8. PROBLEMS FOR SELF-STUDY TRAINING

8.1. Integration of Differential Equations of the Particle Motion under the Action of Constant Forces

A body moves from the point A along a plane AB of length l during τ s. (Fig. 103). Its initial velocity is v_A. Coefficient of sliding friction is f. At the point B the body leaves a plane with a velocity v_B and then falls with a velocity v_C at a point C moving in the air T s.

Determine specified quantities. Consider the body as a material particle neglecting the resistance of the air.

Variants 1—5 (Fig. 103 , scheme 1).

Variant 1. Given are: $\alpha = 30°$; $v_A = 0$; $f = 0{,}2$; $l = 10\ m$; $\beta = 60°$. Determine τ and h.

Variant 2. Given are: $\alpha = 15°$; $v_A = 2\ m/s$; $f = 0{,}2$; $h = 4\ m$; $\beta = 45°$. Determine l and equation of the path along BC.

Variant 3. Given are: $\alpha = 30°$; $v_A = 2{,}5\ m/s$; $f \neq 0$; $l = 8\ m$; $d = 10\ m$; $\beta = 60°$. Determine v_B and τ.

Variant 4. Given are: $v_A = 0$; $\tau = 2\ s$; $l = 9{,}8\ m$; $\beta = 60°$; $f = 0$. Determine α and T.

Variant 5. Given are: $\alpha = 30°$; $v_A = 0$; $l = 9{,}8\ m$; $\tau = 3\ s$; $\beta = 45°$. Determine f and v_C.

Variants 6—10 (Fig. 103, scheme 2).

Variant 6. Given are: $\alpha = 20°$; $f = 0{,}1$; $\tau = 0{,}2\ s$; $h = 40\ m$; $\beta = 30°$. Determine l and v_C.

Variant 7. Given are: $\alpha = 15°$; $f = 0{,}1$; $v_A = 16\ m/s$; $l = 5\ m$; $\beta = 45°$. Determine v_B and T.

Variant 8. Given are: $v_A = 21\ m/s$; $f = 0$; $\tau = 0{,}3\ s$; $v_B = 20 m/$c; $\beta = 60°$. Determine α and d.

Variant 9. Given are: $\alpha = 15°$; $\tau = 0{,}3\ s$; $f = 0{,}1$; $h = 30\sqrt{2}\ m$; $\beta = 45°$. Determine v_B and v_A.

Variant 10. Given are: $\alpha = 15°$; $f = 0$; $v_A = 12\ m/s$; $d = 50\ m$; $\beta = 60°$. Determine τ and equation of the path along BC.

Variants 11—15 (Fig. 103, scheme 3, $f = 0$, M is a mass of a body).

Variant 11, Given are: $\alpha = 30°$; $P \neq 0$; $l = 40\ m$; $v_A = 0$; $v_B = 4{,}5\ m/s$; $d = 3\ m$. Determine τ and h.

Variant 12. Given are: $\alpha = 30°$; $P = 0$; $l = 40\ m$; $v_B = 4{,}5\ m/s$; $h = 1{,}5\ m$. Determine v_A and d.

Variant 13. Given are: $\alpha = 30°$; $M = 400\ kg$; $v_A = 0$; $\tau = 20\ s$; $d =$

$3\ m;\ h = 1{,}5\ m$. Determine P and l.

Variant 14. Given are: $\alpha = 30°;\ M = 400\ kg;\ P = 2{,}2\ kN;\ v_A = 0;\ l = 40\ m;\ d = 5\ m$. Determine v_B and v_C.

Variant 15. Given are: $\alpha = 30°;\ v_A = 0;\ P = 2\ kN;\ l = 50\ m;\ h = 2\ m;\ d = 4\ m$. Determine T and M.

Variants 16—20 (Fig. 103, scheme 4).

Variant 16. Given are: $\alpha = 30°;\ v_A = 1\ m/s;\ l = 3\ m;\ f = 0{,}2;\ d = 2{,}5\ m$. Determine h and T.

Variant 17. Given are: $\alpha = 45°;\ l = 6\ m;\ v_B = 2v_A;\ \tau = 1\ s;\ h = 6\ m$. Determine d and f.

Variant 18. Given are: $\alpha = 30°;\ l = 2\ m;\ v_A = 0;\ f = 0{,}1;\ d = 3\ m$. Determine h and τ.

Variant 19. Given are: $\alpha = 15°;\ l = 3\ m;\ v_B = 3\ m/s; f \neq 0;\ \tau = 1{,}5\ s;\ d = 2\ m$. Determine v_A and h.

Variant 20. Given are: $\alpha = 45°;\ v_A = 0;\ f = 0{,}3;\ d = 2\ m;\ h = 4\ m$. Determine l and τ.

Variants 21—25 (Fig. 103, scheme 5).

Variant 21. Given are: $\alpha = 30°;\ f = 0{,}1;\ v_A = 1\ m/s; \tau = 1{,}5\ s;\ h = 10\ m$. Determine v_B and d.

Variant 22. Given are: $v_A = 0;\ \alpha = 45°;\ l = 10\ m;\ \tau = 2\ s$. Determine f and equation of the path along BC.

Variant 23. Given are: $f = 0;\ v_A = 0;\ l = 9{,}81\ m;\ \tau = 2\ s;\ h = 20\ m$. Determine α and T.

Variant 24. Given are: $v_A = 0;\ \alpha = 30°;\ f = 0{,}2;\ l = 10\ m;\ d = 12\ m$. Determine τ and h.

Variant 25. Given are: $v_A = 0;\ \alpha = 30°;\ f = 0{,}2;\ l = 6\ m;\ h = 4{,}5\ m$. Determine τ and v_C.

Variants 26—30 (Fig. 103, scheme 6).

Variant 26. Given are: $v_A = 7\ m/s;\ f = 0{,}2;\ l = 8\ m;\ h = 20\ m$. Determine d and v_C.

Variant 27. Given are: $v_A = 4\ m/s;\ f = 0{,}1;\ \tau = 2\ s;\ d = 2\ m$. Determine v_B and h.

Variant 28. Given are: $v_B = 3\ m/s;\ f = 0{,}3;\ l = 3\ m;\ h = 5\ m$. Determine v_A and T.

Variant 29. Given are: $v_A = 3\ m/s;\ v_B = 1 m/s;\ l = 2{,}5\ m; h = 20\ m$. Determine f and d.

Variant 30. Given are: $f = 0{,}25;\ l = 4\ m;\ d = 3\ m;\ h = 5\ m$. Determine v_A and τ.

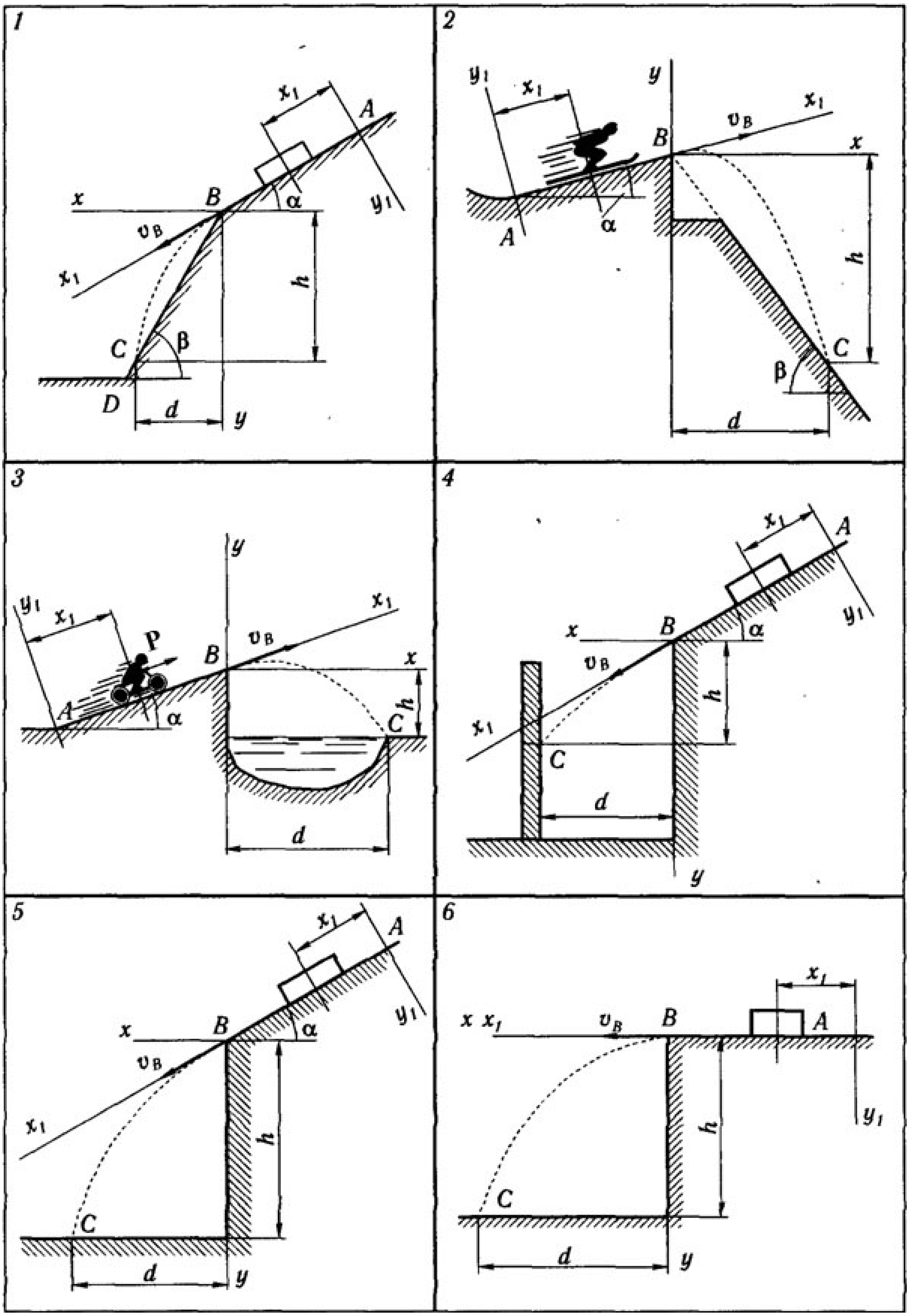
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Fig. 103

Example

In order to protect ditches from the slide-rocks there is a ledge DC in the railway rocky cuttings. Taking into account the possibility of the motion of a stone from the highest point A and assuming its initial velocity to be $v_0 = 0$, determine the minimal width of the ledge b and falling velocity v_C. The stone moves along a slope AB of length l during τ s. The angle α is given. Coefficient of sliding friction f is constant. Neglect the resistance of the air.

Given are: $v_A = 0$; $\alpha = 60°$; $l = 4\ m$; $\tau = 1\ s$; $f \neq 0$; $h = 5\ m$; $\beta = 75°$. Determine b and $v_C = 0$.

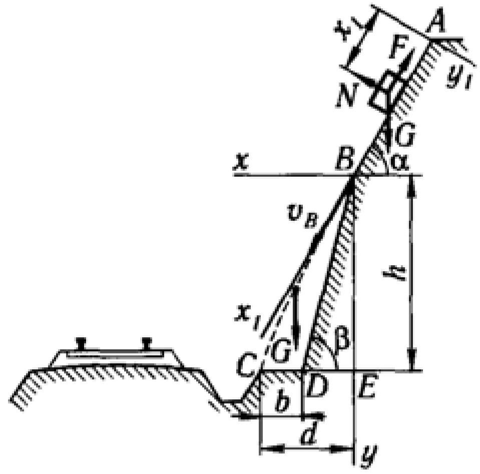


Fig. 104

Solution. Consider the motion of a stone along AB. Assuming the stone as a material particle, show all acting forces: weight $\boldsymbol{G}$, normal reaction $\boldsymbol{N}$ and a force of sliding friction $\boldsymbol{F}$. Work out deferential equation of motion of the stone on section AB (Fig. 104):

$$m\ddot{x}_1 = \sum X_{i1}; \qquad m\ddot{x}_1 = G\sin\alpha - F.$$

The force of friction is

$$F = fN,$$

where $N = G\cos\alpha$.

Therefore,

$$m\ddot{x}_1 = G\sin\alpha - fG\cos\alpha \text{ or } \ddot{x}_1 = g\sin\alpha - fg\cos\alpha.$$

Integrating this deferential equation twice, we obtain

$$\dot{x}_1 = g(\sin\alpha - f\cos\alpha)t + C_1;$$

$$x_1 = \left[\frac{g(\sin\alpha - f\cos\alpha)}{2}\right]t^2 + C_1 t + C_2.$$

In order to determine constants of integration, make use of initial conditions: at $t = 0$, $x_{10} = 0$ and $\dot{x}_{10} = 0^*$.

Compose equations by integrating, for $t = 0$:

$$\dot{x}_{10} = C_1; \quad x_{10} = C_2.$$

Determine constants:

$$C_1 = 0, \quad C_2 = 0.$$

Then

$$\dot{x}_1 = g(\sin\alpha - f\cos\alpha)t; \quad x_1 = \left[\frac{g(\sin\alpha - f\cos\alpha)}{2}\right]t^2.$$

For instant τ, when the stone leaves rectilinear section,

$$\dot{x}_1 = v_B; \quad x_1 = l,$$

i.e.,

$$v_B = g(\sin\alpha - f\cos\alpha)\tau;$$

$$l = \left[\frac{g(\sin\alpha - f\cos\alpha)}{2}\right]\tau^2,$$

whence

$$v_B = \frac{2l}{\tau},$$

i.e.,

$$v_B = \frac{2\cdot 4}{1} = 8\,{}^{m}/{}_{s}.$$

Consider the motion of a stone on curvilinear section BC. There is only a force of weight $\boldsymbol{G}$ acting on a stone here. Derive deferential equations of its motion:

$$m\ddot{x} = 0;\quad m\ddot{y} = G.$$

Initial conditions of the problem: at $t = 0$,

$$x_0 = 0;\quad y_0 = 0;$$
$$\dot{x_0} = v_B\cos\alpha;\quad \dot{y_0} = v_B\sin\alpha.$$

Integrating these deferential equations twice, we obtain

$$\dot{x} = C_3;\quad \dot{y} = gt + C_4;$$
$$x = C_3t + C_5;\quad y = gt^2/2 + C_4t + C_6.$$

Write these equations for $t = 0$:

$$\dot{x_0} = C_3;\quad \dot{y_0} = C_4;$$
$$x_0 = C_5;\quad y_0 = C_6.$$

Whence,

$$C_3 = v_B\cos\alpha;\quad C_4 = v_B\sin\alpha;$$
$$C_5 = 0;\quad C_6 = 0.$$

Equations for the projections of velocity of a stone are

$$\dot{x} = v_B\cos\alpha;\quad \dot{y} = gt + v_B\sin\alpha,$$

and equations of its motion have the following form:

$$x = v_Bt\cos\alpha;\quad y = gt^2/2 + v_Bt\sin\alpha.$$

By excluding parameter t from equations of motion one can derive equation of the path of the stone. Determine t from the first equation and then substitute its value into the second one. We have

$$y = gx^2/(2v_B^2\cos^2\alpha) + xtg\alpha.$$

At point C, $y = h = 5\,m,\ x = d.$

Determining d from equation of the path we have

$$d_1 = 2{,}11\,m,\quad d_2 = -7{,}75\,m.$$

Since equation of the path is a branch of parabola with positive abscissas of its points, then $d = 2{,}11\,m$. The minimal width *CD* is

$$b = d - ED = d - h/\text{tg}75°,\quad \text{or } b = 0{,}77\,m.$$

By making use of equation of motion $x = v_Bt\cos\alpha$, we determine the time *T* of motion of the stone from the point *B* to the point *C:*

$$T = 0{,}53\ s.$$

Velocity of the stone at the point C may be determined by calculating their projections on coordinate axes

$$\dot{x} = v_B \cos\alpha; \quad \dot{y} = gt + v_B \sin\alpha.$$

Finally we have

$$v = \sqrt{\dot{x}^2 + \dot{y}^2}.$$

For instant, when the stone is at the point *C,* we have $t = T = 0{,}53\ s$ and

$$v_C = \sqrt{(v_B \cos\alpha)^2 + (gT + v_B \sin\alpha)^2}, \text{ or } v_C = 12{,}8\ m/\text{s}.$$

* The constants of integration $C_1 - C_6$ can be determined by considering initial conditions for the first and the second sections of the motion. Nevertheless, sometimes it is easier to use boundary conditions for different instants.

8.2. Application of the Theorem of the Change in Kinetic Energy to Study of the Motion of a System

Mechanical system starts moving from the state of rest under the action of the forces of weight; initial state of the system is shown in Figs. 106-108. Determine acceleration of a body *1* and its velocity when it has travelled a distance *s.* Take into account a force of sliding friction (variants 1-3, 5, 6, 8-12, 17—23, 28-30) and the rolling friction of a body *3* (variants 2, 4, 6-9, 11, 13-15, 20, 21, 24, 27, 29). Neglect other forces of resistance and masses of the strings. Assume the strings as inextensible. Make use of the following designations: m_1, m_2, m_3, m_4 are the masses of the bodies *1, 2, 3, 4*; R_2, r_2, R_3, r_3— radii of respective circumferences; i_2, i_3— radii of gyration of the bodies *2* and *3* with respect to axes perpendicular to the plane of figure trough their centers of gravity; α, β— angles of inclination of planes to horizon; *f*—coefficient of sliding friction; δ — coefficient of rolling friction.

