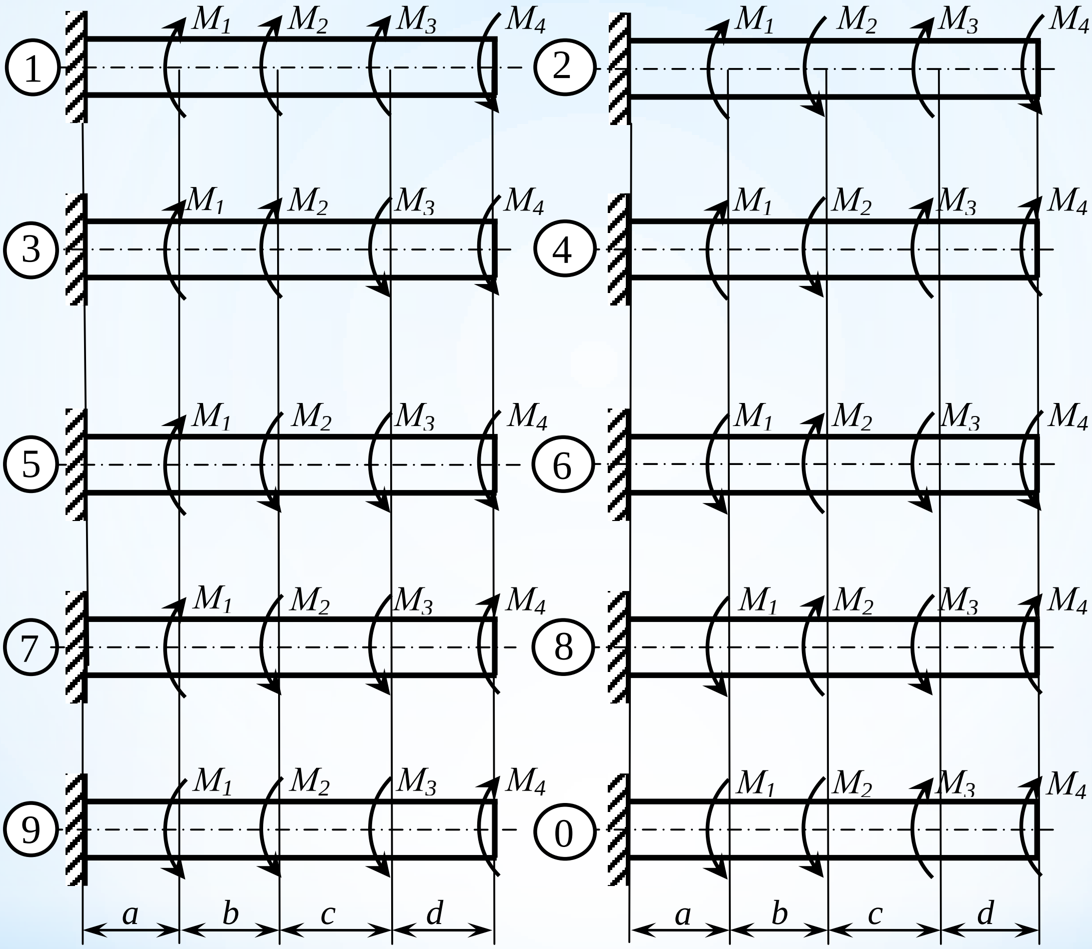


Fig.1

№	M_1, кN·м	M_2, кN·м	M_3, кN·м	M_4, кN·м	*a*, м	*b*, м	*c*, м	*d*, м
1	1,0	2,0	1,0	1,0	1,0	1,2	1,4	1,6
2	1,0	2,0	1,0	0,8	1,2	1,4	1,6	1,9
3	2,0	4,0	1,0	1,0	1,4	1,6	1,0	1,2
4	3,0	5,0	1,6	1,4	1,6	1,0	1,2	1,4
5	4,0	6,0	1,8	1,4	1,1	1,1	1,8	1,5
6	2,0	4,0	1,2	1,2	1,3	1,3	1,5	1,1
7	2,0	3,0	1,2	1,0	1,5	1,5	1,3	1,3
8	3,0	4,0	1,0	1,0	1,7	1,7	1,5	1,4
9	4,0	5,0	1,8	1,6	1,9	1,9	1,7	1,3
0	5,0	6,0	2,0	1,6	1,2	1,4	1,4	1,2

Problem sample

Given:

$M_1 = 1{,}5$ kN·м; $M_2 = 5{,}5$ kN·м; $M_3 = 3{,}2$ kN·м; $M_4 = 1{,}8$ kN·м; $a = 1{,}5$ м; $b = 2$ м, $c = 1$ м, $d = 1{,}2$ м; $G = 0{,}8 \cdot 10^4$ kN/cm^2; $[\tau] = 8$ kN/cm^2

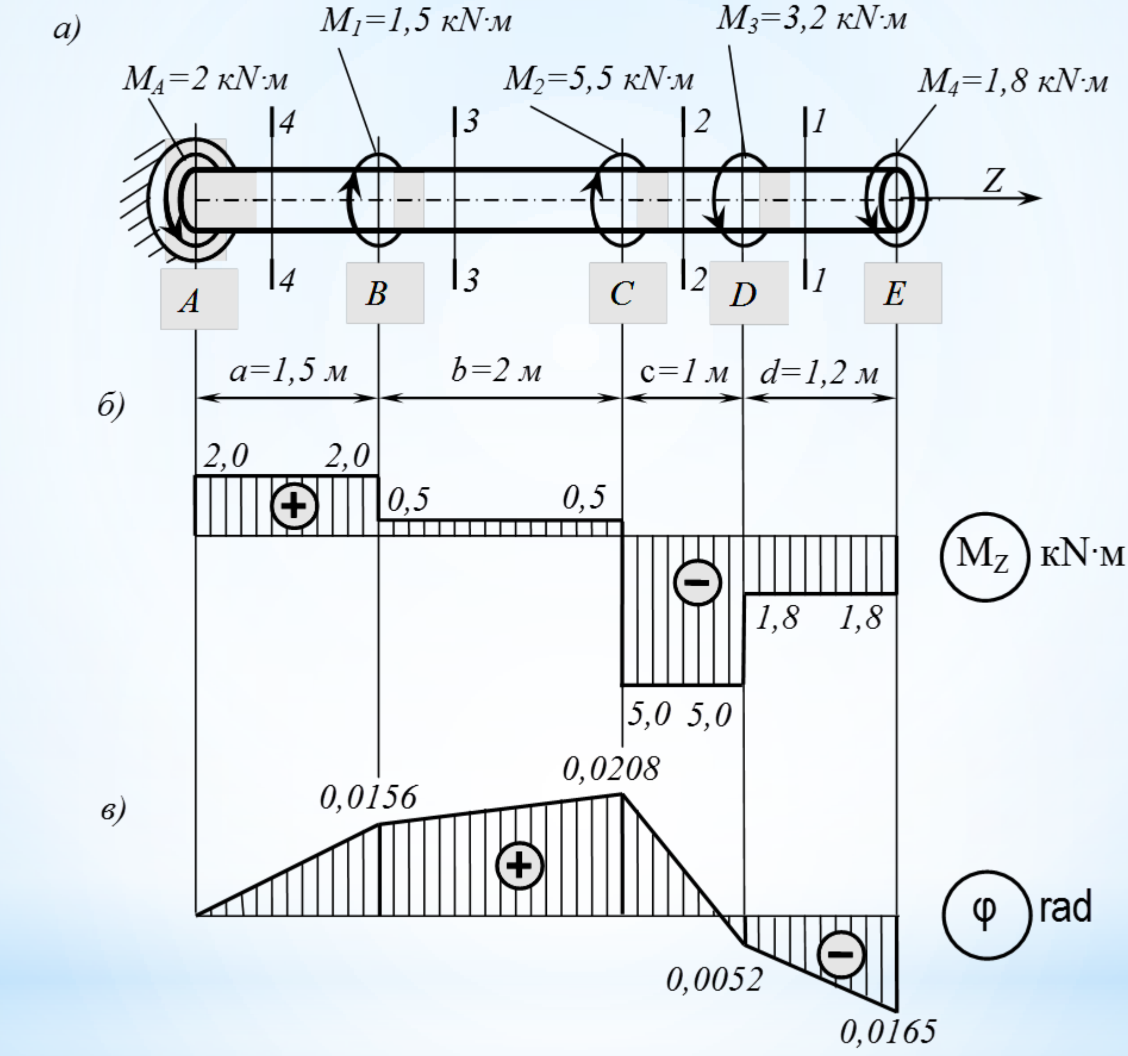


Fig.2

Solution

1. Define torque in fixed support.

Equilibrium equation

$$\sum M_z = 0; \quad M_A - M_1 - M_2 + M_3 + M_4 = 0;$$

$$M_A = M_1 + M_2 - M_3 - M_4 = 1{,}5 + 5{,}5 - 3{,}2 - 1{,}8 = 2 \text{ kN·м}.$$

2. Draw torque diagram.

For section 1 – 1**:** $M_{z_1} = -M_4 = -1{,}8$ кN·м.

By analogy for sections *2 – 2* и *3 – 3*:

$$M_{z_2} = -M_4 - M_3 = -1{,}8 - 3{,}2 = -5{,}0 \text{ кN·м};$$

$$M_{z_3} = -M_4 - M_3 + M_2 = -1{,}8 - 3{,}2 + 5{,}5 = +0{,}5 \text{ кN·м}.$$

$$M_{z_4} = +M_A = +2 \text{ кN·м}.$$

3. Determine diameter of the shaft from strength condition.

$$\tau_{\max} = \frac{M_{z\max}}{W_\rho} \le [\tau],$$

where $W_\rho = \pi d^3/16 \approx 0{,}2d^3$ -section modulus.

$$M_{z\max} = \left|M_{z_2}\right| = 500 \text{ кN·cm}.$$

Then

$$d \ge \sqrt[3]{\frac{\left|M_{z_2}\right|}{0{,}2[\tau]}} = \sqrt[3]{\frac{500}{0{,}2 \cdot 8}} = 6{,}79 \text{ cm}.$$

Rounding we have $d = 70$ мм.