The necessary data are represented in tab.1. Assume pulleys and blocks for which the radii of gyration are not given as homogeneous disks. Inclined strings are parallel to respective inclined planes.

Example. Given are: m_1 is a mass of the weight *1,* $m_2 = 2m_1$, $m_3 = m_1$, $m_4 = 0{,}5m_1$, $m_5 = 20m_1$, $R_2 = R_3 = 12\ cm$, $r_2 = 0{,}5R_2$, $r_3 = 0{,}75R_3$, $R_5 = 20\ cm$, $AB = l = 4R_3$, $i_2 = 8\ cm$, $i_3 = 10\ cm$, $\alpha = 30°$, $f = 0{,}1$, $\delta = 0{,}2\ cm$, $s = 0{,}06\pi\ m$.

Neglect rolling friction of a body *2*, masses of the link BC_5 and the slide *B*.

Connecting rod *4* is a thin homogeneous rod. Roller *5* is a homogeneous solid cylinder. Initial state of a system is shown in Fig. 105a.

Determine v_1 —velocity of the weight in a final position and its acceleration.

Solution. Apply the theorem of the change in kinetic energy of a system:

$$T - T_0 = \sum A_i^E + \sum A_i^J,$$

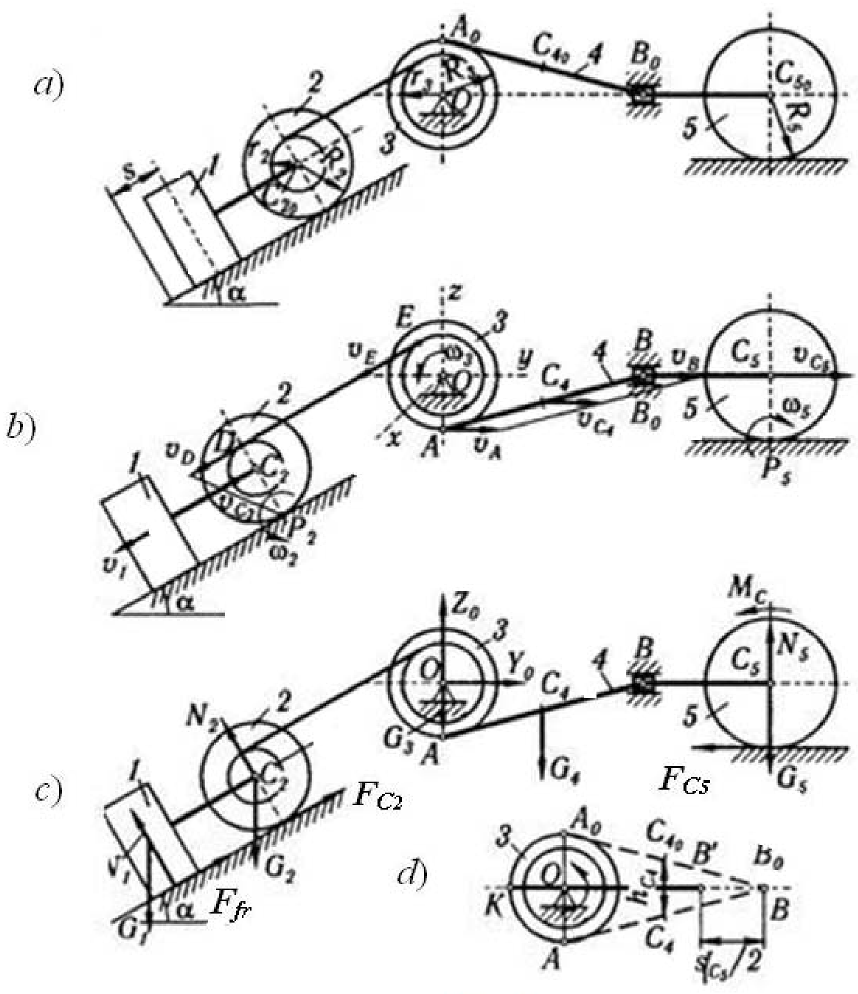


Fig. 105

where T and T_0 are kinetic energy of a system in a final and initial position; $\sum A_i^E$ is a sum of the work done by all the external forces acting on the system during its displacement from initial to final position; $\sum A_i^J$ is a sum of the work done by all the internal forces in that displacement.

For systems in question which consist of solids connected by inextensible thread and rods

$$\sum A_i^J = 0.$$

Since the system in initial position is in a state of rest, $T_0 = 0$.

Hence, we have $T = \sum A_i^E$.

In order to define kinetic energy T and the sum of the work done by all the external forces, depict the system in a final position (Fig. 105*b,c*).

Write down relations between velocities and displacements of the points of a system. Express all velocities and displacements through velocity and displacement of the weight *1*.

Velocity of the center of mass of the roller 2 equals velocity of the weight *1*:$v_{C2} = v_1$.

The instantaneous center of zero velocity of the roller *2* is located at point P_2. Its angular velocity is

$$\omega_2 = \frac{v_{C2}}{C_2P_2} \text{ or } \omega_2 = \frac{v_1}{R_2}.$$

Velocity of the point D is

$$v_D = \omega_2 DP_2, \text{ i.e,}$$

$$v_D = \frac{v_1(R_2+r_2)}{R_2}.$$

It is obvious that $v_E = v_D$. But $v_E = \omega_3 r_3$, hence,

$$\omega_3 r_3 = \frac{v_1}{R_2}(R_2 + r_2).$$

Since $R_2 = 2r_2$, then

$$\omega_3 r_3 = \frac{3}{2} v_1,$$

whence $\omega_3 = \frac{3}{2}\frac{v_1}{r_3}$.

Taking into account that

$$\omega_3 = \frac{d\varphi_3}{dt}, v_1 = \frac{ds}{dt},$$

we have

$$\frac{d\varphi_3}{dt} = \frac{3}{2r_3}\frac{ds}{dt}, \text{ or } d\varphi_3 = \frac{3}{2r_3} ds.$$

By integrating one can obtain

$$\varphi_3 = \frac{3}{2}\frac{s}{r_3}.$$

When the weight *1* travels a distance $s = 0{,}06\pi\ m$, the pulley *3* turns trough the angle φ_3:

$$\varphi_3 = \frac{3}{2}\frac{s}{r_3} = \frac{3}{2}\frac{0{,}06\pi}{0{,}09} = \pi.$$

At this angle of rotation of the pulley *3* on $180°$ its point A_0 moves to the final location *A,* and connecting rod *4* moves from initial location A_0B_0 to the final location *AB.*

The roller *5* moves to the left at the angle of rotation of the pulley *3* equal to $\frac{\pi}{2}$, and it moves to the right at the angle of rotation equal to π. Hence, the final location of the roller *5* coincides with its initial location.

Thus, the final location of all the parts of a system is defined completely (Fig. 105b).

Determine kinetic energy of a system in the final position as a sum of kinetic energy of the bodies *1, 2, 3, 4, 5:*

$$T = T_1 + T_2 + T_3 + T_4 + T_5.$$

Kinetic energy of the load *1,* which is in translational motion, is

$$T_1 = \frac{m_1 v_1^2}{2}.$$

Kinetic energy of the roller *2,* which is in a plane motion, is

$$T_2 = \frac{m_2 v_{C2}^2}{2} + \frac{J_{2\xi}\omega_2^2}{2},$$

where $J_{2\xi}$ is a moment of inertia of the roller *2* with respect to its longitudinal central axis $C_{2\xi}$:

$$J_{2\xi} = m_2 i_2^2.$$

Then we find

$$T_2 = \frac{m_2 v_1^2}{2} + \frac{m_2 i_2^2}{2R_2^2} v_1^2 = \frac{1}{2} m_2 \left(1 + \frac{i_2^2}{R_2^2}\right) v_1^2.$$

Kinetic energy of the body *3,* which rotates around the axis *Ox,* is

$$T_3 = \frac{1}{2} J_{3x}\omega_3^2,$$

where J_{3x} is a moment of inertia of the block *3* with respect to the axis *Ox:*

$$J_{3x} = m_3 i_3^2.$$

Then for the body *3* we obtain

$$T_3 = \frac{m_3 i_3^2}{2}\left(\frac{3}{2}\frac{v_1}{r_3}\right)^2 = \frac{9}{8} m_3 \frac{i_3^2}{r_3^2} v_1^2.$$

Kinetic energy of the connecting rod *4,* which is in a plane motion, is

$$T_4 = \frac{m_4 v_{C4}^2}{2} + \frac{J_{4\xi}\omega_4^2}{2},$$

where v_{C4} is a velocity of the center of mass of the connecting rod *4*; $J_{4\xi}$ is its moment of inertia with respect to the central axis $C_{4\xi}$.

In order to determine v_{C4} and ω_4, find location of the instantaneous center of zero velocity of the connecting rod *4.* Since points *A* and *B* at this instant are parallel, the instantaneous center of zero velocity of the connecting rod *4* lies in infinity. Hence, its angular velocity at the given instant is $\omega_4 = 0$, and velocities of all the points are parallel and equal. Thus, kinetic energy of the connecting rod *4* is

$$T_4 = \frac{m_4 v_{C4}^2}{2},$$

where $v_{C4} = v_A$.

Linear velocity of the point *A* of the body *3* is

$$v_A = \omega_3 R_3, \text{or } v_A = \frac{3}{2} R_3 v_1 / r_3.$$

Since $r_3 = 3/4 R_3$, we have $v_A = 2v_1$.

But $v_{C4} = v_A$, $v_{C4} = 2v_1$.

So, the expression for kinetic energy of the connecting rod *4* has the following form:

$$T_4 = \frac{1}{2} m_4 (2v_1)^2 = 2 m_4 v_1^2.$$

Kinetic energy of the roller *5,* which is in a plane motion, is

$$T_5 = \frac{m_5 v_{C5}^2}{2} + \frac{J_{5\xi}\omega_5^2}{2},$$

where v_{C5} is a velocity of the center of mass C_5 of the roller 5; $J_{5\xi}$ — its moment of inertia (as a homogeneous solid cylinder) with respect to its longitudinal central axis $C_{5\xi}$, $J_{5\xi} = \frac{m_5 R_5^2}{2}$; ω_5 — its angular velocity.

Since the roller moves without slipping, its instantaneous center of zero velocity is at point P_5. Then

$$\omega_5 = \frac{v_{C5}}{R_5}.$$

Hence,

$$T_5 = \frac{m_5 v_{C5}^2}{2} + \frac{m_5 R_5^2 v_{C5}^2}{2 \cdot 2 R_5^2} = \frac{3}{4} m_5 v_{C5}^2.$$

As far as the link BC_5 is in a plane motion, $v_{C5} = v_B$. But $v_B = v_{C4} = 2v_1$, then $v_{C5} = 2v_1$.

Therefore, kinetic energy of the roller 5 is

$$T_5 = \frac{3}{4} m_5 (2v_1)^2 = 3m_5 v_1^2.$$

The total kinetic energy of the system will be

$$T = \frac{m_1 v_1^2}{2} + m_2(1 + i_2^2/R_2^2)v_1^2/2 + \frac{9}{8} m_3 v_1^2 i_3^2/r_3^2 + 2m_4 v_1^2 + 3m_5 v_1^2.$$

Substituting values of the given masses, we obtain

$$T = m_1 v_1^2 \left[1 + 2(1 + i_2^2/R_2^2) + \frac{9}{4} i_3^2/r_3^2 + 2 + 120\right]/2, \text{ or}$$

$$T = 129 m_1 v_1^2/2.$$

Define the sum of the work done by all the external forces acting on a system in its specified displacement. Depict all the external forces (Fig. 105c).

Work done by the weight $\boldsymbol{G}_1$ is

$$A_{G_1} = G_1 h_1 = m_1 g s \sin\alpha.$$

Work done by frictional force $\boldsymbol{F}_{fr}$ is

$$A_{F_{fr}} = -F_{fr\mathrm{p}} s.$$

As far as $F_{fr} = f N_1 = f G_1 \cos\alpha$, then

$$A_{F_{fr}} = -f m_1 g s \cos\alpha.$$

Work done by the weight $\boldsymbol{G}_2$ is

$$A_{G_2} = G_2\, h_{C2} = m_2 g s \sin\alpha.$$

Work done by forces of traction $\boldsymbol{F}_{C2}, \boldsymbol{F}_{C5}$ of the rollers *2* and *5* is zero since these forces are applied at their instantaneous centers of zero velocity.

Work done by the weight $\boldsymbol{G}_4$ is

$$A_{G_4} = G_4 h_{C4},$$

where h_{C4} is a vertical displacement of the center of gravity C_4 of the connecting rod *4* from initial location to its final position (Fig. 105d*):*

$$h_{C4} = R_3,\ A_{G_4} = m_4 g R_3.$$

Work done by the rolling friction of the roller *5* is

$$A_{M_C} = -M_C \varphi_5,$$

where $M_C = \delta N_5 = \delta G_5$ is a moment of a couple of the resisting forces to rolling of the roller 5; φ_5 is an angle of its rotation.

Since roller *5* moves without slipping, angle of its rotation is

$$\varphi_5 = s_{C5}/R_5,$$

where s_{C5} is a displacement of the center of gravity C_5 of the roller 5.

In this example the work of the mentioned couple is calculated as a sum of the work done by this couple at the rotation of the body 3 on the angle $\pi/2$ to the left and when the body *3* turns on angle $\pi/2$ to the right once more.

The displacement of the center of gravity C_5 of the roller 5 equals displacement of the slider *B* to the left and to the right:

$$s_{C5} = 2(B_0B').$$

Determine the displacement B_0B' at the rotation of body *3* on angle $\pi/2$. Choose fixed point *K* of the plane as a reference (Fig. 105d*).* At this rotation of the body *3* the connecting rod will move from position A_0B_0 to position KB'. Then

$$B_0B' = KB_0 - KB',$$

where $KB_0 = KO + OB_0 = R_3 + \sqrt{(A_0B_0)^2 - (A_0O)^2} = R_3 + \sqrt{l^2 - R_3^2},$

$$KB' = l = 4R_3.$$

Hence,

$$B_0B' = R_3 + \sqrt{l^2 - R_3^2} - l = R_3 + \sqrt{(4R_3)^2 - R_3^2} - 4R_3 = 0{,}88R_3.$$

The total angle of rotation of the roller *5* is

$$\varphi_5 = 1{,}76R_3/R_5.$$

Then $A_{M_C} = -\delta m_5 g \cdot 1{,}76R_3/R_5.$

The total sum of the work done by all the external forces is

$$\sum A_i^E = m_1 g s\sin\alpha - f m_1 g s\cos\alpha + m_2 g s\sin\alpha + m_4 g R_3 - \delta m_5 g \cdot 1{,}76R_3/R_5.$$

By substituting the given values of masses we obtain

$$\sum A_i^E = m_1 g s(\sin\alpha - f\cos\alpha + 2\sin\alpha + \frac{R_3}{2s} - \frac{\delta \cdot 20 \cdot 1{,}76R_3}{R_5 s})$$

or $\sum A_i^E = 1{,}51 m_1 g s.$

According to the theorem of the change in kinetic energy of a system equate the values $T$and $\sum A_i^E$:

$$129 \cdot \frac{m_1 v_1^2}{2} = 1{,}51 m_1 g s,$$

whence

$$v_1 = 0{,}21\ m/s.$$

In order to define acceleration of the weight, make use of the theorem in differential form:

$$dT = \sum d\, A_i^E.$$

Therefore,

$$129 \cdot \frac{m_1}{2} 2 v_1 d v_1 = 1{,}51 m_1 g d s,$$

whence $\frac{v_1 d v_1}{ds} = \frac{1{,}51}{129} g$ or $a_1 = 0{,}115\ {}^{m}/_{s^2}.$