4. Calculate angles of twist and draw twist angles diagram.

$I_\rho = \pi d^4/32 \approx 0{,}1 d^4$

$$GI_\rho = 0{,}8 \cdot 10^4 \cdot 0{,}1 \cdot 7^4 = 192 \cdot 10^4 \text{ кN·cm}^2.$$

$$\varphi_{AB} = \frac{M_{z_4} a}{GI_\rho} = \frac{200 \cdot 150}{192 \cdot 10^4} = 0{,}0156 \text{ rad};$$

$$\varphi_{BC} = \frac{M_{z_3} b}{GI_\rho} = \frac{50 \cdot 200}{192 \cdot 10^4} = 0{,}0052 \text{ rad};$$

$$\varphi_{CD} = \frac{M_{z_2} c}{GI_\rho} = \frac{-500 \cdot 100}{192 \cdot 10^4} = -0{,}0260 \text{ rad};$$

$$\varphi_{DE} = \frac{M_{z_1} d}{GI_\rho} = \frac{-180 \cdot 120}{192 \cdot 10^4} = -0{,}0113 \text{ rad}.$$

$$\varphi_A = 0.$$

$$\varphi_B = \varphi_A + \varphi_{AB} = 0 + 0{,}0156 = 0{,}0156 \text{ rad};$$

$$\varphi_C = \varphi_B + \varphi_{BC} = 0{,}0156 + 0{,}0052 = 0{,}0208 \text{ rad};$$

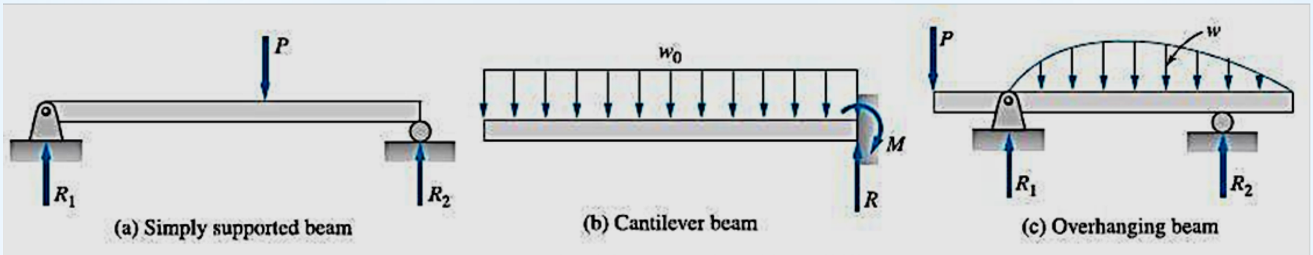
$$\varphi_D = \varphi_C + \varphi_{CD} = 0{,}0208 - 0{,}0260 = -0{,}0052 \text{ rad};$$

$$\varphi_E = \varphi_D + \varphi_{DE} = -0{,}0052 - 0{,}0113 = -0{,}0165 \text{ rad}.$$

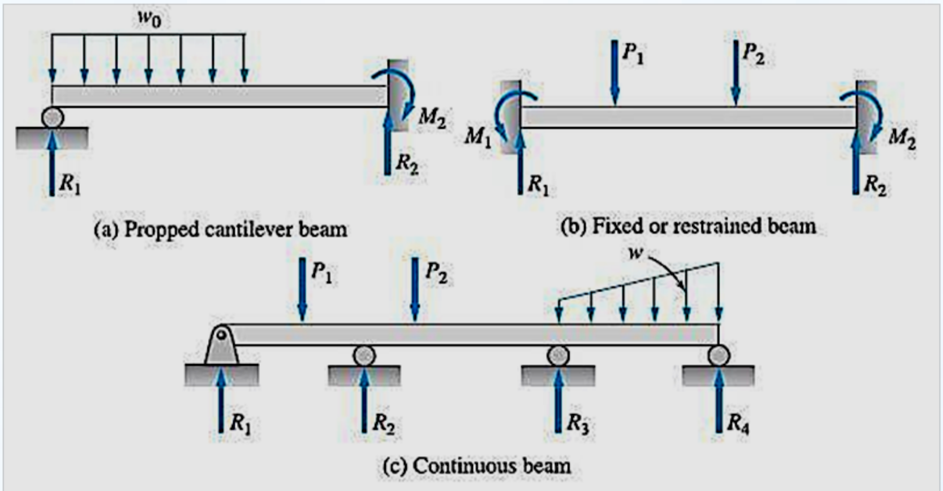
BENDING

Shear Forces and Bending Moments in Beams

Supports and Loads



Statically determinate beams.



Statically indeterminate beams.

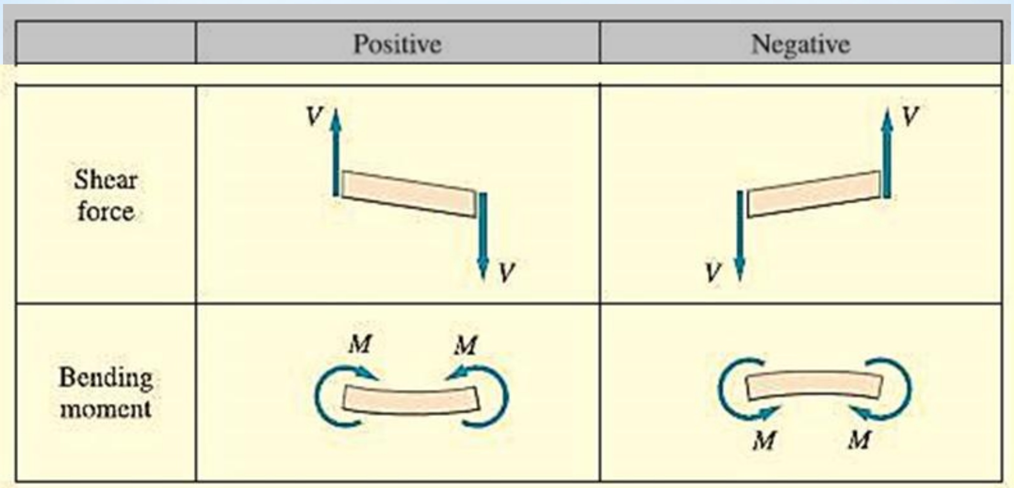
Shear-Moment Equations and Shear-Moment Diagrams

Sign conventions

Conventions, which assume the following to be positive:

• Shear forces that tend to rotate a beam element clockwise.

• Bending moments that tend to bend a beam element concave upward (the beam "smiles").



Procedure for determining shear force and bending moment diagrams:

• Compute the support reactions from the FBD of the entire beam.

• Divide the beam into segments so that the loading within each segment is continuous.

Perform the following steps for each segment of the beam:

• Introduce an imaginary cutting plane within the segment, located at a distance *x* from either end of the beam, that cuts the beam into two parts.

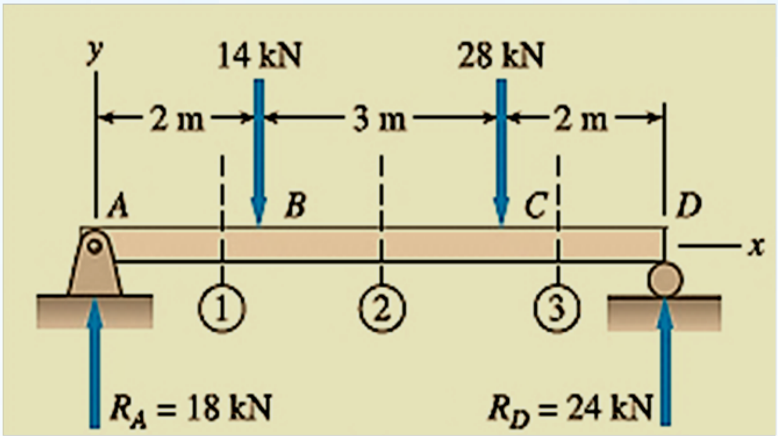
• Draw a FBD for the part of the beam lying either to the left or to the right of the cutting plane, whichever is more convenient.

• Determine the expressions for *V* and *M* from the equilibrium equations obtainable from the FBD.

• Plot the expressions for *V* and *M* for the segment.

Sample Problem.

The simply supported beam carries two concentrated loads. (1) Derive the expressions for the shear force and the bending moment for each segment of the beam. (2) Draw the shear force and bending moment diagrams. Neglect the weight of the beam. Note that the support reactions at A and D have been computed and are shown in Fig. (a).



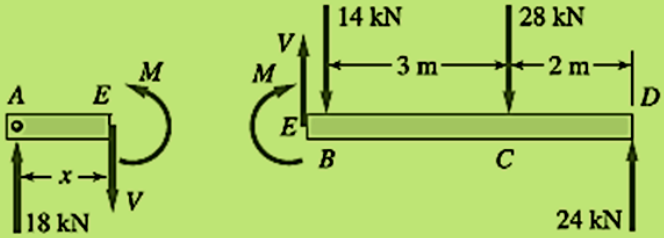
Segment AB (0 < x < 2 m)

$$\Sigma F_y = 0 \quad +\uparrow \quad 18 - V = 0$$

$$V = +18 \text{ kN} \qquad \textit{Answer}$$

$$\Sigma M_E = 0 \quad +\circlearrowleft \quad -18x + M = 0$$

$$M = +18x \text{ kN}\cdot\text{m} \qquad \textit{Answer}$$



(b) FBDs

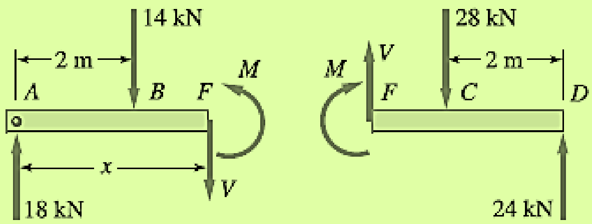
Segment BC (2 m < x < 5 m)

$$\Sigma F_y = 0 \quad +\uparrow \quad 18 - 14 - V = 0$$

$$V = +18 - 14 = +4 \text{ kN} \qquad \textit{Answer}$$

$$\Sigma M_F = 0 \quad +\circlearrowleft \quad -18x + 14(x-2) + M = 0$$

$$M = +18x - 14(x-2) = 4x + 28 \text{ kN} \cdot \text{m} \qquad \textit{Answer}$$