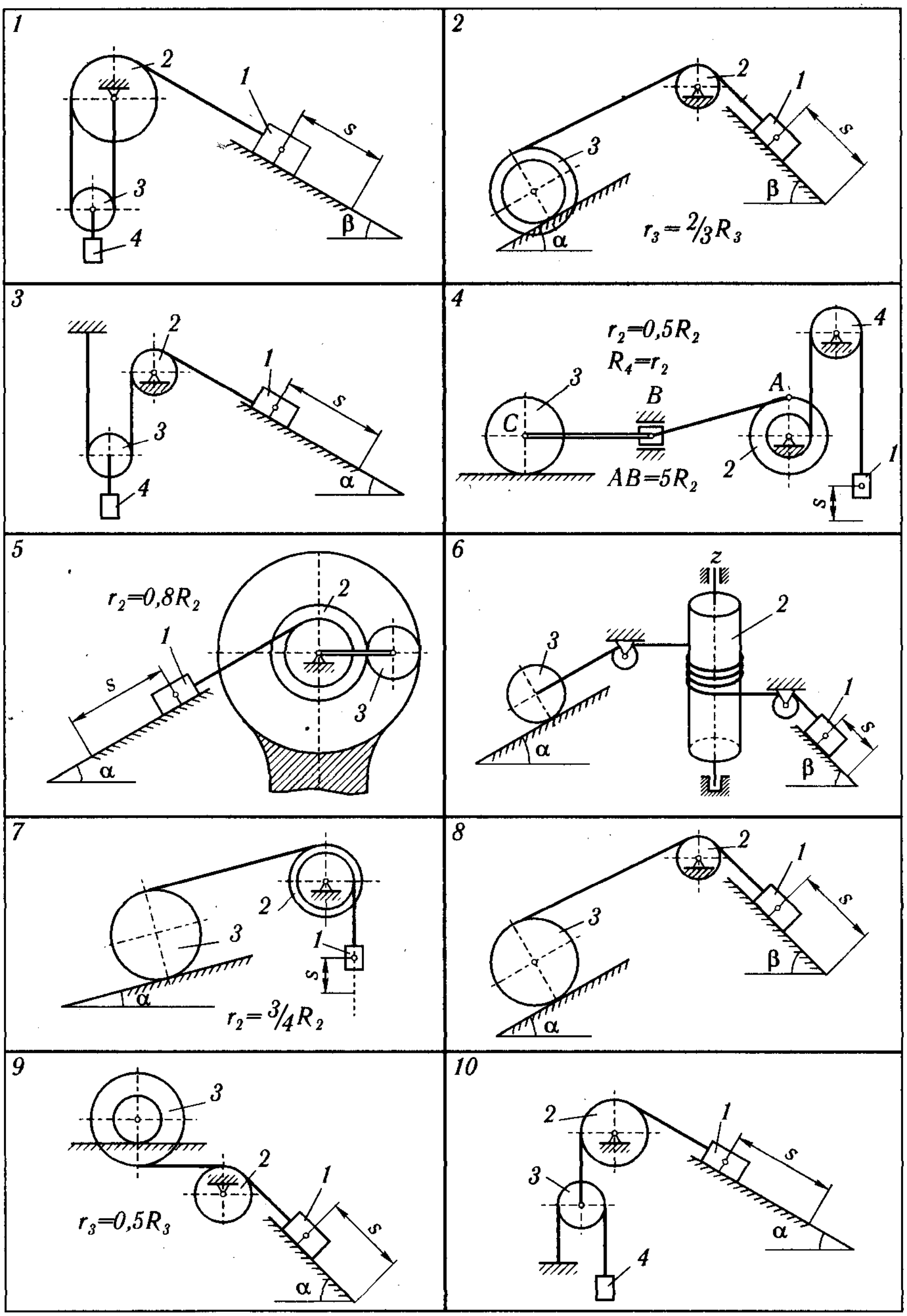

1
2
3
4
5
6
7
8
9
10
r3 = 2/3 R3
r2=0,5R2
R4=r2
AB=5R2
A
B
C
r2=0,8R2
z
r2 = 3/4 R2
r3=0,5R3
s
α
β

Fig. 106

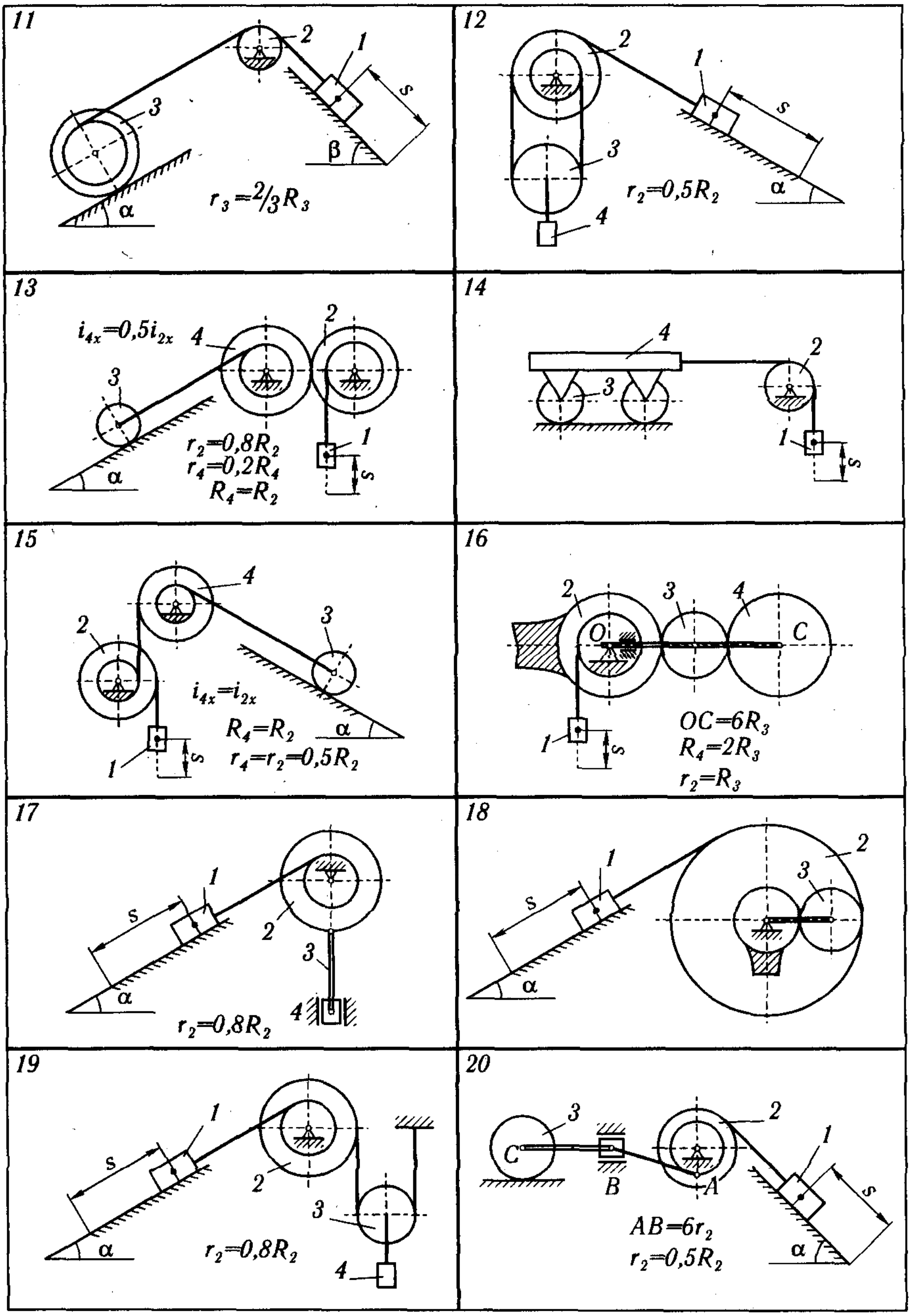

11
2
1
3
s
β
α
$r_3 = 2/3 R_3$
12
2
1
3
s
α
4
$r_2 = 0,5R_2$
13
$i_{4x} = 0,5 i_{2x}$
4
2
3
1
$r_2 = 0,8R_2$
$r_4 = 0,2R_4$
$R_4 = R_2$
α
s
14
4
2
3
1
s
15
4
2
3
$i_{4x} = i_{2x}$
$R_4 = R_2$
$r_4 = r_2 = 0,5R_2$
α
1
s
16
2
3
4
O
C
1
s
$OC = 6R_3$
$R_4 = 2R_3$
$r_2 = R_3$
17
1
s
2
3
α
4
$r_2 = 0,8R_2$
18
1
2
3
s
α
19
1
s
2
3
α
$r_2 = 0,8R_2$
4
20
3
2
1
C
B
A
s
$AB = 6r_2$
$r_2 = 0,5R_2$
α

Fig. 107

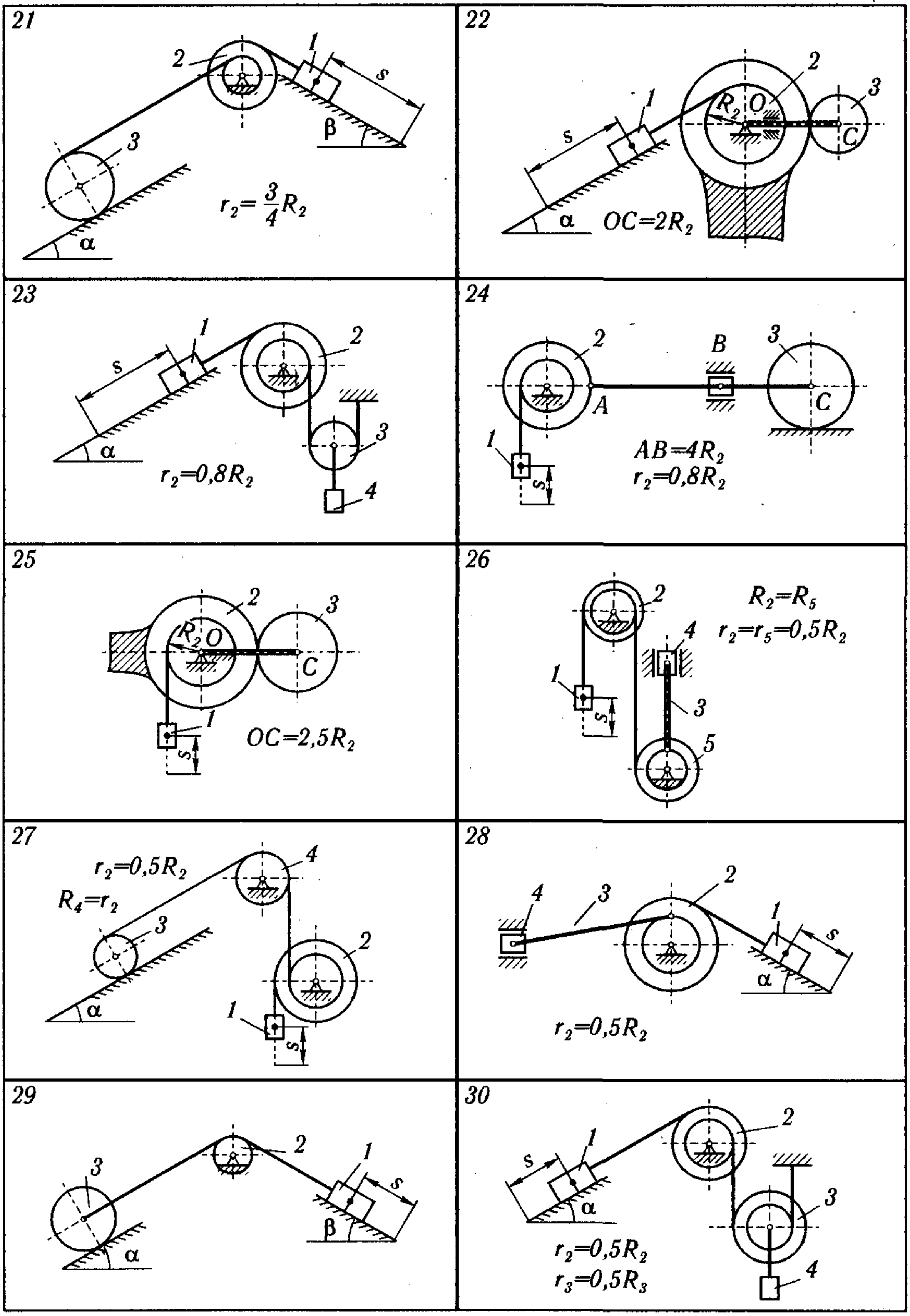

21
r2= 3/4 R2
22
OC=2R2
23
r2=0,8R2
24
AB=4R2
r2=0,8R2
25
OC=2,5R2
26
R2=R5
r2=r5=0,5R2
27
r2=0,5R2
R4=r2
28
r2=0,5R2
29
30
r2=0,5R2
r3=0,5R3

Fig. 108

Tab. 1

Variant number (Fig.106-108)	m_1	m_2	m_3	m_4	R_2	R_3	i_2	i_3	α	β	f	δ, cm	s, m	Notes
	kg				cm		cm		$degree$					
1	m	$4m$	$1/5m$	$4/3m$	–	–	–	–	–	60	0,10	–	2	
2	m	$1/2m$	$1/3m$	–	–	30	–	20	30	45	0,22	0,20	2	
3	m	m	$1/10m$	m	–	–	–	–	45	–	0,10	–	2	
4	m	$2m$	$40m$	m	20	40	18	–	–	–	–	0,30	$0,1\pi$	Neglect the masses of the slider *B*, links *AB* and *BC*
5	m	$2m$	m	–	20	15	18	–	60	–	0,12	–	0.28π	Neglect the mass of the cage
6	m	$3m$	m	–	–	28	–	–	30	45	0,10	0,28	1,5	
7	m	$2m$	$2m$	–	16	25	14	–	30	–	–	0,20	2	
8	m	$1/2m$	$1/3m$	–	–	30	–	–	30	45	0,15	0,20	1,75	
9	m	$2m$	$9m$	–	–	30	–	20	30	–	0,12	0,25	1,5	
10	m	$1/4m$	$1/4m$	$1/5m$	–	–	–	–	60	–	0,10	–	3	
11	m	$1/2m$	$1/4m$	–	–	30	–	25	30	45	0,17	0,20	2,5	
12	m	$1/2m$	$1/5m$	m	30	–	20	–	30	–	0,20	–	2,5	
13	m	$2m$	$5m$	$2m$	30	20	26	–	30	–	–	0,24	2	
14	m	$1/2m$	$5m$	$4m$	–	25	–	–	–	–	–	0,20	2	The masses of all the wheels are equal
15	m	$1/2m$	$4m$	$1/2m$	20	15	18	–	60	–	–	0,25	1,5	

Continuation of the tab. 1

Variant number (Fig.106-108)	m_1	m_2	m_3	m_4	R_2	R_3	i_2	i_3	α	β	f	δ, cm	s, m	Notes
	kg				cm		m		$degree$					
16	m	$1/10m$	$1/20m$	$1/10m$	10	12	–	–	–	–	–	–	$0{,}05\pi$	Neglect the mass of the cage
17	m	$1/4m$	$1/5m$	$1/10m$	20	–	15	–	60	–	0,10	–	$0{,}16\pi$	Consider connecting rod 3 as a thin homogeneous rod
18	m	$3m$	m	–	35	15	32	–	60	–	0,15	–	$0{,}2\pi$	Neglect the mass of the cage
19	m	$1/3m$	$1/10m$	m	24	–	20	–	60	–	0,15	–	1,5	
20	m	$2m$	$20m$	–	20	15	16	–	30		0,10	0,20	$0{,}2\pi$	Neglect the masses of the slider *B*, links *AB* and *BC*
21	m	m	$2m$	–	20	20	16	–	30	45	0,20	0,32	1,2	
22	m	$1/2m$	$1/4m$	–	20	10	–	–	60	–	0,17	–	$0{,}1\pi$	Neglect the mass of the cage
23	m	m	$1/10m$	$4/5m$	20	–	18	–	30	–	0,10	–	1	
24	m	$3m$	$20m$	–	20	30	18	–	–	–	–	0,60	$0{,}08\pi$	Neglect the masses of the slider *B*, links *AB* and *BC*
25	m	$1/3m$	$1/4m$	–	16	20	–	–	–	–	–	–	$0{,}04\pi$	Neglect the mass of the cage
26	m	$1/2m$	m	$1/3m$	30	–	20	–	–	–	–	–	$0{,}6\pi$	The masses and moments of inertia of pulleys *2* and *5* are equal. Consider connecting rod *3* as a thin homogeneous rod
27	m	m	$6m$	$1/2m$	20	20	16	–	30	–	–	0,20	2	
28	m	$2m$	$3m$	–	20	–	14	–	60	–	0,10	–	$0{,}1\pi$	Consider connecting rod *3* as a thin homogeneous rod
29	m	$1/4m$	$1/8m$	–	–	35	–	–	15	30	0,20	0,20	2,4	
30	m	$1/2m$	$3/10m$	$3/2m$	26	20	20	18	30	–	0,12	–	2	

8.3. Application of Virtual Work Principle to the Static Problems

Mechanisms are in a state of equilibrium. Their schemas are shown in Fig. 111-113. The necessary data are given in tab. 2. By making use of virtual work principle determine quantity specified in table 2. Neglect the forces of resistance. Mechanisms are located in a vertical plane for variants 3, 6, 10, 14, 16, 18, 19, 25 and 30, the rest of mechanisms are in a horizontal plane.

Example. Given are: $Q = 100\ N;\ c = 5\ {}^{N}/_{cm};\ r_1 = 20\ cm;\ r_2 = 40\ cm;\ r_3 = 10\ cm;\ OA = l = 50\ cm;\ \alpha = 30°;\ \beta = 90°$ (Fig. 109)

Determine deformation of the spring h in a state of equilibrium neglecting the weight of the links *OA* and *AB*.

Solution. Mechanism is under the action of the following balanced force system: elastic force $\boldsymbol{F},\ \boldsymbol{G}_1 -$ weight of the shaft *1* with gear *2*, $\boldsymbol{G}_3 -$ weight of the gear *3*, $\boldsymbol{G}_4 -$ weight of the slider B, $\boldsymbol{Q} -$ weight of the load, and the reactions of constraints (supports) which are not shown in Fig. 109.

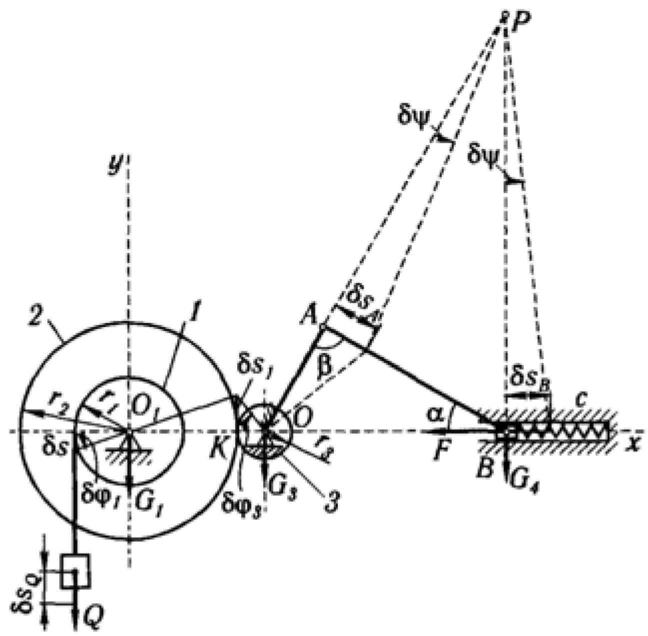


Fig. 109

Make up equation of virtual work principle taking into account that all constraints are ideal (6.6):

$$\sum \delta A_k^a = 0.$$

There are following virtual displacements consistent with the constraints of mechanism in this problem: rotation of the shaft *1* with the gear *2* on angle $\delta\varphi_1$, rotation of the gear *3* on angle $\delta\varphi_3$ and vertical translation of the load δs_Q. The slider *B* has a virtual displacement δs_B (along piston rod guide), and point *A* has a displacement δs_A (δs_A is perpendicular to *OA*). Equation of the virtual work principle has a form:

$$Q\delta s_Q - F\delta s_B = 0.$$

Define virtual displacements relation. Since the load Q is fasten to inextensible string and there is no sliding between the string and the shaft, the displacement of the load Q equals the displacement of the points of the rim of the rod *1*. Therefore, the angle of rotation of the shaft with gear *2* is

$$\delta\varphi_1 = \delta s_Q / r_1.$$

The displacement of the point *K* is

$$\delta s_1 = r_2\delta\varphi_1 = (r_2/r_1)\delta s_Q.$$

Since there is no sliding between the rods *2* and *3,* the virtual displacements of their points of contact are equal and the angle of rotation of the gear *3* is

$$\delta\varphi_3 = \delta s_1/r_3 = [r_2/(r_1r_3)]\delta s_Q.$$

The crankshaft *OA* is rigidly connected with the gear *3* and so

$$\delta s_A = OA\delta\varphi_3 = [r_2 l/(r_1r_3)]\delta s_Q.$$

In order to determine the dependency between virtual displacements δs_B and δs_A, find the position of the instantaneous rotation center of the link *AB*, i.e., point *P.*

Then

$$\delta s_B/\,\delta s_A = PB/PA; \qquad \delta s_B = (PB/PA)\delta s_A.$$

From the ΔAPB

$$PB/PA = 1/\cos 30°.$$

Hence,

$$\delta s_B = [r_2 l/(r_1r_3\cos 30°)]\delta s_Q.$$

Elastic force of the spring is proportional to its deformation: $F = ch$. Then from equation of the virtual work principle we have

$$Q\delta s_Q - ch[r_2 l/(r_1r_3\cos 30°)]\delta s_Q = 0,$$

whence

$$h = \frac{Qr_1r_3\cos 30°}{cr_2 l}; \quad h = 1{,}74\ cm.$$

Consequently the spring is compressed on 1,74 *cm*.

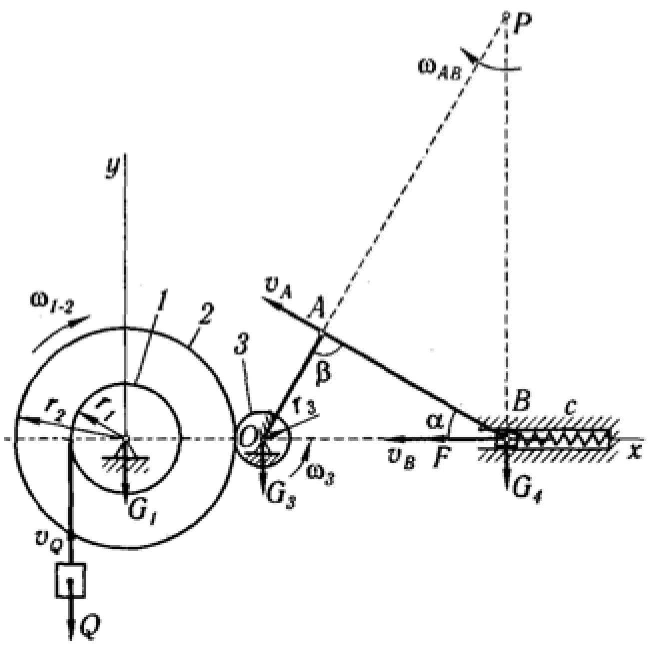


Fig.110

Let us resolve this problem by composing equation of power which expresses virtual velocities principle:

$$\sum \boldsymbol{P}_i \cdot \boldsymbol{v}_i = 0 \text{ or}$$
$$\sum P_i v_i \cos\left(\widehat{\boldsymbol{P}_i, \boldsymbol{v}_i}\right) = 0,$$

where $\boldsymbol{P}_i$ are all the active forces applied to a system, $\boldsymbol{v}_i$ are velocities of their points of application.

Give the shaft *1* with the gear *2* a virtual angular velocity ω_{1-2} around its axis of rotation, let it be clockwise (Fig.110). Then the load Q gets vertical velocity v_Q. The gear *3* with rigidly connected crankshaft *OA* acquires angular velocity ω_3 around *O.*

The link *AB* will have angular velocityω_{AB} which can be represented around instantaneous velocity center *P*. This center is located at point of intersection of perpendiculars erected to velocities $\boldsymbol{v}_A$

and $\boldsymbol{v}_B$ (velocity $\boldsymbol{v}_A$ belonging to the crankshaft OA is perpendicular to OA, and velocity $\boldsymbol{v}_B$ belonging to the slider is parallel to the piston rod guide).

Compose equation of virtual velocity principle:

$$-Qv_Q + Fv_B = 0, \text{ or } -Qv_Q + chv_B = 0.$$

There are three unknown quantities here: deformation of the spring h, velocities v_Q and v_B.

Velocity of the load equals velocities of the points of the rim since the string is inextensible, and so

$$v_Q = \omega_{1-2} r_1.$$

Velocities of the point of contact *K* of the gears *2* and *3* are

$$v_K = \omega_{1-2} r_2, \quad v_K = \omega_3 r_3.$$

They are equal since there is no sliding between the gears.

As far as a point *A* belongs simultaneously to the crankshaft *OA* and to the link *AB,* we have

$$v_A = \omega_3 \cdot OA; \qquad v_A = \omega_{AB} \cdot AP.$$

Velocity of the point *B* of the link *AB* is

$$v_B = \omega_{AB} \cdot BP.$$

Therefore,

$$\omega_{1-2} r_2 = \omega_3 r_3; \qquad \omega_3 \cdot OA = \omega_{AB} \cdot AP,$$

whence

$$\omega_3 = \frac{\omega_{1-2} r_2}{r_3}; \quad \omega_{AB} = \frac{\omega_3 \cdot OA}{AP} = \frac{\omega_{1-2} r_2 \cdot OA}{r_3 \cdot AP}.$$

Then

$$v_B = \omega_{AB} \cdot BP = \frac{\omega_{1-2} r_2 \cdot OA \cdot BP}{r_3 \cdot AP}.$$

From the ΔAPB

$$AP = BP \cdot \cos 30°.$$

So

$$v_B = \frac{\omega_{1-2} r_2 \cdot OA}{r_3 \cos 30°}.$$

Thus, equation of the virtual velocity principle acquires the following form:

$$-Q\omega_{1-2} r_1 + ch \frac{\omega_{1-2} r_2 l}{r_3 \cos 30°} = 0.$$

Dividing this equation by ω_{1-2}, we find deformation of the spring

$$h = \frac{Q r_1 r_3 \cos 30°}{c r_2 l}.$$

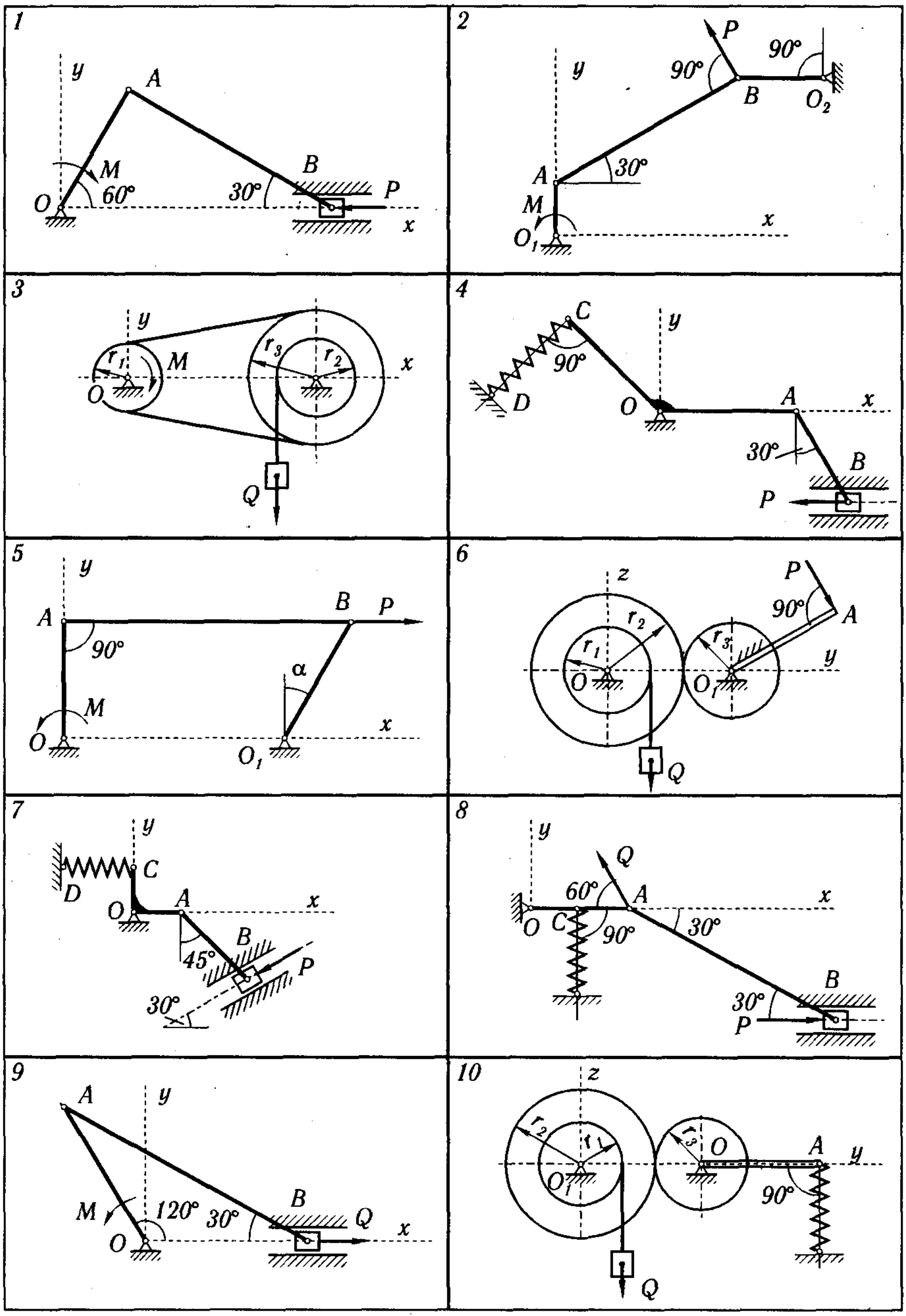

1
y
A
M
60°
O
B
30°
P
x
2
P
90°
90°
y
B
O2
A
30°
M
O1
x
3
y
r1
M
O
r3
r2
x
Q
4
C
y
90°
D
O
A
x
30°
B
P
5
y
A
B
P
90°
α
M
O
O1
x
6
z
P
90°
A
r2
r1
r3
y
O
O1
Q
7
y
C
D
O
A
x
B
45°
P
30°
8
y
Q
60°
A
x
O
C
90°
30°
B
30°
P
9
A
y
M
120°
30°
B
Q
x
O
10
z
r2
r1
r3
O
A
y
O1
90°
Q

Fig. 111

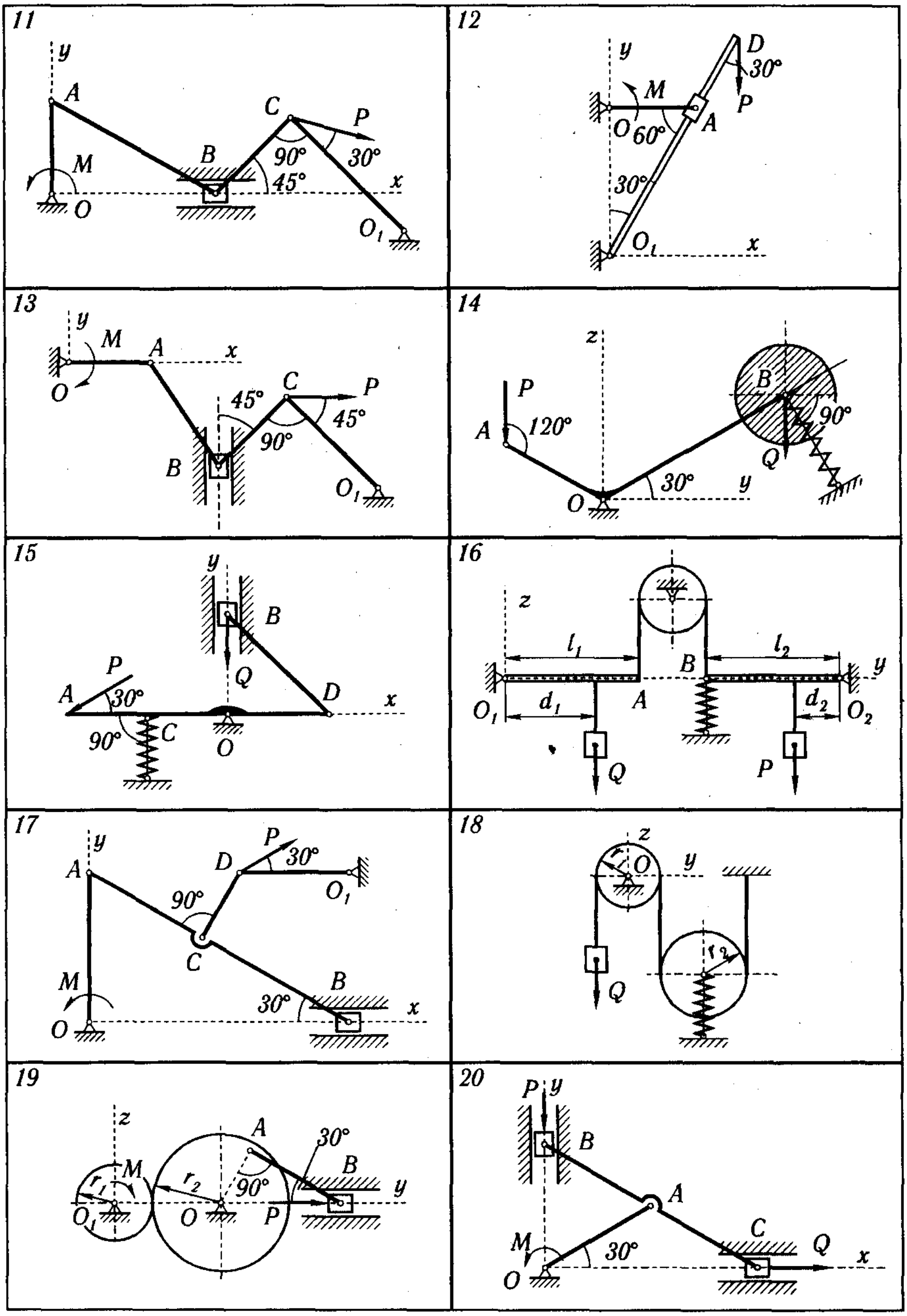

11
y
A
M
O
B
C
P
90°
30°
45°
x
O_1
12
y
D
30°
M
P
O
60°
A
30°
O_1
x
13
y
M
A
x
O
C
P
45°
45°
90°
B
O_1
14
z
P
B
A
120°
90°
Q
30°
y
O
15
y
B
Q
P
A
30°
D
x
90°
C
O
16
z
l_1
l_2
B
y
O_1
d_1
A
d_2
O_2
Q
P
17
y
P
A
D
30°
O_1
90°
C
M
B
30°
x
O
18
z
r
O
y
r_2
Q
19
z
A
30°
M
r_1
r_2
90°
B
y
O_1
O
P
20
P
y
B
A
M
30°
C
Q
x
O