(c) FBDs

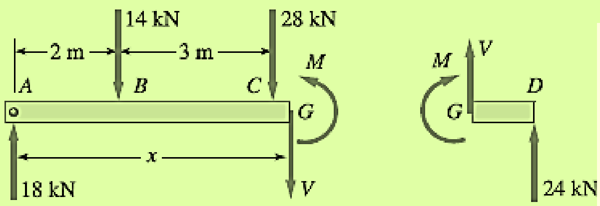
Segment CD (5 m < x < 7 m)

$$\Sigma F_y = 0 \quad +\uparrow \quad 18 - 14 - 28 - V = 0$$

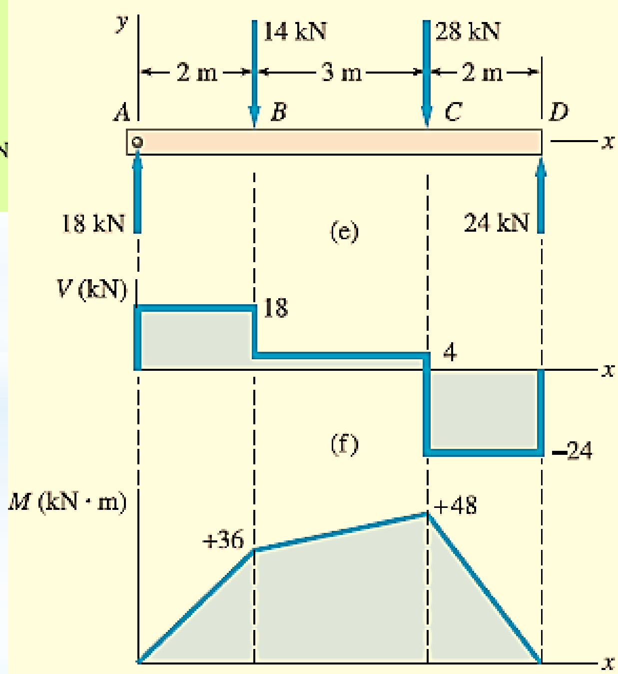
$$V = +18 - 14 - 28 = -24 \text{ kN} \qquad \textit{Answer}$$

$$\Sigma M_G = 0 \quad +\circlearrowleft \quad -18x + 14(x-2) + 28(x-5) + M = 0$$

$$M = +18x - 14(x-2) - 28(x-5) = -24x + 168 \text{ kN} \cdot \text{m} \qquad \textit{Answer}$$



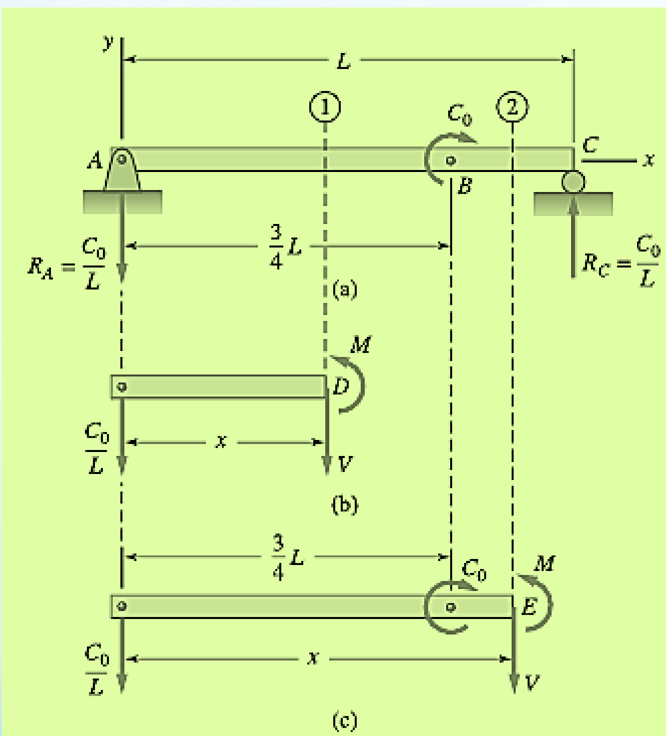
(d) FBDs



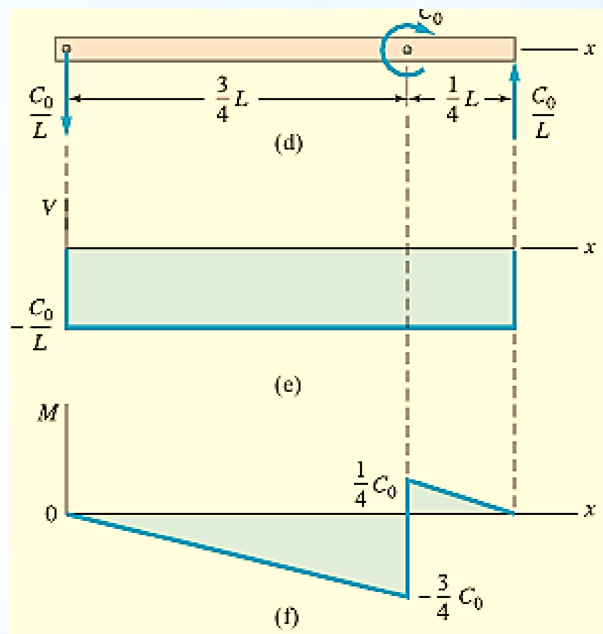
Shear force and bending moment diagrams

Sample Problem

The simply supported beam in Fig (a) is loaded by the clockwise couple C_0 at *B*. (1) Derive the shear force and bending moment equations, and (2) draw the shear force and bending moment diagrams. Neglect the weight of the beam. The support reactions *A* and *C* have been computed, and their values are shown in Fig. (a).



FBDs



Shear force and bending moment diagrams

Segment AB ($0 < x < \frac{3}{4}L$)

$$\Sigma F_y = 0 \quad +\uparrow \quad -\frac{C_0}{L} - V = 0 \qquad V = -\frac{C_0}{L}$$

$$\Sigma M_D = 0 \quad +\circlearrowleft \quad \frac{C_0}{L}x + M = 0 \qquad M = -\frac{C_0}{L}x$$

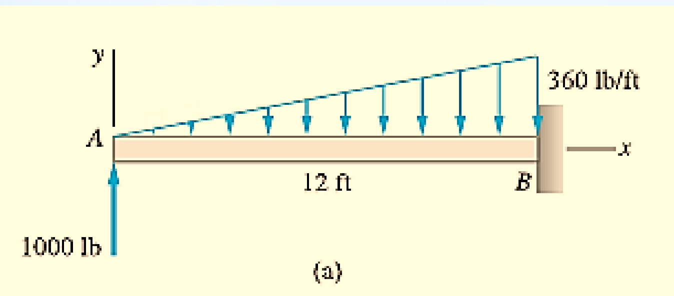
Segment BC ($3L/4 < x < L$)

$$\Sigma F_y = 0 \quad +\uparrow \quad -\frac{C_0}{L} - V = 0 \qquad V = -\frac{C_0}{L}$$

$$\Sigma M_E = 0 \quad +\circlearrowleft \quad \frac{C_0}{L}x - C_0 + M = 0 \qquad M = -\frac{C_0}{L}x + C_0$$

Sample Problem

The cantilever beam in Fig (a) carries a triangular load, the intensity of which varies from zero at the left end to 360 lb/ft at the right end. In addition, a 1000-lb upward vertical load acts at the free end of the beam. (1) Derive the shear force and bending moment equations, and (2) draw the shear force and bending moment diagrams. Neglect the weight of the beam.



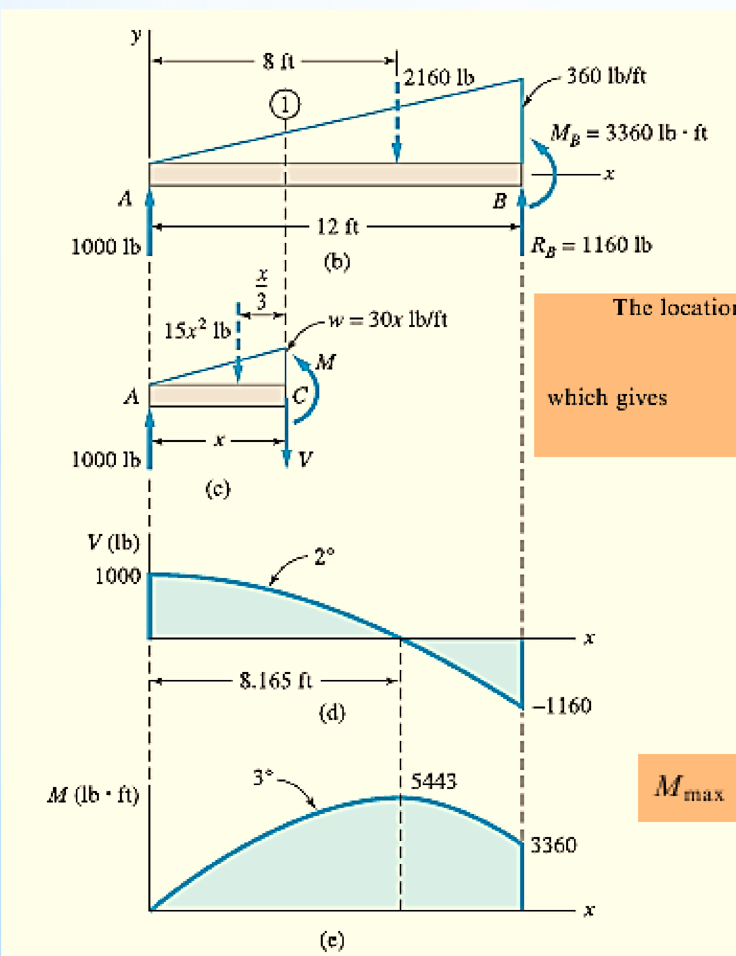
(a)