Fig. 112

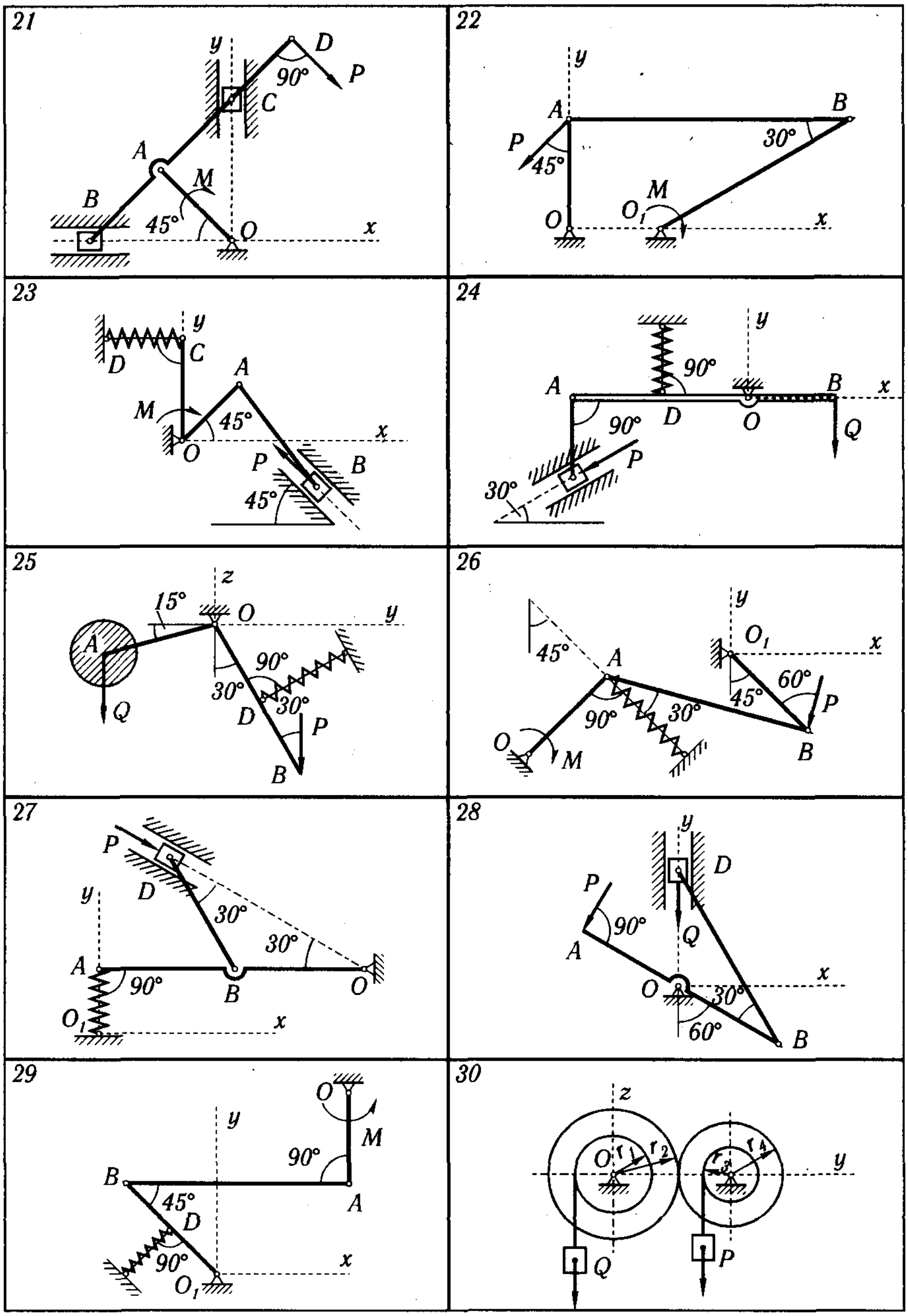

21
y
D
90°
P
C
A
M
B
45°
O
x
22
y
A
B
P
30°
45°
M
O
O_1
x
23
y
D
C
A
M
45°
O
x
P
B
45°
24
y
90°
A
B
x
D
O
90°
Q
P
30°
25
z
15°
O
y
A
90°
Q
30°
30°
D
P
B
26
y
O_1
x
45°
A
60°
45°
P
90°
30°
O
M
B
27
P
y
D
30°
30°
A
90°
B
O
O_1
x
28
y
D
P
90°
Q
A
x
O
30°
60°
B
29
O
y
M
90°
B
A
45°
D
90°
x
O_1
30
z
O
r_1
r_2
r_3
r_4
y
Q
P

Fig. 113

Tab. 2

Variant number (Fig.111-113 -)	Linear dimensions	Forces, N		Moment of a couple M, $N \cdot m$	Stiffness factor $c, N/cm$	Spring deflection h, cm	Determine	Notes
		Q	P					
1	$OA = 10\ cm$	–	–	20	–	–	P	
2	$O_1A = 20\ cm$	–	100	–	–	–	M	
3	$r_1 = 20\ cm, r_2 = 30\ cm,$ $r_3 = 40\ cm$	–	–	100	–	–	Q	
4	$OC:OA = 4:5$	–	200	–	–	4	c	
5	$OA = 100\ cm$	–	–	10	–	–	P	
6	$r_1 = 15\ cm, r_2 = 50\ cm,$ $r_3 = 20\ cm, O_1A = 80\ cm$	200	–	–	–	–	P	Neglect the weight of the handle O_1A
7	$OC = OA$	–	–	–	10	3	P	Spring is compressed
8	$OC = AC$	–	200	–	10	2	Q	The same
9	$OA = 20\ cm$	200	–	–	–	–	M	
10	$r_1 = 15\ cm, r_2 = 40\ cm,$ $r_3 = 20\ cm, OA = 100\ cm$	$2 \cdot 10^3$	–	–	–	4	c	Neglect the weight of the handle OA
11	$OA = 20\ cm$	–	–	300	–	–	P	
12	$O_1D = 60\ cm, OA = 20\ cm$	–	–	100	–	–	P	
13	$OA = 40\ cm$	–	–	200	–	–	P	
14	$OB = 2 \cdot OA$	20	–	–	25	3	P	Neglect the weight of OA and OB; spring is stretched
15	$AC = OC = OD$	$3 \cdot 10^3$	–	–	250	3	P	Spring is compressed

Continuation of the tab. 2

Variant number (Fig.111-113)	Linear dimensions	Forces, N		Moment of a couple M, $N\cdot m$	Stiffness factor $c, N/cm$	Spring deflection h, cm	Determine	Notes
		Q	P					
16	$d_1 = 80\ cm, d_2 = 25\ cm,$ $l_1 = 100\ cm, l_2 = 50\ cm$	$5 \cdot 10^3$	–	–	100	4	P	Neglect the weight of O_1A and O_2B. Spring is compressed
17	$OA = 20\ cm$	–	–	200	–	–	P	
18	$r_1 = 20\ cm,$ $r_2 = 30\ cm$	200	200	–	100	–	h	P — weight of the block of the radius r_2
19	$r_1 = 20\ cm,$ $r_2 = 30\ cm, OA = 25\ cm$	–	–	100	–	–	P	Neglect the weight of the link AB
20	$OA = AB = AC = 50\ cm$	50	100	–	–	–	M	
21	$OA = AB = AC = DC =$ $= 25\ cm$	–	200	–	–	–	M	
22	$OA = 40\ cm$	–	–	400	–	–	P	
23	$OC = 2OA = 100\ cm$	–	200	50	50	–	h	
24	$AD = OD = OBcm$	–	250	–	150	2,5	Q	Spring is compressed
25	$OD = DB = 0,8AO$	400	–	–	120	3	P	Neglect the weight of AO and BO. Spring is stretched
26	$OA = 25cm$	–	500	120	—	2	c	Spring is stretched
27	$OB = AB$	–	–	–	180	2	P	
28	$OB = (5/4)OA$	–	450	–	–	–	Q	
29	$AO = 30\ cm, BD = O_1D$	–	–	120	100	–	h	
30	$r_1 = 15cm, r_2 = 36\ cm,$ $r_3 = 10\ cm, r_4 = 20\ cm$	–	600	–	–	–	Q	

8.4. Application of General Equation of Dynamics to Study of Motion of Mechanical System with One Degree of Freedom

For the given mechanical system define accelerations of weights and a tension in threads to which weights are attached. Neglect mass of threads, friction of a rolling and force of resistance in bearings. The system moves from a state of rest.

Variants of mechanical systems are shown in Fig. 116-118, and the necessary data are represented in tab.3. Radii of gyration are given with respect to central axes perpendicular to the plane of figure.

Assume coefficient of friction identical both for body's sliding along the plane and for braking shoe (variants 9—12).

Assume pulleys and blocks for which the radii of gyration are not given as solid homogeneous disks.

Example. It is given: $G_1 = G_2 = 2G$; $G_3 = G_4 = G$; $R = 2r$; $i_{2x} = r\sqrt{2}$; $f = 0{,}2$.

The block 3 is solid homogeneous cylinder (Fig. 114). Define accelerations of weights *1* and *4* and tension of branches of a thread *1—2* and *3—4.*

Solution. Let us apply to the problem solution the general equation of dynamics. As the system starts moving from a state of rest, directions of accelerations of bodies correspond to directions of their motion.

Whereas among the forces acting on bodies of system, there is a force of a friction, it is expedient to find a true direction of motion according to initial data in order to show correct direction of force of friction.

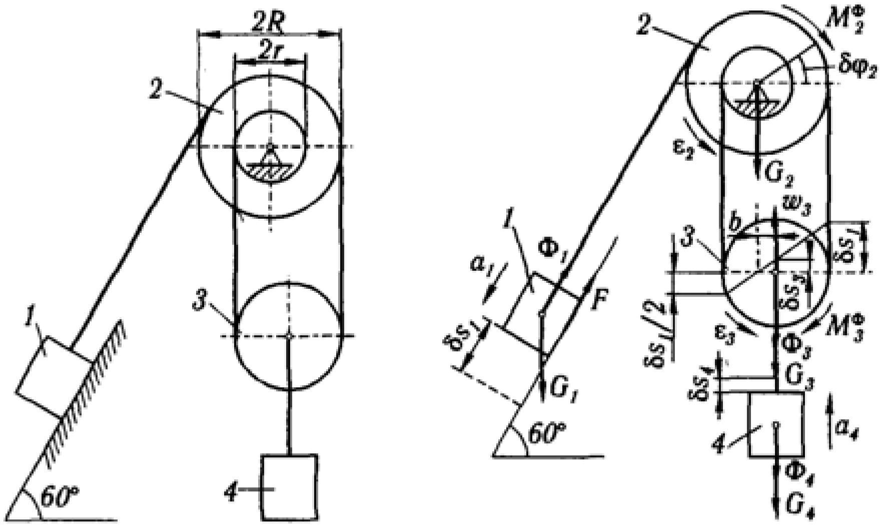


Fig. 114

If the direction of motion of system is chosen wrongly, required acceleration turns out with a sign«—». In this case it is necessary to change directions of force of friction and forces of inertia and to make corresponding corrections in the general equation of dynamics.

In the given example motion of a system is that weight *1* falls.

Let's show force system: a gravity $\boldsymbol{G}_1$— load *1,* $\boldsymbol{G}_2$— the block *2*, $\boldsymbol{G}_3$— the block *3* and $\boldsymbol{G}_4$— load *4*, and also $\boldsymbol{F}$— force of a sliding friction of load *1* on an inclined plane (Fig. 114).

Let's apply forces of inertia. Force of inertia of load *1* making translational motion with acceleration $\boldsymbol{a}_1$ is expressed by a vector

$$\boldsymbol{\Phi}_1 = -m_1\boldsymbol{a}_1.$$

Inertial forces of the block *2* rotating around fixed axis with angular acceleration ε_2 are reduced to a couple. Its moment is

$$M_2^{\Phi} = J_{2x}\varepsilon_2.$$

Inertial forces of the block *3* making a plane motion are reduced to a force

$$\boldsymbol{\Phi}_3 = -m_3\boldsymbol{a}_3,$$

where $\boldsymbol{a}_3$ — acceleration of the center of mass of the block *3*, and to a couple of forces, which moment is

$$M_3^{\Phi} = J_{3x}\varepsilon_3,$$

where ε_3 — angular acceleration of the block.

Inertial force of a load *4* making translational motion with acceleration $\boldsymbol{a}_4$ is

$$\boldsymbol{\Phi}_4 = -m_4\boldsymbol{a}_4.$$

Let's give the system a virtual displacement to a direction of its true motion (Fig. 114) (it is possible to give virtual displacement in the opposite direction).

Writing the general equation of dynamics, we obtain

$$G_1\delta s_1\sin60° - F\delta s_1 - \Phi_1\delta s_1 - M_2^{\Phi}\delta\varphi_2 - G_3\delta s_3 - \Phi_3\delta s_3 - M_3^{\Phi}\delta\varphi_3 - G_4\delta s_4 - \Phi_4\delta s_4 = 0,$$

where $\delta\varphi_2$ and $\delta\varphi_3$— angles of rotation of blocks *2* and *3*.

Relations between virtual displacements are the same, as for relations between corresponding velocities.

Let's express velocities of the centers of mass and angular velocities of bodies of system as a function of velocity of a body *1*.

As is shown in Fig. 114, the instantaneous center of zero velocity of the block *3* is on one vertical with the center of the block *2*. Distance between the instantaneous center of zero velocity and the center of the block *3* is

$$b = \frac{3r}{2} - r = \frac{r}{2}.$$

Now we find

$$\left.\begin{array}{c}\omega_2 = \omega_3 = v_1/R = v_1/2r;\\ v_3 = v_4 = \omega_3 b = v_1/4.\end{array}\right\}$$

The same dependences are between virtual displacements

$$\left.\begin{array}{c}\delta\varphi_2 = \delta\varphi_3 = \delta s_1/(2r);\\ \delta s_3 = \delta s_4 = \delta s_1/4.\end{array}\right\}$$

General equation taking into account these formulas becomes

$$G_1\sin60° - F - \Phi_1 - M_2^{\Phi}/(2r) - G_3/4 - \Phi_3/4 - M_3^{\Phi}/(2r) - G_4/4 - \Phi_4/4 = 0.$$

The same equation can be obtained, if to work out the equation of power, having gave system virtual velocities. The relationships resulted for the real velocities of bodies of system are the same, as for any virtual velocities.

Considering that

$$G_1 = G_2 = 2G = 2mg; \qquad G_3 = G_4 = G = mg,$$

we have

$$\begin{cases} F = fG\cos 60° = fmg; \\ \Phi_1 = m_1 a_1 = 2ma_1; \\ M_2^{\Phi} = J_{2x}\varepsilon_2 = m_2 i_{2x}^2 \varepsilon_2 = 4mr^2\varepsilon_2; \\ \Phi_3 = m_3 a_3 = ma_3; \\ M_3^{\Phi} = J_{3x}\varepsilon_3 = [m_3(1{,}5r)^2/2]\varepsilon_3 = 9mr^2\varepsilon_3/8; \\ \Phi_4 = m_4 a_4 = ma_4. \end{cases}$$

Dependences between accelerations are

$$\left.\begin{aligned} \varepsilon_2 = \varepsilon_3 = a_1/(2r); \\ a_3 = a_4 = a_1/4. \end{aligned}\right\}$$

Then we will obtain

$$g\sqrt{3} - fg - 2a_1 - a_1 - g/4 - a_1/16 - 9a_1/32 - g/4 - a_1/16 = 0,$$

whence

$$a_1 = \frac{g(\sqrt{3} - f - 0{,}5)}{3{,}41}; \quad a_1 = 2{,}96\, m/sec^2;$$

$$a_4 = \frac{a_1}{4}; \qquad a_4 = 0{,}74\, m/sec^2.$$

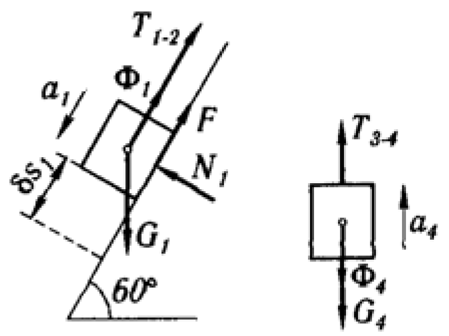


Fig. 115

In order to define tension in a thread *1—2,* we will mentally cut this thread and replace its action on a load *2* by reaction T_{1-2} (Fig. 115).

Then general equation of dynamics will be

$$G_1\delta s_1 \sin 60° - F\delta s_1 - \Phi_1 \delta s_1 - T_{1-2}\delta s_1 = 0,$$

whence $T_{1-2} = G_1 \sin 60° - F - \Phi_1 = 2G\sin 60° - 2Gf\cos 60° - 2(G/g)a_1$; $T_{1-2} = 0{,}93G$.

In order to define tension in a thread *3—4,* we will mentally cut this thread and replace its action on a load *4* by reaction T_{3-4} (Fig. 115).

Without making the general equation of dynamics, on the basis of D'Alembert's principle we have

$$T_{3-4} = G_4 + \Phi_4 = G + (G/g)a_4; \qquad T_{3-4} = 1.08G.$$

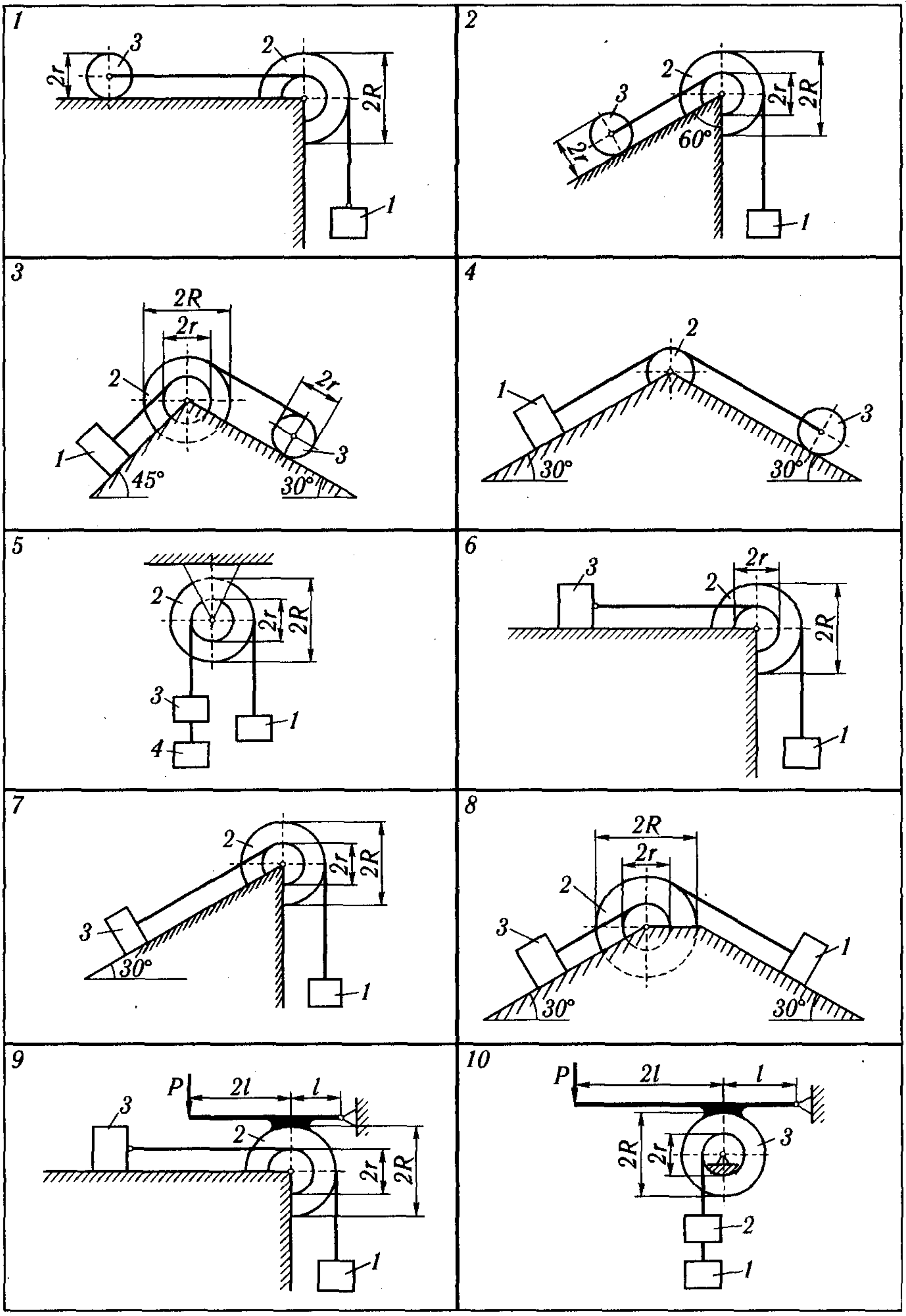

1
2r
3
2
2R
1
2
2
3
2r
2R
2r
60°
1
3
2R
2r
2
1
3
45°
30°
4
2
1
3
30°
30°
5
2
2r
2R
3
1
4
6
3
2r
2
2R
1
7
2
2r
2R
3
30°
1
8
2R
2r
2
3
1
30°
30°
9
P
2l
l
3
2
2r
2R
1
10
P
2l
l
2R
2r
3
2
1

Fig. 116

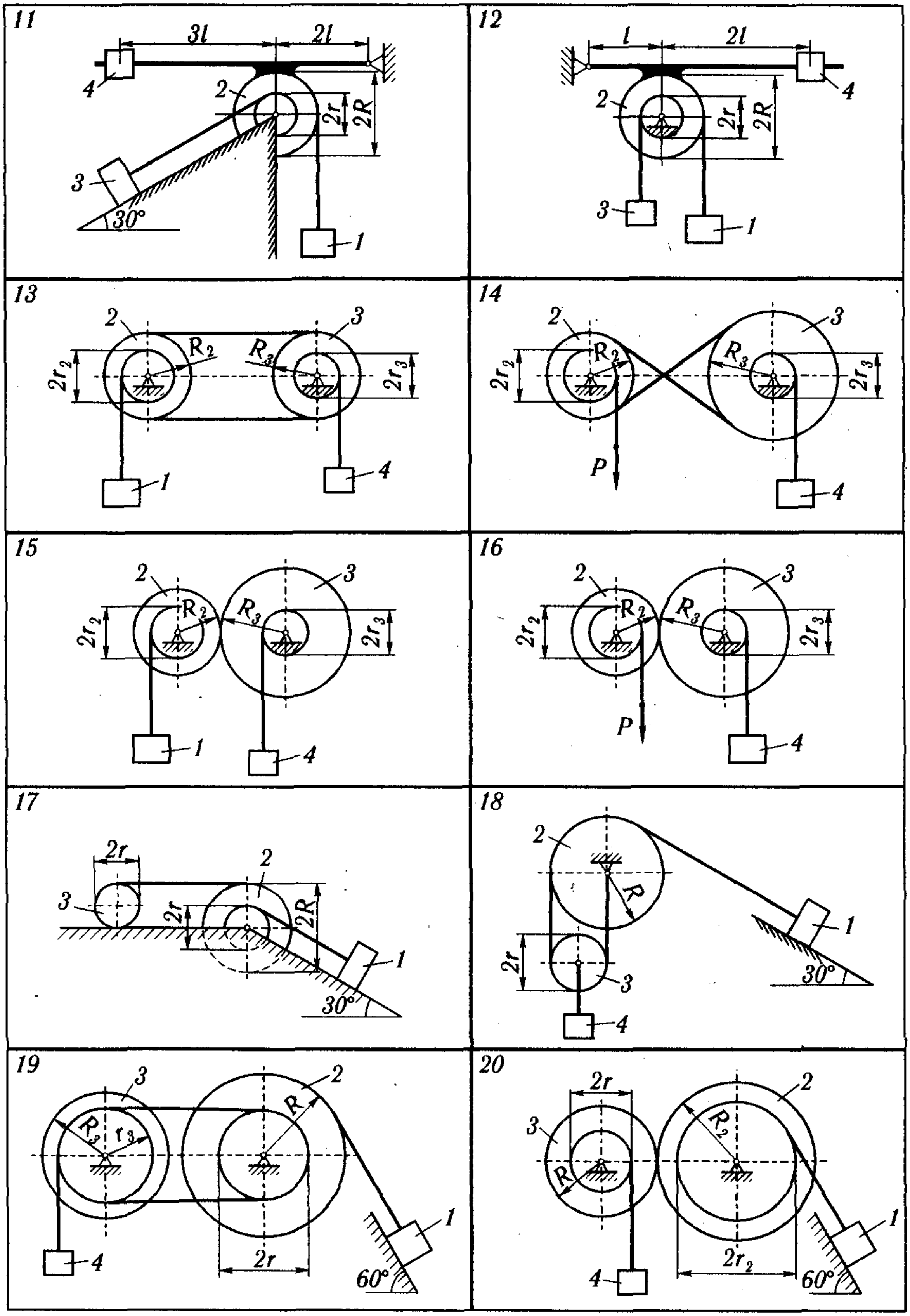

11
3l
2l
4
2
2r
2R
3
30°
1
12
l
2l
4
2
2r
2R
3
1
13
2
3
R_2
R_3
$2r_2$
$2r_3$
1
4
14
2
3
R_2
R_3
$2r_2$
$2r_3$
P
4
15
2
3
R_2
R_3
$2r_2$
$2r_3$
1
4
16
2
3
R_2
R_3
$2r_2$
$2r_3$
P
4
17
2r
2
3
2r
2R
1
30°
18
2
R
1
2r
3
30°
4
19
3
2
R
R_3
r_3
1
4
2r
60°
20
2r
2
3
R_2
R
1
4
$2r_2$
60°

Fig. 117

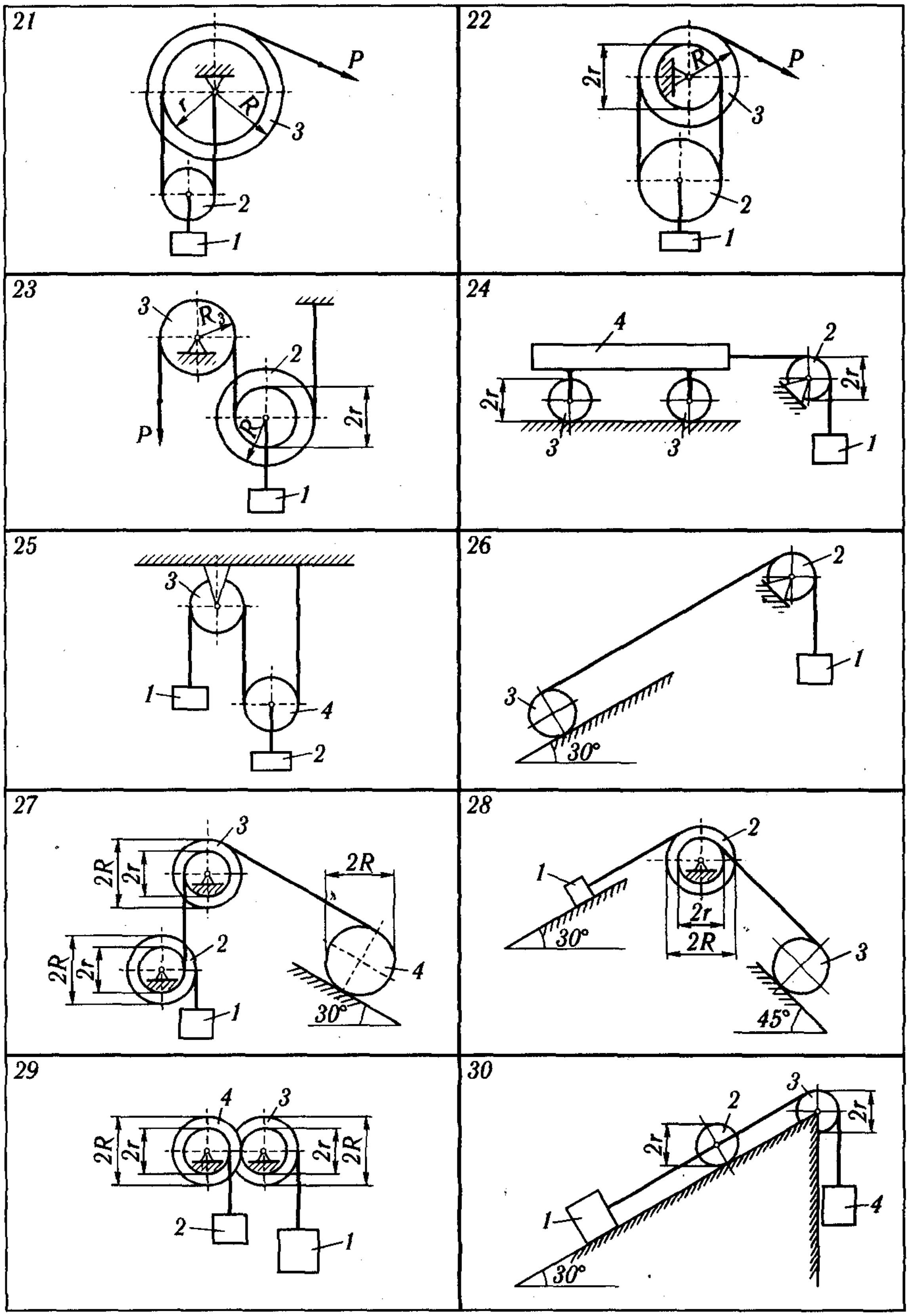

21
P
r
R
3
2
1
22
2r
R
P
3
2
1
23
3
R3
2
2r
P
R
1
24
4
2
2r
2r
3
3
1
25
3
1
4
2
26
2
1
3
30°
27
3
2R
2r
2R
2
2R
2r
4
1
30°
28
1
2
30°
2r
2R
3
45°
29
4
3
2R
2r
2r
2R
2
1
30
3
2
2r
2r
1
4
30°

Fig. 118

Tab. 3

Variant number (Fig.116-118)	Gravity force				$\frac{R}{r}$	Radii of gyration		P	Coefficient of friction f	Supplementary data
	G_1	G_2	G_3	G_4		i_{2x}	i_{3x}			
1	G	G	$3G$	–	2	$r\sqrt{2}$	–	–	–	
2	G	G	G	–	2	$r\sqrt{2}$	–	–	–	
3	$3G$	G	G	–	2	$r\sqrt{2}$	–	–	0,1	
4	G	G	$2G$	–	–	–	–	–	0,2	$r_2 = r_3$
5	$2G$	G	G	G	3	$2r$	–	–	–	
6	$2G$	G	$2G$	–	3	$2r$	–	–	0,2	
7	$2G$	G	$2G$	–	3	$2r$	–	–	0,2	
8	$2G$	G	$2G$	–	3	$2r$	–	–	0,2	
9	$2G$	G	$2G$	–	3	$2r$	–	$0{,}2G$	0,2	
10	$2G$	$2G$	G	–	4	–	$2r$	$G/3$	0,4	
11	$2G$	G	$2G$	$0{,}2G$	3	$2r$	–	–	0,2	
12	$2G$	G	$2G$	$0{,}2G$	3	$2r$	–	–	0,2	
13	$4G$	$2G$	G	$4G$	–	$r_2\sqrt{2}$	$2r_3$	–	–	$r_2 = 2r_3;\ R_2 = R_3$
14	–	G	G	$4G$	–	$r_2\sqrt{2}$	$2r_3$	$8G$	–	$r_2 = 2r_3;\ R_3 = 1{,}5R_2$
15	$4G$	G	$2G$	$4G$		$r_2\sqrt{2}$	$2r_3$	–	–	$r_2 = 2r_3;\ R_3 = 1{,}5R_2$
16	–	G	$2G$	$4G$	–	$r_2\sqrt{2}$	$2r_3$	$4G$	–	$r_2 = 2r_3;\ R_3 = 1{,}5R_2$
17	$2G$	G	G	–	2	$r\sqrt{2}$	–	–	0,1	
18	$3G$	$0{,}2G$	$0{,}1G$	$0{,}5G$	2	–	–	–	0,4	
19	$4G$	$0{,}3G$	$0{,}2G$	$3G$	3	$2r$	$1{,}2r$	–	0,1	$r_3 = 1{,}2r;\ R_3 = 1{,}2r_3$
20	$4G$	$0{,}2G$	$0{,}1G$	$3G$	2	$1{,}6r$	$r\sqrt{2}$	–	0,2	$r_2 = 1{,}5r;\ R_2 = 1{,}2r_2$
21	$5G$	$0{,}1G$	$0{,}2G$	–	3	–	$r\sqrt{2}$	G	–	
22	G	$0{,}2G$	$0{,}3G$	–	2	–	$r\sqrt{2}$	G	–	
23	G	$0{,}2G$	$0{,}1G$	–	1,5	$1{,}2r$	–	$2G$	–	$R_3 = 1{,}2r$
24	$2G$	G	G	$8G$	–	–	–	–	–	Masses of the wheels are equal
25	$6G$	$2G$	$2G$	G	–	–	–	–	–	$r_3 = r_4$
26	$6G$	G	$2G$	–	–	–	–	–	–	$r_3 = r_2$
27	G	G	G	$4G$	2	$r\sqrt{2}$	$r\sqrt{2}$	–	–	
28	$3G$	G	G	–	2	$r\sqrt{2}$	–	–	0,1	
29	$6G$	$3G$	G	G	2	–	$r\sqrt{2}$	–	–	$i_{4x} = i_{3x}$
30	$8G$	G	G	$2G$	–	–	–	–	0,1	

8.5. Research of Free Vibrations of Mechanical Systems with One Degree of Freedom

Define frequency and period of small free vibrations of mechanical system with one degree of freedom neglecting forces of resistance and masses of threads.

Derive the equation of motion of a load *1*, $y = y(t)$, having accepted for origin position of rest of a load *1* (at a static elongation of springs). Determine also amplitude of vibrations of a load *1*.

Schemes of systems are shown in Fig. 120—122 , and the necessary data are represented in tab. 4.