$$\Sigma F_y = 0 \quad +\uparrow \quad 1000 - 15x^2 - V = 0$$

$$V = 1000 - 15x^2 \text{ lb}$$

$$\Sigma M_C = 0 \quad +\circlearrowleft \quad -1000x + 15x^2\left(\frac{x}{3}\right) + M = 0$$

$$M = 1000x - 5x^3 \text{ lb} \cdot \text{ft}$$



The location of the section where the shear force is zero is found from

$$V = 1000 - 15x^2 = 0$$

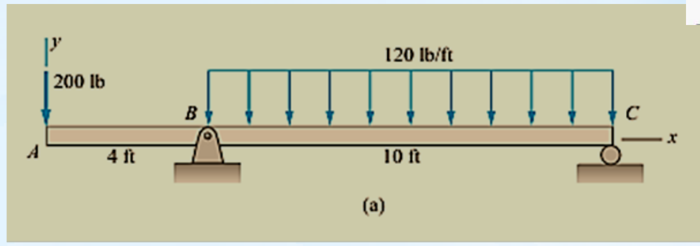
which gives

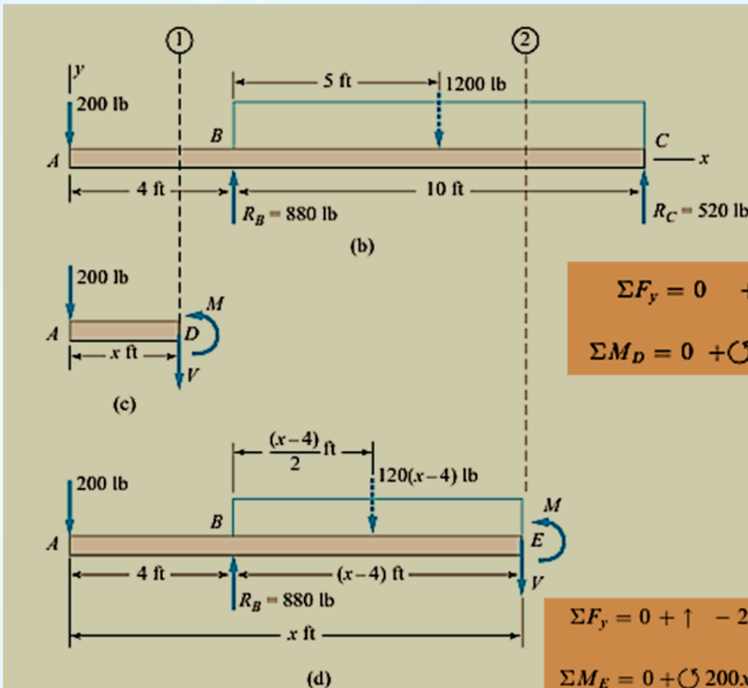
$$x = 8.165 \text{ ft}$$

$$M_{\max} = 1000(8.165) - 5(8.165)^3 = 5443 \text{ lb} \cdot \text{ft}$$

Sample Problem

The same.





Segment *AB* (0 < x < 4 ft)

$$\Sigma F_y = 0 \quad +\uparrow \quad -200 - V = 0 \qquad V = -200 \text{ lb}$$

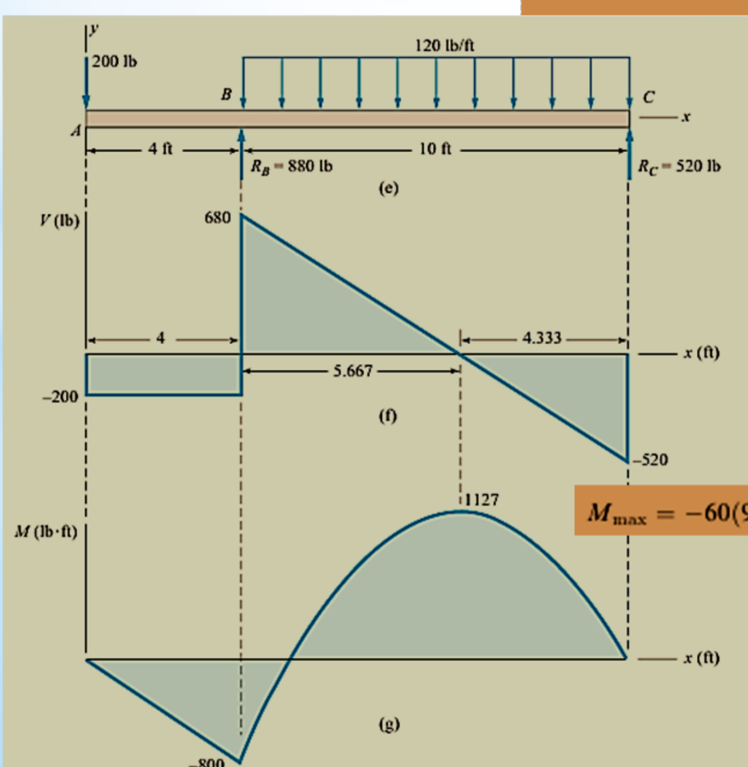
$$\Sigma M_D = 0 \quad +\circlearrowleft \quad -200x + M = 0 \qquad M = -200x \text{ lb}\cdot\text{ft}$$

Segment *BC* (4 ft < x < 14 ft)

$$\Sigma F_y = 0 +\uparrow \quad -200 + 880 - 120(x-4) - V = 0 \qquad V = 1160 - 120x \text{ lb}$$

$$\Sigma M_E = 0 +\circlearrowleft 200x - 880(x-4) + 120(x-4)\frac{(x-4)}{2} + M = 0$$

$$M = -60x^2 + 1160x - 4480 \text{ lb}\cdot\text{ft}$$



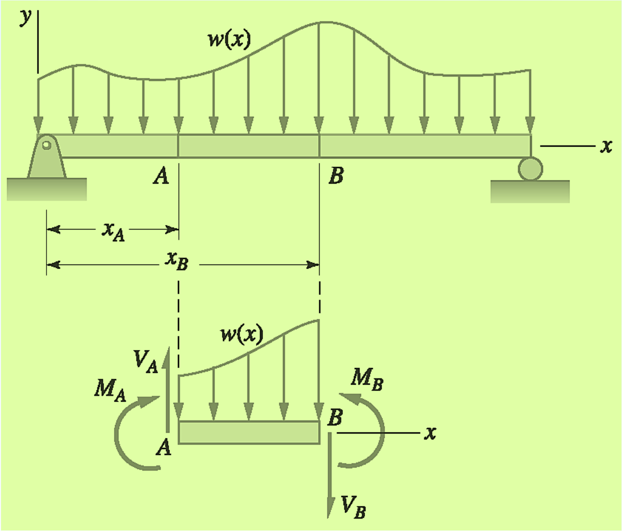
$$V = 1160 - 120x = 0$$

$$x = 9.667 \text{ ft}$$

$$M_{max} = -60(9.667)^2 + 1160(9.667) - 4480 = 1127 \text{ lb}\cdot\text{ft}$$

DIFFERENTIAL EQUATIONS OF EQUILIBRIUM FOR BEAMS

Consider the beam that is subjected to a distributed load of intensity *w(x)*, where *w(x)* is assumed to be a continuous function.



The force equation of equilibrium for the element is

$$\sum F_y = 0 \quad \uparrow \quad V - wdx - (V + dv) = 0.$$

From which get

$$\boxed{w = -\frac{dV}{dx}.}$$

The moment equation of equilibrium yields

$$\sum M_O = 0 \qquad -M - Vdx + (M + dM) + wdx\frac{dx}{2} = 0.$$

After canceling *M* and dividing by *dx*, we get

$$-V + \frac{dM}{dx} + \frac{wdx}{2} = 0.$$

Because *dx* is infinitesimal, the last term can be dropped (this is not an approximation), yielding

$$\boxed{V = \frac{dM}{dx}.}$$

BENDING STRESS

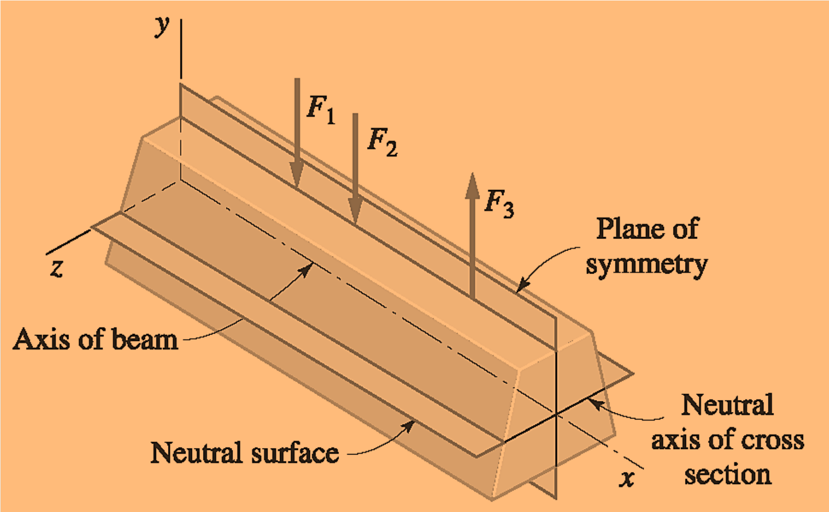
The stresses caused by the bending moment are known as bending stresses, or *flexure stresses*. The relationship between these stresses and the bending moment is called the *flexure formula*

Simplifying assumptions

- The beam has an axial plane of symmetry, which we take to be the *xy*-plane.
- The applied loads lie in the plane of symmetry and are perpendicular to the axis of the beam (the *x*-axis).
- The axis of the beam bends but does not stretch (the axis lies somewhere in the plane of symmetry; its location will be determined later).
- Plane sections of the beam remain plane (do not warp) and perpendicular to the deformed axis of the beam.
- Changes in the cross-sectional dimensions of the beam are negligible.

These assumptions lead us to the following conclusion: Each cross section of the beam rotates as a rigid entity about a line called the neutral axis of the cross section. The neutral axis passes through the axis of the beam and is perpendicular to the plane of symmetry. The *xz*-plane that contains the neutral axes of all the cross sections is known as the neutral surface of the beam.

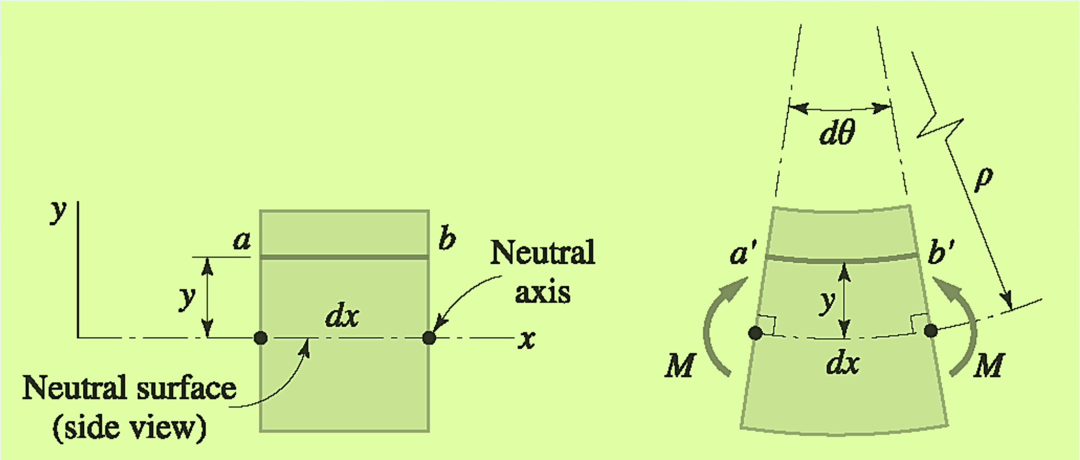
We are limiting our discussion here to the deformations caused by the bending moment alone. However, it can be shown that the deformations due to the vertical shear force are negligible in slender beams compared to the deformations caused by bending.



Symmetrical beam with loads lying in the plane of symmetry.

Compatibility

A segment of the beam bounded by two cross sections that are separated by the infinitesimal distance *dx*.



Deformation of an infinitesimal beam segment.

The distance between the cross sections, measured along the neutral surface, remains unchanged at *dx*. The longitudinal fibers lying on the neutral surface are undeformed, whereas the fibers above the surface are compressed and the fibers below are stretched.

$$a'b' = (\rho - y)d\theta .$$

The original length of this fiber is $ab = dx = \rho d\theta$.

$$\varepsilon = \frac{a'b' - ab}{ab} = \frac{(\rho - y)d\theta - \rho d\theta}{\rho d\theta} = -\frac{y}{\rho}.$$

From Hooke's law $\sigma = E\varepsilon = -\dfrac{E}{\rho}y$

Equilibrium

The normal force acting on the infinitesimal area *dA* of the cross section is $dP = \sigma dA$.

Substituting $\sigma = -(E/\rho)y$, we obtain

$$dP = -\frac{E}{\rho}ydA \quad ,$$

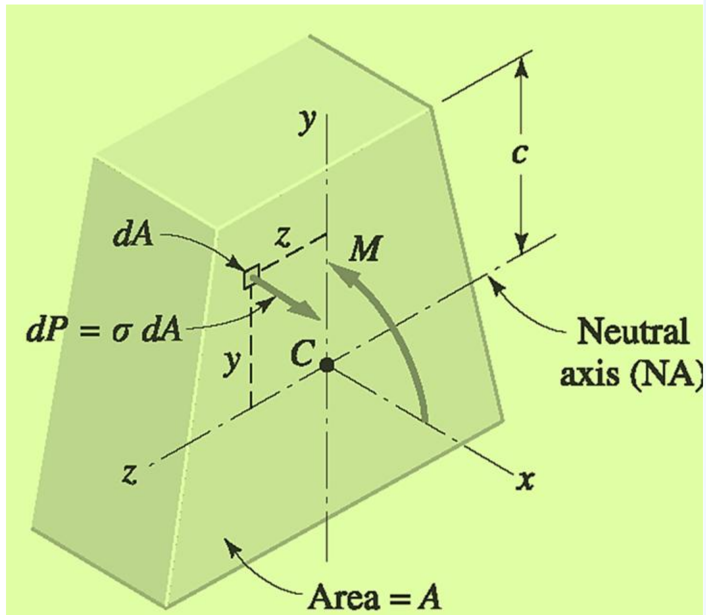
where *y* is the distance of *dA* from the neutral axis.

Equilibrium requires that $-\int_A ydP = M$, $-\int_A ydP = 0$ and $-\int_A zdP = 0$.

Resultant Axial Force Must Vanish

The condition for zero axial force is $\int_A zdP = -\frac{E}{\rho}\int_A yzdA = 0$.

Because $E/\rho \neq 0$, the last equation can be satisfied only if $\int_A zdP = -\frac{E}{\rho}\int_A yzdA = 0.$



Resultant is a couple equal to the internal bending moment *M*.

Resultant Moment About y-Axis Must Vanish

The integral $\int_A zydA$ ***is called the product of inertia of the cross-sectional area.***
According to our assumptions, the *y*-axis is an axis of symmetry for the cross section, in which case this integral is zero.

Resultant Moment About the Neutral Axis Must Equal M

$$-\int_A ydP = \frac{E}{\rho}\int_A y^2 dA = M.$$

$\int_A y^2 dA = I$ ***is the moment of inertia of the cross-sectional area about the neutral axis (the z-axis).*** Hence

$$M = \frac{EI}{\rho} \quad \text{or} \quad \frac{1}{\rho} = \frac{M}{EI}.$$

Flexure formula; section modulus

Substituting the expression for $1/\rho$, we get the flexure formula:

$$y = '\ \frac{My}{I}$$

The maximum value of bending stress is

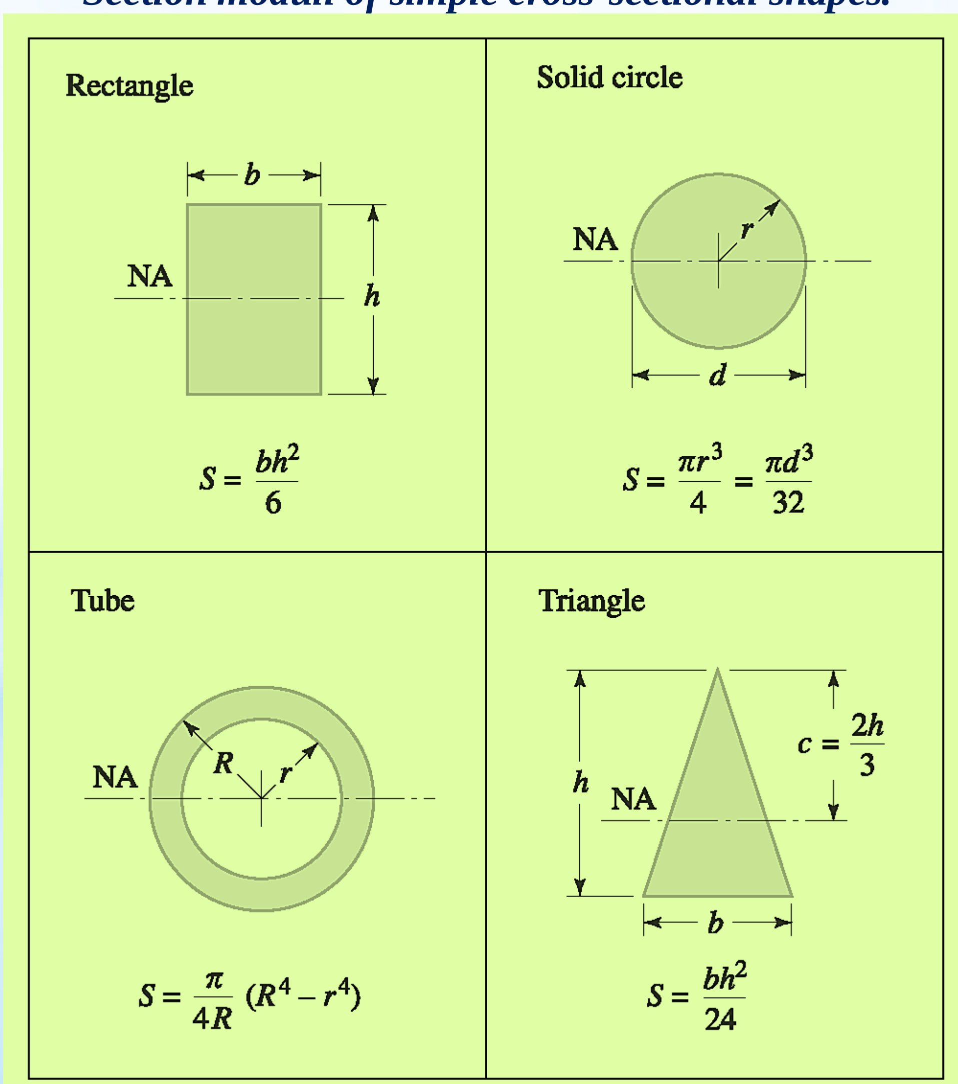
$$\sigma_{\max} = -\frac{|M|_{\max}\, c}{I}$$

where $|M|_{\max}$ is the largest bending moment in the beam regardless of sign, and ***c*** is the distance from the neutral axis to the outermost point of the cross section or

$$\sigma_{\max} = -\frac{|M|_{\max}}{S},$$

where ***S = I/c* is called the section modulus of the beam**.

Section moduli of simple cross-sectional shapes.



Procedures for determining bending stresses

•Use the method of sections to determine the bending moment *M* (with its correct sign) at the cross section containing the given point.

•Determine the location of the neutral axis.

•Compute the moment of inertia *I* of the cross-sectional area about the neutral axis. (If the beam is a standard structural shape, its cross- sectional properties are tabulated.)

• Determine the *y*-coordinate of the given point.

•Compute the bending stress from $\sigma = -My / I$. If correct signs are used for *M* and *y*, the stress will also have the correct sign (tension positive, compression negative).

Maximum Bending Stress: Symmetric Cross Section

If the neutral axis is an axis of symmetry of the cross section, the maximum tensile and compressive bending stresses in the beam are equal in magnitude and occur at the section of the largest bending moment.