In the problem following designations are accepted: *1* is a load of mass m_1; 2 is a block of mass m_2 and radius r_2 (a solid homogeneous disk); 3 is a block of mass m_3 and gyration radius i_x; 4 is a solid homogeneous disk of mass m_4 and radius r_4; 5 is a disk of mass m_5 and gyration radius i'_x; 6 is a thin homogeneous rod of mass m_6 and length l; 7 is a rod, which mass is neglected; c is a spring stiffness factor; y_0 is an initial deflection of a load *1* on a vertical from position of the rest corresponding to a static elongation of a spring; $\dot{y}_0$ is a projection of initial speed $\boldsymbol{v_0}$ of a load *1* on a vertical axis.

In Fig. 120—122 systems of bodies *1—7* are represented in rest position (at a static elongation of springs).

In variants 5, 6, 14 and 23 rod *6* is rigidly connected to a disk *4*.

Example. It is given: $m_1 = 1kg$; $m_2 = 2kg$; $m_4 = 1kg$; $m_6 = 3kg$; $l = 0{,}6\,m$; $c = 20\,{}^{N}/_{cm}$; $y_0 = 0{,}2\,cm$; $\dot{y}_0 = 8cm/sec$ (Fig. 119).

Define cyclic frequency k, the period T of small free vibrations of system, amplitude a and derive equation of motion of load *1*

Solution. We will take advantage of the Lagrange's equations for conservative systems. Having accepted for system generalized coordinate a vertical deflection y of the weight *1* from the rest position, which corresponds to the static deflection of a spring, we have

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{y}}\right) - \left(\frac{\partial T}{\partial y}\right) = -\frac{\partial \Pi}{\partial y},$$

where T is a system kinetic energy; Π is a system potential energy.

Let's calculate kinetic energy T with the second order infinitesimal accuracy relatively $\dot{y}$, and potential energy Π define with the second order infinitesimal accuracy relatively generalized coordinate y. Determine a kinetic energy of the system as a sum of kinetic energy of bodies *1, 2, 6* and *4:*

$$T = T_1 + T_2 + T_6 + T_4.$$

Let's express velocity of the centre of mass of a body *4* and angular velocities of bodies *2,* 4 and 6 as a function of the generalized velocity $\dot{y}$:

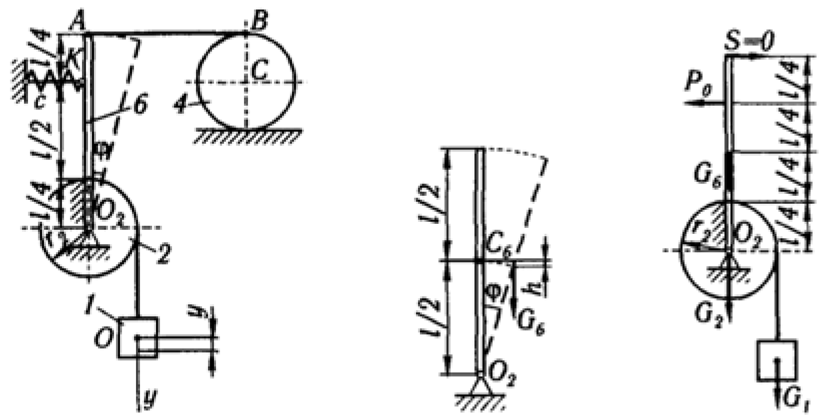


Fig. 119

$$v_1 = \dot{y}; \qquad \omega_2 = \dot{y}/r_2\,; \qquad \omega_6 = \omega_2 = \dot{y}/r_2.$$

As we consider small oscillations, then $v_B = v_A$. Disk *4* rolls without sliding, so $v_C = v_B/2$. Hence,

$$v_C = v_A/2 = \omega_6 l/2 = \omega_2 l/2 = \dot{y}l/(2r_2) = 2\dot{y};$$

$$\omega_4 = v_C/r_4 = 2\dot{y}/r_4.$$

Moment of inertia of the body *4* with respect to the central axis is

$$J_C = m_4 r_4^2/2.$$

Moments of inertia of bodies *2* and *6* with respect to rotation axis are

$$J_2 = m_2 r_2^2/2\,; \qquad J_6 = m_6 l^2/3.$$

Kinetic energy of bodies *1, 2, 4* and *6* is:

$$T_1 = \frac{m_1 v_1^2}{2} = \frac{m_1 \dot{y}^2}{2}; \qquad T_2 = \frac{J_2\omega_2^2}{2} = \frac{m_2\dot{y}^2}{4};$$

$$T_4 = \frac{m_4 v_C^2}{2} + \frac{J_C\omega_4^2}{2} = 3m_4\dot{y}^2; \qquad T_6 = \frac{J_6\omega_6^2}{2} = \frac{8m_6\dot{y}^2}{3}.$$

Thus, kinetic energy of a system is

$$T = T_1 + T_2 + T_6 + T_4 = m_1\dot{y}^2/2 + m_2\dot{y}^2/4 + 8/3m_6\dot{y}^2 + 3m_4\dot{y}^2$$
$$= 1/2[m_1 + m_2/2 + 16/3m_6 + 6m_4]\,\dot{y}^2.$$

Let's calculate a potential energy of a system which equals work of its gravity force and work of elastic force of a spring on a displacement from a deflected location, when load has coordinate *y*, to zero position which is a position of rest of a system:

$$\Pi = \Pi_{\rm I} + \Pi_{\rm II}.$$

Potential energy corresponding to gravity forces on mentioned displacement is

$$\Pi_1 = -G_1 y - G_6 h,$$

where h is a vertical displacement of a center of mass of rod 6, which is computed with the second order infinitesimal accuracy relatively generalized coordinate y.

It follows from Fig. 119

$$h = l/2 - (l/2)\cos\varphi = (l/2)(1 - \cos\varphi).$$

Decomposition formula for $\cos\varphi$ is

$$\cos\varphi = 1 - \varphi^2/2! + \varphi^4/4! - \cdots$$

Restricting this formula by the two first members and considering that

$$\varphi = y/r_2 = 4y/l,$$

we have

$$\Pi_{\mathrm{I}} = -G_1 y - G_6 \cdot 4y^2/l.$$

The potential energy of the deformed spring is

$$\Pi_{\mathrm{II}} = c\,(f_{st} + \lambda_K)^2/2 - cf_{st}^2/2,$$

where f_{st} is a static deflection of a spring; λ_K is a displacement of a point of an attachment of a spring K corresponding to coordinate y.

As, (Fig. 119),

$$\frac{\lambda_K}{y} = \frac{3/4l}{1/4l},$$

i.e., $\lambda_K = 3y$, then

$$\Pi_{\mathrm{II}} = \frac{c(f_{st} + \lambda_K)^2}{2} - \frac{cf_{st}^2}{2} = 3f_{st}y + \frac{9}{2}cy^2.$$

System potential energy is

$$\Pi = \Pi_{\mathrm{I}} + \Pi_{\mathrm{II}} = -G_1 y - (4G_6/l)y^2 + 3cf_{st}y + 9/2cy^2.$$

Since in the position of rest corresponding to a static elongation of a spring

$$(\partial\Pi/\partial y)_{y=0} = 0,$$

then

$$-G_1 + 3cf_{\text{ст}} = 0.$$

This equation can be obtained also having worked out the equation of moments of forces for the state of static equilibrium of system (Fig. 119):

$$\sum M_{i02} = P_0 \cdot 3/4l - G_1 r_2 = 0,$$

or

$$cf_{st} \cdot 3/4l - G_1 l/4 = 0, \qquad \text{i.e.}, 3cf_{st} - G_1 = 0.$$

Thus, potential energy of considered mechanical systems is

$$\Pi = 9/2cy^2 - \frac{4G_6}{l}y^2 = 1/2\left(9C - \frac{8G_6}{l}\right)y^2.$$

Further,

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{y}}\right) = \left(m_1 + \frac{m_2}{2} + \frac{16}{3}m_6 + 6m_4\right)\ddot{y};$$

$$\frac{\partial T}{\partial y} = 0;\ \frac{\partial \Pi}{\partial y} = \left(9C - \frac{8G_6}{l}\right)y.$$

The Lagrange's equation takes a form

$$\left(m_1 + \frac{m_2}{2} + \frac{16}{3}m_6 + 6m_4\right)\ddot{y} + \left(9C - \frac{8G_6}{l}\right)y = 0,$$

or

$$\ddot{y} + \frac{9C - 8G_6/l}{m_1 + m_2/2 + 16m_6/3 + 6m_4}y = 0.$$

Let's introduce designation

$$k^2 = \frac{9C - 8G_6/l}{m_1 + m_2/2 + 16m_6/3 + 6m_4}.$$

Then we shall have the following equation:

$$\ddot{y} + k^2 y = 0.$$

Hence, cyclic frequency of free vibrations is

$$k = \sqrt{\frac{9C - 8G_6/l}{m_1 + m_2/2 + 16m_6/3 + 6m_4}}, \qquad k = 27{,}1\ sec^{-1}.$$

The period of vibrations is

$$T = 2\,\pi/k = 2 \cdot 3{,}14/27{,}1 = 0{,}23 sec.$$

Integrating the differential equation, we obtain the law of motion of a load *1*

$$y = C_1 \cos kt + C_2 \sin kt.$$

In order to define constants C_1 and C_2, determine the equation of velocity of a load

$$\dot{y} = -kC_1 \sin kt + kC_2 \cos kt.$$

Use initial conditions of problem. From the equations $y = y(t)$ and $\dot{y} = \dot{y}(t)$ at $t = 0$ we have

$$y_0 = C_1;\ \dot{y} = kC_2,$$

whence,

$$C_1 = y_0;\ C_2 = \dot{y}_0/k.$$

Hence,

$$y = y_0 \cos kt + (\dot{y}/k) \sin kt,$$
$$y = 0{,}2 \cos 27{,}1t + 0{,}3 \sin 27{,}1\,t.$$

It is possible to obtain this equation in other form if introduce constants of integration a a and β having designated

$$C_1 = a \sin\beta\,;\ C_2 = a \cos\beta.$$

Then

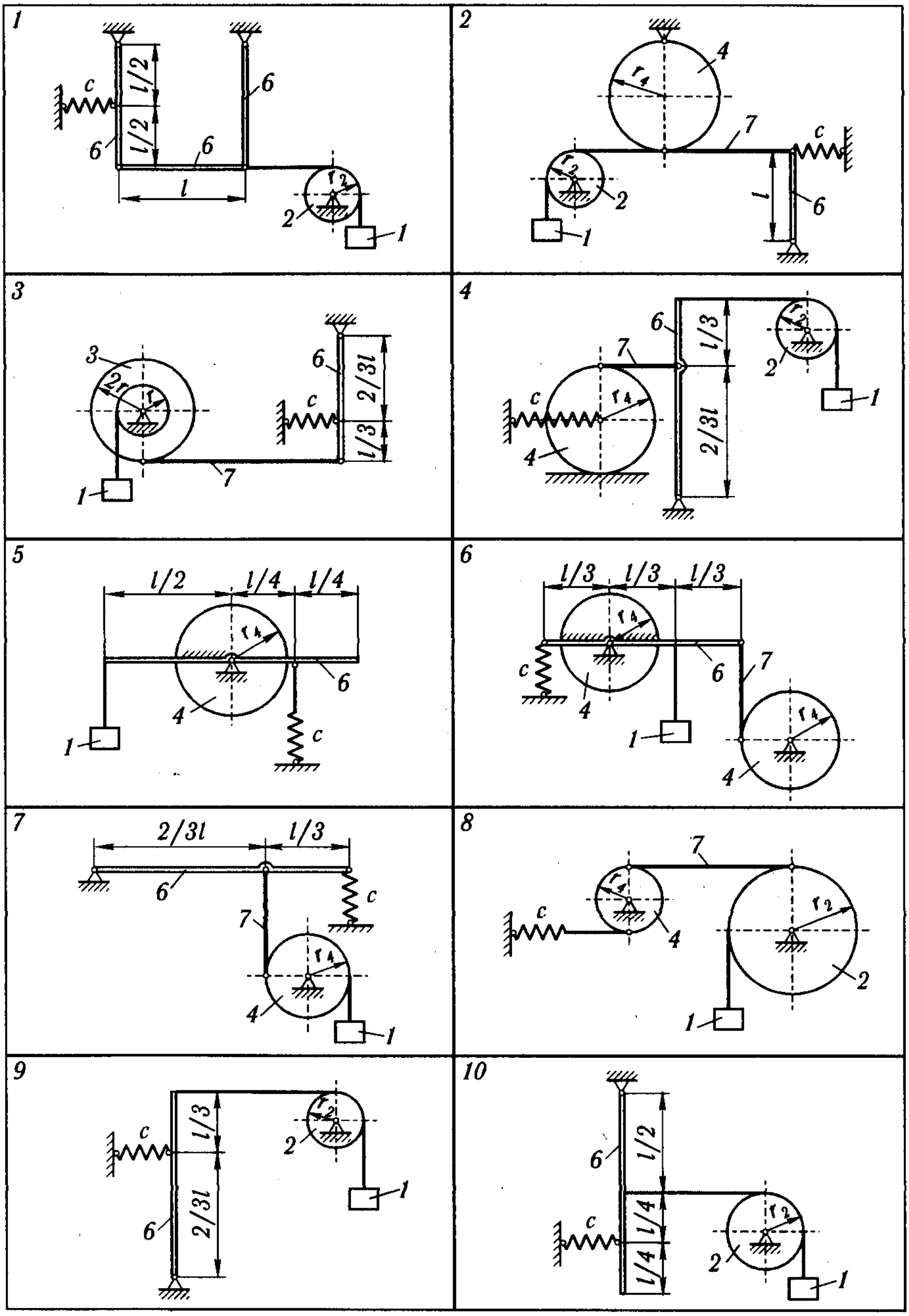
$$y = a \sin(kt + \beta),$$

where $a = \sqrt{C_1^2 + C_2^2}$, $\beta = \tan(C_1/C_2)$, or

$$a = \sqrt{y_0^2 + (\dot{y}_0/k)^2};\ \beta = \tan(ky_0/\dot{y}_0).$$

Let's calculate numerical values a and β: $a = 3{,}6 \cdot 10^{-2} m$, $\beta = \tan 0{,}68$. As $\sin\beta > 0$ $(C_1 > 0)$, then $\beta = 34\,°12' = 0{,}597\ radian$.

Finally $$y = 3{,}6 \cdot 10^{-2} \sin(27{,}1t + 0{,}597)\ m.$$



1
c
l/2
l/2
l
6
6
6
r2
2
1
2
4
r4
7
c
r2
2
l
6
1
3
3
2r
r
6
2/3l
c
l/3
1
7
4
6
7
l/3
r2
2
1
c
r4
2/3l
4
5
l/2
l/4
l/4
r4
6
4
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c
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l/3
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l/3
r4
c
6
7
4
1
r4
4
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2/3l
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6
c
7
r4
4
1
8
7
r4
c
4
r2
1
2
9
c
l/3
r2
2
1
6
2/3l
10
6
l/2
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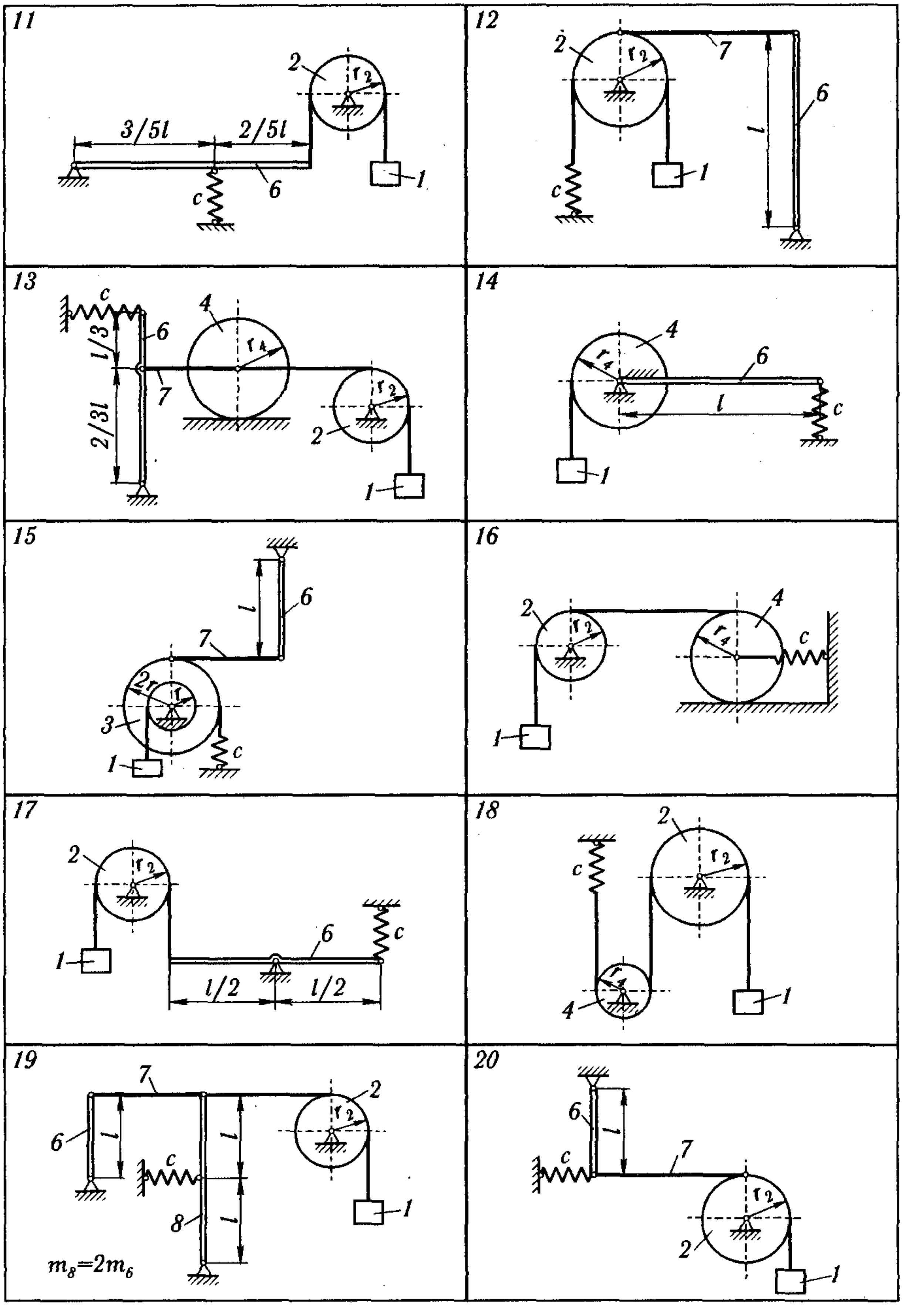
c
l/4
l/4
r2
2
1