Procedure for determining the maximum bending stress in a prismatic beam

• Draw the bending moment diagram Identify the bending moment $M_{\max}$.

• Compute the moment of inertia *I* of the cross-sectional area about the neutral axis.

• Calculate the maximum bending stress from $\sigma_{\max} = |M|_{\max} c / I = |M|_{\max} / S$, where *c* is the distance from the neutral axis to the top or bottom of the cross section.

Procedure for determining Maximum Tensile and Compressive Bending Stressesfor Unsymmetrical Cross Section

•Draw the bending moment diagram. Identify the largest positive and negative bending moments.

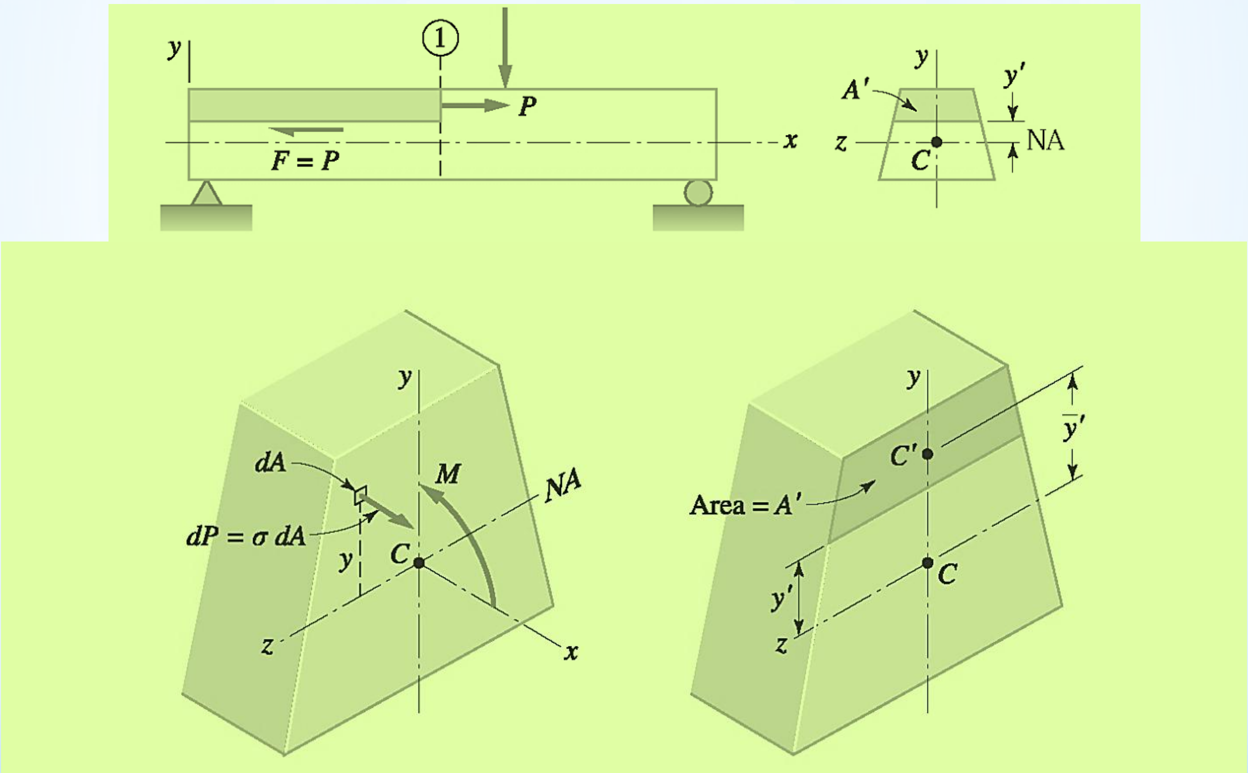
•Determine the location of the neutral axis and record the distances c_{top} and c_{bot} from the neutral axis to the top and bottom of the cross section.

•Compute the moment of inertia *I* of the cross section about the neutral axis.

•Calculate the bending stresses at the top and bottom of the cross section where the largest positive bending moment occurs from $\sigma = -My / I$. At the top of the cross section, where $y = c_{top}$, we obtain $\sigma_{top} = -Mc_{top} / I$. At the bottom of the cross section, we have $y = -c_{bot}$, so that $\sigma_{bot} = Mc_{bot} / I$. Repeat the calculations for the cross section that carries the largest negative bending moment. Inspect the four stresses thus computed to determine the largest tensile (positive) and compressive (negative) bending stresses in the beam.

Analysis of flexure action

Isolate the shaded portion of the beam by using two cutting planes: a vertical cut along section *1* and a horizontal cut located at the distance y' above the neutral axis. The isolated portion is subjected to the two horizontal forces *P* and *F* (vertical forces are not shown). The axial force *P* is due to the bending stress acting on the area A' of section *1* , whereas *F* is the resultant of the shear stress acting on the horizontal surface. Equilibrium requires that *F* = *P*.



Calculating the resultant force of the normal stress over a portion of the cross-sectional area.

The bending stress is $\sigma = -My/I$, where *y* is the distance of the element from the neutral axis, and *I* is the moment of inertia of the *entire cross-sectional area* of the beam about the neutral axis. Therefore,

$$dP = -\frac{My}{I}dA.$$

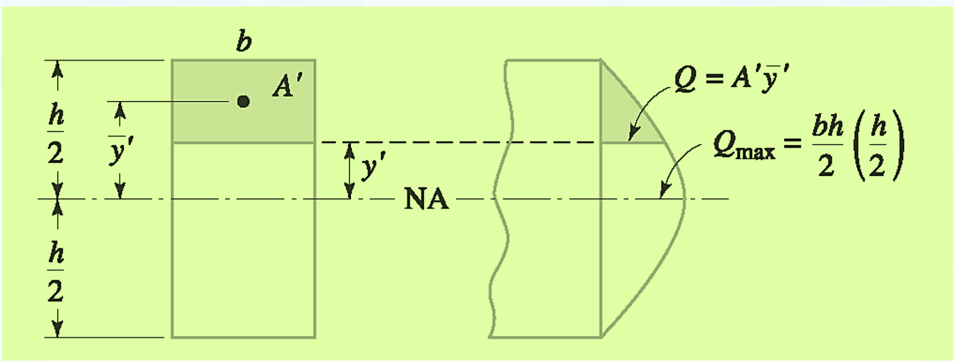
Integrating over the area A' we get

$$P = \int_{A'} dP = -\frac{M}{I}\int_{A'} y dA = -\frac{MQ}{I},$$

where $Q = \int_{A'} dA$ is the first moment of area A' about the neutral axis.

Denoting the distance between the neutral axis and the centroid C' of the area A' by $\overline{y'}$ we can write $Q = A'\overline{y'}$. Q represents the first moment of the cross-sectional area that lies *above* y'. Because the first moment of the total cross-sectional area about the neutral axis is zero, the first moment of the area *below* y' is -Q. The magnitude of Q can be computed by using the area either above or below y', whichever is more convenient. The maximum value of Q occurs at the neutral axis where y' = 0. It follows that the horizontal shear force F is largest on the neutral surface.

Horizontal shear stress



Variation of the first moment Q of area A' about the neutral axis for a rectangular cross section.

The resultant force acting on face *1* of the body is $P = -M\dfrac{Q}{I}$.

The bending moment acting at section *2* is $M + dM$. The resultant normal force acting on face *2* of the body is

$$P + dP = -(m + dM)\frac{Q}{I}.$$

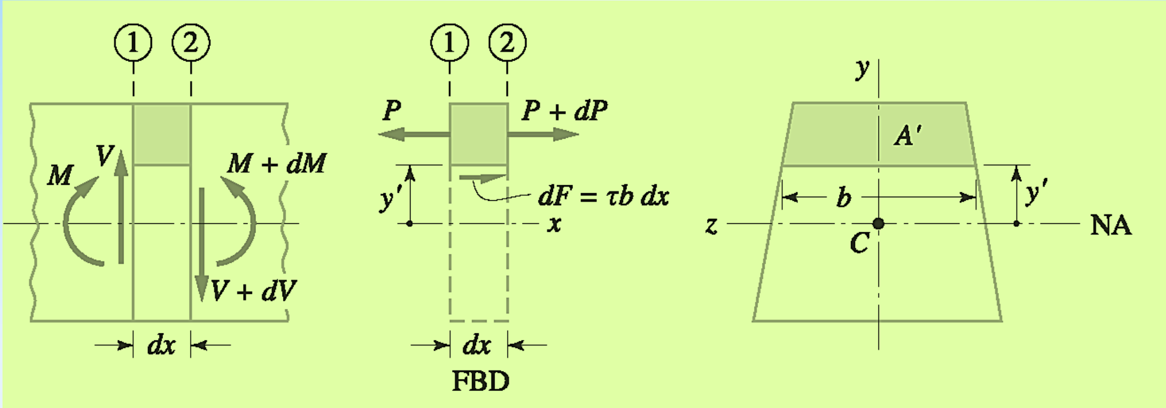
Because these two forces differ by

$$(P + dP) - P = -(\mathrm{M} + dM)\frac{Q}{I} - (-M\frac{Q}{I}) = -dM\frac{Q}{I}$$

equilibrium can exist only if there is an equal and opposite shear force dF acting on the horizontal surface.

If we let τ be the average shear stress acting on the horizontal surface, its resultant is $dF = \tau b\,dx$, where b is the width of the cross section at $y = y'$. The equilibrium requirement for the horizontal forces is

$$(P + dP) - P + \tau b\,dx = 0.$$



Determining the longitudinal shear stress from the free-body diagram of a beam element.

Substituting for $(P+dP)-P$ we get

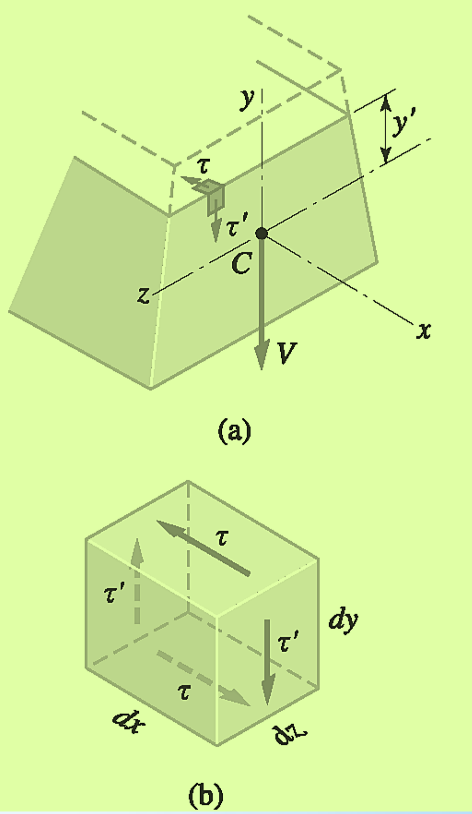
$$-dM\frac{Q}{I}+\tau b\,dx=0 \text{ which gives } \tau=\frac{dM}{dx}\frac{Q}{Ib}.$$

Recalling the relationship $V = dM/dx$ we obtain

$$\boxed{\tau=\frac{VQ}{Ib}}.$$

Vertical shear stress

A shear stress is always accompanied by a ***complementary shear stress of equal magnitude.*** In a beam, the complementary stress τ' is a vertical shear stress that acts on the cross section of the beam. Because $\tau=\tau'$, the last Eq. can be used to compute the vertical as well as the horizontal shear stress at a point in a beam. The resultant of the vertical shear stress on the cross-sectional area A of the beam is the shear force $V=\int_A \tau dA$.



The vertical stress τ' acting at a point on a cross section equals the longitudinal shear stress τ acting at the same point.

Rectangular and wide-flange sections

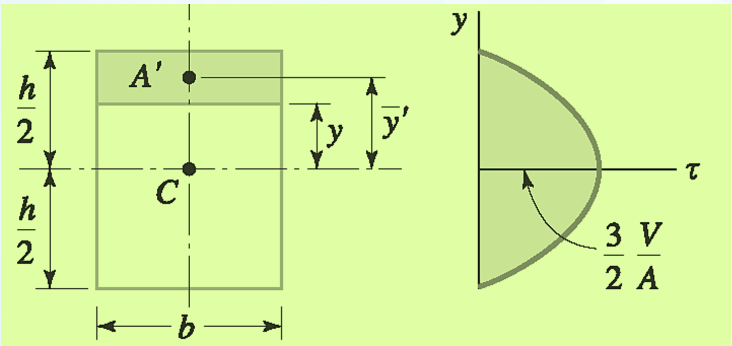
The shaded area is $A' = b\left[(h/2) - y\right]$, its centroidal coordinate being $\overline{y'} = \left[(h/2) + y\right]/2$. Thus,

$$Q = A'\overline{y'} = \left[b\left(\frac{h}{2} - y\right)\right]\left[\frac{1}{2}\left(\frac{h}{2} + y\right)\right] = \frac{b}{2}\left(\frac{h^2}{4} - y^2\right).$$ Then

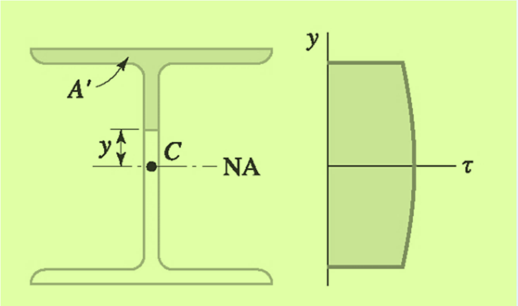
$$\tau = \frac{VQ}{Ib} = \frac{V}{2I}\left(\frac{h^2}{4} - y^2\right).$$

The shear stress is distributed ***parabolically*** across the depth of the section. ***The maximum shear stress occurs at the neutral axis***. If we substitute $y = 0$ and $I = bh^3/12$, we obtain

$$\tau_{max} = \frac{3}{2}\frac{V}{bh} = \frac{3}{2}\frac{V}{A}.$$



In wide-flange sections, most of the bending moment is carried by the flanges, whereas the web resists the bulk of the vertical shear force.



Procedure for analysis of shear stress

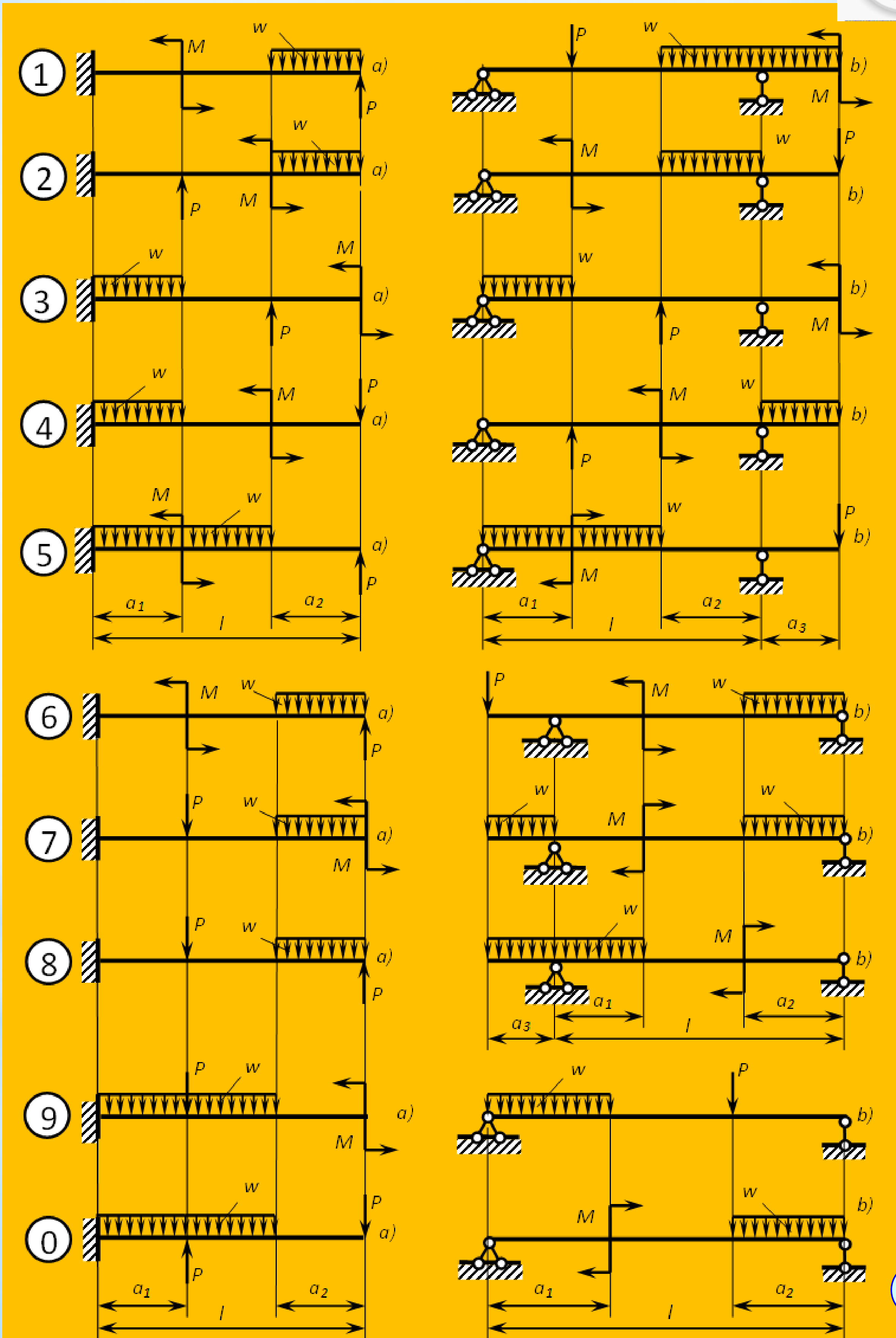
- **Determine the vertical shear force *V* acting on the cross section containing the specified point.**
- **Locate the neutral axis and compute the moment of inertia *I* of the cross-sectional area about the neutral axis.**
- **Compute the first moment *Q* of the cross-sectional area that lies above (or below) the specified point.**
- **Calculate the shear stress from τ = *VQ/Ib*, where *b* is the width of the cross section at the specified point. Note that τ is the actual shear stress only if it is uniform across *b*; otherwise, τ should be viewed as the average shear stress.**

The maximum shear stress τ_{max} on a given cross section occurs where *Q=b* is largest. If the width *b* is constant, then τ_{max} occurs at the neutral axis because that is where *Q* has its maximum value. If *b* is not constant, it is necessary to compute the shear stress at more than one point in order to determine its maximum value.

COMPLEMENTARY PROBLEMS

For two given schemes of beams is required:
• draw the diagrams of shear forces and bending moments;
• on the basis of the strength condition by the normal stresses ($\sigma_w = 16$ kN кN/cm^2), select the beam of the circular cross section for the scheme *a,* and the I-beam cross-section for the scheme *b*;
• check the strength of the selected beams by shear stresses ($\tau_w =$ kN/ cm^2).
Take the data from the following table.

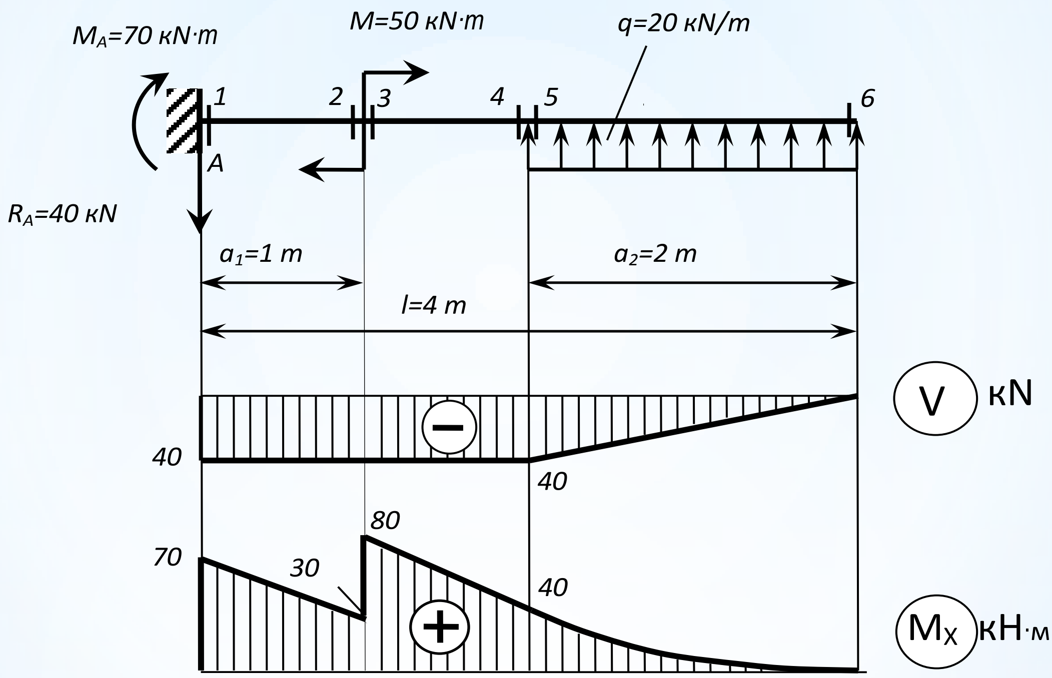
Scheme number	*l*, m	a_1/l	a_2/l	a_3/l	*M*, кN·m	*P*, кN	*q*, кН/m
1	3	0,2	0,6	0,2	8	5	10
2	4	0,3	0,5	0,3	7	6	11
3	5	0,4	0,4	0,3	6	7	12
4	6	0,5	0,3	0,2	5	8	13
5	3	0,6	0,7	0,2	4	9	14
6	4	0,7	0,5	0,3	8	10	9
7	5	0,8	0,4	0,6	7	5	10
8	6	0,2	0,6	0,3	6	6	11
9	3	0,3	0,5	0,4	5	7	12
0	4	0,4	0,4	0,2	4	8	8


1
2
3
4
5
6
7
8
9
0
M
w
P
a)
b)
a1
a2
a3
l

Sample Problem 1

Given: $q=20$ кN/m, $M=50$ кN·m $[\sigma]=16$ кN/cm^2, $[\tau]=8$ кN/cm^2, $a_1=1$ m; $a_2=2$ m; $l=4$ m.

Draw the shear force and bending moment diagrams.



Determine the required cross-sectional diameter of the beam.

The strength condition by the normal stresses has the form

$$\sigma_{\max}=\frac{M_{x\max}}{S_x}\le[\sigma],$$

where - the section modulus in bending. For the circular cross-section beams it is $S_x=\dfrac{\pi d^3}{32}\approx 0{,}1d^3$.

The maximum absolute value of the bending moment occurs in the third section of the beam $M_{x\max}=|M_{x3}|=8000$ кN·cm.

Then the required beam diameter determined by the formula

$$d\ge\sqrt[3]{\frac{|M_{x3}|}{0{,}1[\sigma]}}=\sqrt[3]{\frac{8000}{0{,}1\cdot 16}}=17{,}1 \text{ cm}.$$

Take $d = 170$ mm.

Then

$$\sigma_{\max} = \frac{M_{x\max}}{S_x} = \frac{8000}{\frac{\pi \cdot 17^3}{32}} = 16{,}6 \text{ кN/cm}^2 > [\sigma] = 16 \text{ кN/cm}^2.$$

The overstrain is $\frac{16{,}6 - 16}{16} \cdot 100\% = 3{,}75\% < 5\%$, which is allowed.

Check the strength of the beam by the maximum shear stresses.

$$\tau_{\max} = \frac{4V_{y\max}}{3A},$$

where $A = \pi d^2/4$.

The maximum absolute value of the shear force is $V_{y\max} = \left|V_{y_{1-5}}\right| = 40$ кN

Therefore

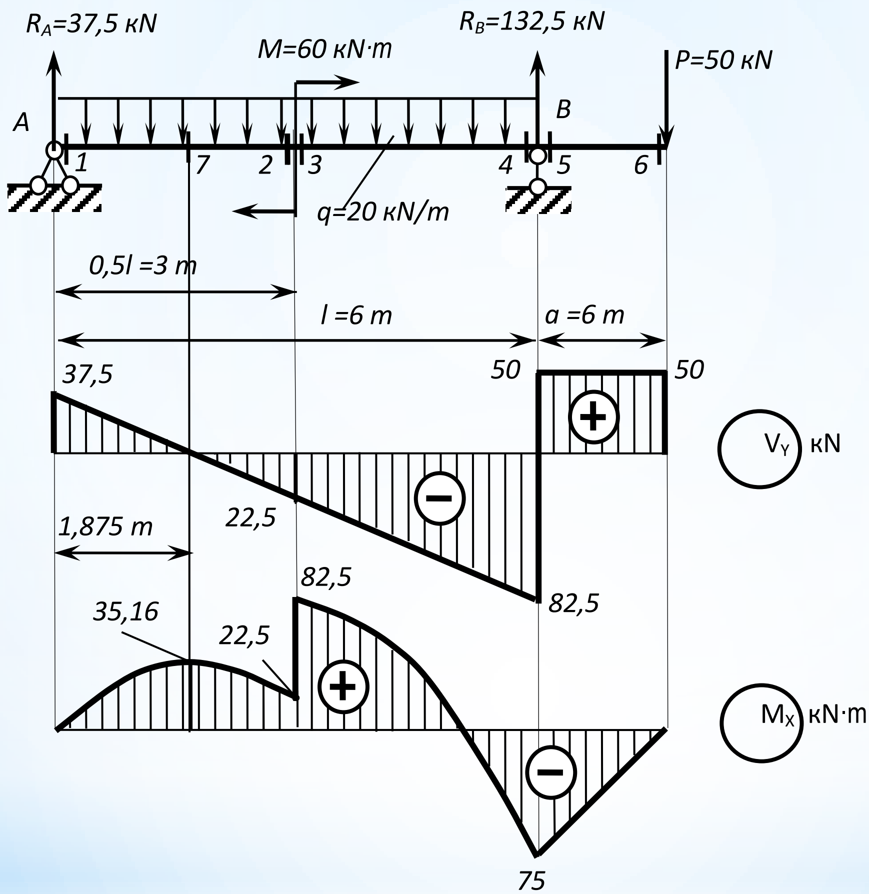
$$\tau_{\max} = \frac{4V_{y\max}}{3A} = \frac{4 \cdot 40}{3 \cdot \frac{\pi \cdot 17^2}{4}} = 0{,}235 \text{ кN/cm}^2 < [\tau] = 8 \text{ кN/cm}^2.$$

The strength condition by shear stress is satisfied.

Sample Problem 2

Given: $q = 20$ кN/m, $P = 50$ кN, $M = 60$ кN·m, $[\sigma] = 16$ кN/cm^2, $[\tau] = 8$ кN/cm^2, $l = 6$ m.

Draw the shear force and bending moment diagrams.



Define the section modulus from the strength condition by the normal stresses.

From the diagram we have $M_{x\max} = |M_{x3}| = 8250$ кN·cm. Whence

$$S_x \geq \frac{M_{x\max}}{[\sigma]} = \frac{8250}{16} = 516 \text{ cm}^3.$$

By the properties of I-beam sections we select № 30*a*, having $S_x = 518$ m^3.

Check the strength of the beam by the maximum shear stresses.

The maximum absolute value of the shear force for the I-beam is

$$\tau_{\max} = \frac{V_{y\max} Q_x}{I_x d}.$$

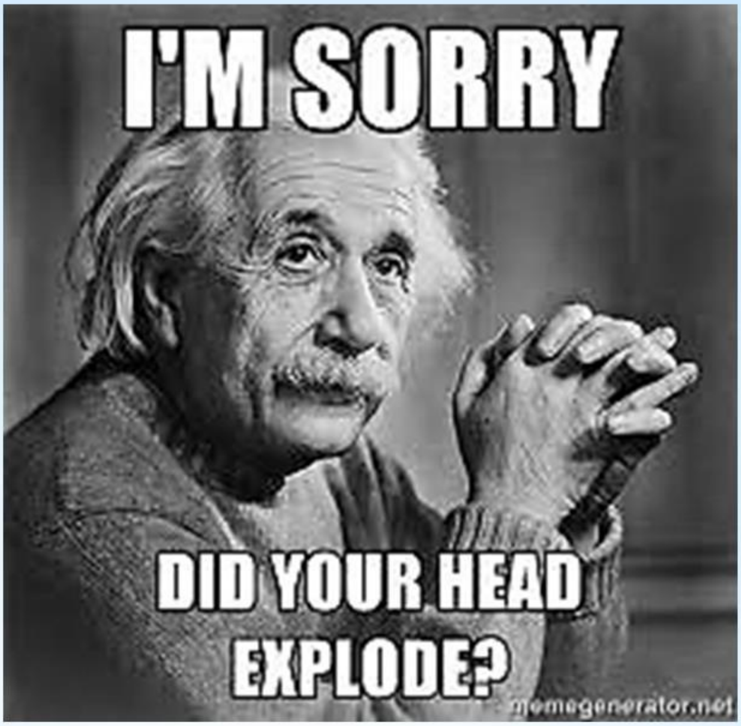
For the selected beam we determine the first moment of a half of the section about the neutral axis $S_x = 292$ cm^3, $I_x = 7780$ cm^4, and the wall thickness $d = 0{,}65$ cm.

From the diagram we have $V_{y\max} = \left|V_{y_4}\right| = 82{,}5$ кN.

Whence

$$\tau_{\max} = \frac{\left|V_y\right|_{\max} Q_x}{I_x d} = \frac{82{,}5 \cdot 292}{7780 \cdot 0{,}65} = 4{,}76 \text{ кN/cm}^2 < [\tau] = 8 \text{ кN/cm}^2,$$

i.e, the strength condition by shear stresses is satisfied.


I'M SORRY
DID YOUR HEAD
EXPLODE?
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