Fig. 120



11
2
r_2
3/5l
2/5l
c
6
1
12
2
r_2
7
6
l
c
1
13
c
l/3
2/3l
6
7
4
r_4
r_2
2
1
14
4
r_4
6
l
c
1
15
l
6
7
2r
r
3
c
1
16
2
r_2
4
r_4
c
1
17
2
r_2
1
6
c
l/2
l/2
18
c
2
r_2
r_4
4
1
19
7
2
r_2
6
l
c
l
8
l
1
$m_8=2m_6$
20
6
l
c
7
r_2
2
1

Fig. 121

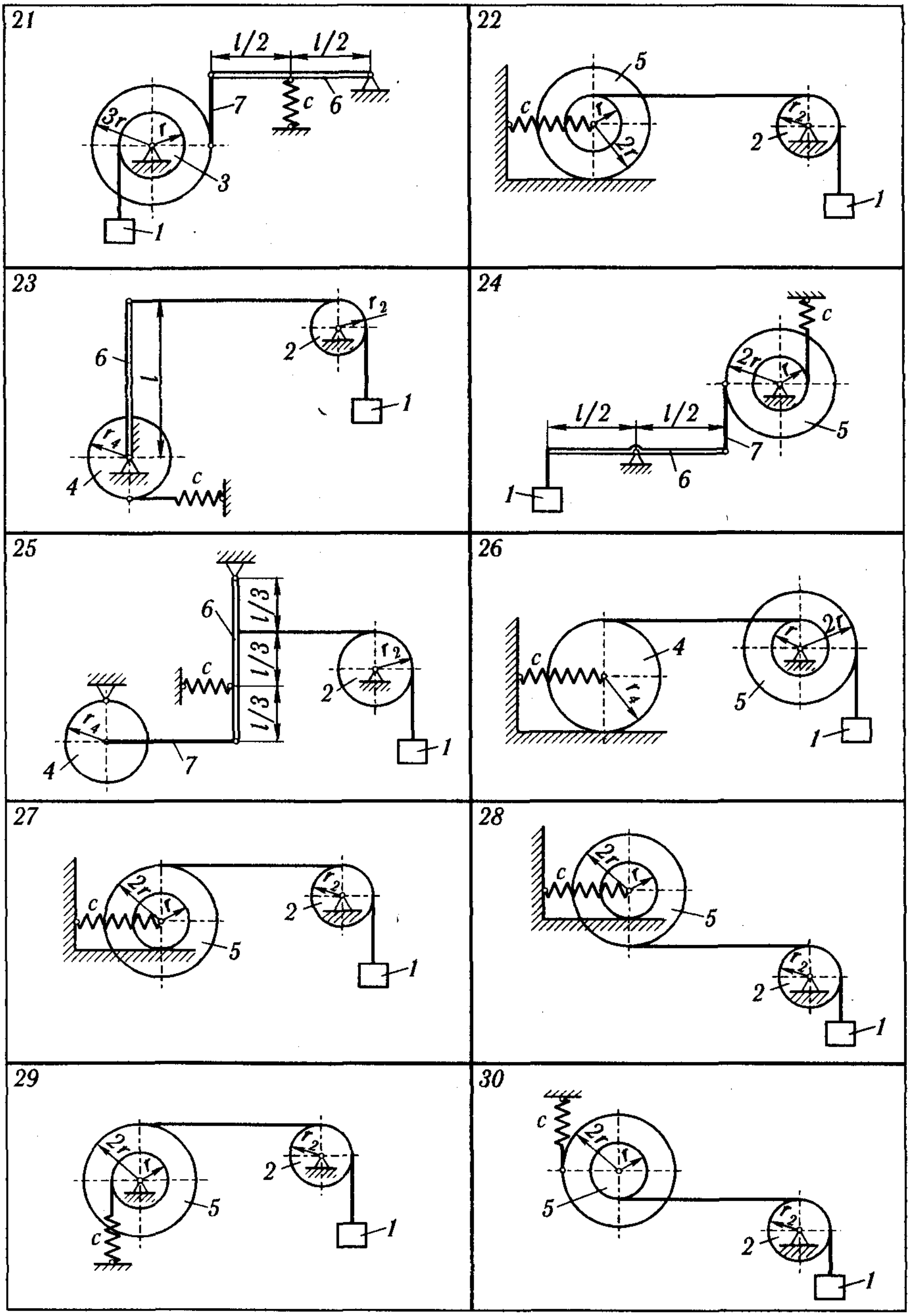

21
l/2
l/2
c
6
7
3r
r
3
1
22
5
c
r
2r
r2
2
1
23
r2
2
6
l
1
r4
4
c
24
c
2r
r
l/2
l/2
7
5
6
1
25
6
l/3
l/3
l/3
c
r2
2
1
r4
4
7
26
c
4
r4
r
2r
5
1
27
c
2r
r
r2
2
5
1
28
2r
c
r
5
r2
2
1
29
2r
r
r2
2
5
c
1
30
c
2r
r
5
r2
2
1

Fig. 122

Tab. 4

Variant number (Fig. 120-122)	l	i_x	i'_x	r_4	m_1	m_2	m_3, m_4, m_5	m_6	c	Initial conditions ($t = 0$)	
	m				kg				N/cm	y_0, cm	$\dot{y}_0$, m/sec
1	0,5	-	-	-	1	2	-	3	40	0,1	5,0
2	0,5	-	-	0,2	1	2	2	3	40	0	6,0
3	0,5	$3/2r$	-	-	1	-	4	3	20	0,2	7,0
4	0,6	-	-	-	1	2	3	2	36	0,2	0
5	0,6	-	-	0,15	1	-	3	3	16	0	8,0
6	0,6	-	-	0,15	1	-	1	1	40	0,3	7,0
7	-	-	-	-	1	-	2	2	40	0,4	0
8	-	-	-	-	1	3	2	-	40	0	6,0
9	0,6	-	-	-	1	2	-	3	38	0,5	5,0
10	0,6	-	-	-	1	2	-	3	32	0	6,0
11	-	-	-	-	1	2	-	3	30	0,4	7,0
12	0,5	-	-	-	1	2	-	3	20	0,2	0
13	0,3	-	-	-	1	1	1	2	32	0	8,0
14	0,4	-	-	0,1	1	-	2	3	20	0	7,0
15	0,4	$r\sqrt{3}$	-	-	1	-	2	2	20	0,1	0
16	-	-	-	-	1	2	3	-	32	0,3	6,0
17	-	-	-	-	1	2	-	2	20	0	5,0
18	-	-	-	-	1	2	1	-	40	0	6,0
19	0,2	-	-	-	1	1	-	1	32	0,1	0
20	0,5	-	-	-	1	2	-	3	20	0,4	7,0
21	-	$2r$	-	-	1	-	2	3	32	0	8,0
22	-	-	$r\sqrt{2}$	-	1	2	4	-	40	0,1	7,0
23	0,4	-	-	0,2	1	2	2	3	40	0,3	0
24	-	-	$r\sqrt{3}$	-	1	-	3	2	40	0	6,0
25	0,3	-	-	0,1	1	2	2	1	40	0,2	5,0
26	-	$r\sqrt{2}$	-	-	1	-	2	-	40	0,3	0
27	-	-	$3r/2$	-	1	2	3	-	40	0	6,0
28	-	-	$r\sqrt{3}$	-	1	2	3	-	40	0,2	0
29	-	-	$4r/3$	-	1	2	3	-	40	0	7,0
30	-	-	$r\sqrt{2}$	-	1	2	3	-	40	0,3	7,0

8.6. Application of the Lagrange's Equations to Research of Motion of Mechanical System with Two Degrees of Freedom

The mechanical system of bodies (Fig. 127—129) moves under the action of constant forces $\boldsymbol{P}$ and couples with the moments M or only under the gravity forces.

Make up the equations of motion of system in generalized coordinates q_1 and q_2 at the specified initial conditions. The necessary data are reduced in tab. 5; in the same place recommended generalized coordinates are specified *(*x and φ are generalized coordinates for absolute motion, and ξ is for relative motion).

Neglect masses of the threads. Take into consideration that the rolling of wheels occurs without a slippage. Neglect rolling friction and forces of resistance in bearings. Wheels for which in the table 5 inertia radii are not specified, consider as solid homogeneous disks. Consider cages (cranks) as thin homogeneous rods. Accept that in variants 6, 9, 11, 20, 22 and 30 mechanism is located in a horizontal plane.

Radii of gyration of bodies 2 and 3 are defined with respect to the central axis perpendicular to the figure. Coefficient of viscosity is a quantity b in expression $\boldsymbol{R} = -b\boldsymbol{v},$ where $\boldsymbol{v}$ is a relative velocity of bodies 1 and 2

Example It is given: masses of bodies of mechanical system (Fig. 123) $m_1 = 3m$; $m_2 = 8m$; $m_3 = m_4 = m_6 = 2m$; $m_5 = 4m$; $m_7 = m$; $\boldsymbol{P}$ is a constant force applied to a body 2; M is a constant moment applied to a block *6; b* is a coefficient of proportionality in expression for force of resistance to motion of a body *5:* $\boldsymbol{R} = — b\boldsymbol{v}_5$($\boldsymbol{v}_5$ is a velocity of a body *5*); L is a length of a thread *3; r* is a radius of the blocks *4* and *6*. Here the thread *3* is accepted ponderable. This is a complicating in comparison with the common condition of the problem. Thread sagging is not considered.

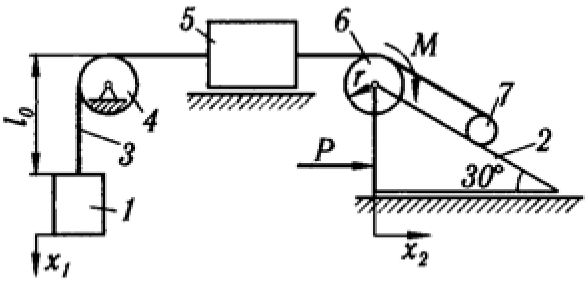


Fig. 123

Consider all wheels as solid homogeneous disks. Neglect sliding friction of a body 2.

Make up the equations of motion of system in generalized coordinates $q_1 = x_1$; $q_2 = x_2$.

Initial conditions: $q_{10} = 0$ (initial distance on a vertical from the lower end of a thread *3* to its horizontal site equals l_0), $q_{20} = 0, \dot{q}_{10} = 0, \dot{q}_{20} = \dot{x}_{20}$.

In Fig. 123 system is figured in initial position.

Solution. In order to resolve problem, we will apply the Lagrange's equations

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{x}_1} - \frac{\partial T}{\partial x_1} = Q_1;$$

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{x}_2} - \frac{\partial T}{\partial x_2} = Q_2,$$

where T is a kinetic energy of a system; Q_1 and Q_2 are generalized forces corresponding to generalized coordinates x_1 and x_2.

For the given system $T = \sum_{i=1}^{7} T_i$.

Let's express velocities of the centers of mass of rigid bodies of system through generalized velocities:

$$\left.\begin{aligned} v_1 = v_5 = \dot{x}_1; \\ v_2 = v_6 = \dot{x}_2; \\ v_7^2 = \frac{(\dot{x}_1 + \dot{x}_2)^2}{4} + \dot{x}_2{}^2 - (\dot{x}_1 + \dot{x}_2)\dot{x}_2 \cos\alpha , \end{aligned}\right\}$$

Taking into account that $\alpha = 30°$,

$$v_7^2 = 0{,}25[\dot{x}_1{}^2 + (5 - 2\sqrt{3})2\dot{x}_2{}^2 - 2(\sqrt{3} - 1)\dot{x}_1\dot{x}_2].$$

It was considered that $(\dot{x}_1 + \dot{x}_2)/2$ is a velocity of the centre of mass of a body 7 concerning a body *2,* i.e., relative velocity and $\dot{x}_2$ is its transport velocity (Fig. 124).

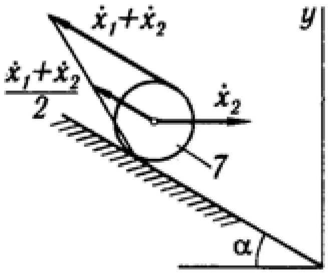


Fig. 124

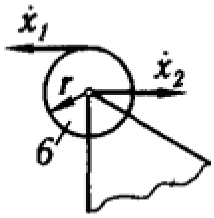


Fig. 125

Angular velocities of bodies (Fig. 123—125)

$$\left.\begin{aligned} \omega_4 = \frac{\dot{x}_1}{r}; \\ \omega_6 = \omega_7 = \frac{\dot{x}_1 + \dot{x}_2}{r}. \end{aligned}\right\}$$

Moments of inertia of wheels with respect to the central axes are $J_4 = J_6 = \frac{2mr^2}{2} = mr^2; J_7 = (m/2)(r/2)^2 = mr^2/8.$

Kinetic energy of bodies *1, 2, 4*—7 is

$$T_1 = m_1 v_1^2/2 = 3m\dot{x}_1{}^2/2;\ T_2 = m_2 v_2^2/2 = 4m\dot{x}_2{}^2;$$

$$T_4 = \frac{J_4\omega_4^2}{2} = \frac{m\dot{x}_1{}^2}{2}; T_5 = \frac{m_5 v_3^2}{2} = 2m\dot{x}_1{}^2;$$

$$T_6 = \frac{m_6 v_6^2}{2} + \frac{J_6\omega_6^2}{2} = 0{,}5m\left(\dot{x}_1{}^2 + 3\dot{x}_2{}^2 + 2\dot{x}_1\dot{x}_2\right);$$

$$T_7 = \frac{m_7 v_7^2}{2} + \frac{J_7\omega_7^2}{2} = (m/16)[3\dot{x}_1^2 + (11 - 4\sqrt{3})\dot{x}_2^2 - 2(2\sqrt{3} - 3)\dot{x}_1\dot{x}_2].$$

Taking into account that all points of a thread *3* have equal velocities $v_{3i} = v_3 = \dot{x}_1$, and that $\sum m_{3i} = m$, we have $T_3 = m_3 v_3^2/2 = m\dot{x}_1^2$.

Substituting all these magnitudes, one can obtain

$$T = \left(\frac{m}{16}\right)\left[75\dot{x}_1^2 + \left(99 - 4\sqrt{3}\right)\dot{x}_2^2 + 2\left(11 - 2\sqrt{3}\right)\dot{x}_1\dot{x}_2\right].$$

And now let’s define generalized forces Q_1 and Q_2 corresponding to generalized coordinates x_1 and x_2. Consider virtual work of all forces on virtual displacements δx_1 and δx_2.

Define virtual work of all the forces done in a virtual displacement δx_1. It should be born in mind that now $\delta x_2 = 0$. We have

$$\delta A_1 = \delta A(m_1\boldsymbol{g}) + \delta A(\boldsymbol{R}) + \delta A(M) + \delta A(m_7\boldsymbol{g}) + \delta A(m_3\boldsymbol{g}).$$

Calculate all of these members:

$$\delta A(m_1\boldsymbol{g}) = m_1 g\delta x_1,\ \delta A(\boldsymbol{R}) = -bv_5\delta x_1 = -b\dot{x}_1\delta x_1\,, \delta A(M) = -\frac{M}{r}\delta x_1,$$
$$\delta A(m_7\boldsymbol{g}) = -\frac{m_7 g}{2}\sin 30^\circ\delta x_1.$$

Let's notice that work of a gravity force $m_7\boldsymbol{g}$ of a thread *3* from position $a'b'$ to position ab, at which $x_1 = 0$, is equal to the work of a gravity force of a site of a thread bb' at its displacement to position aa' (Fig. 126).

Thus,

$$\delta A(m_3\boldsymbol{g}) = \frac{m_3 g}{L}(x_1 + \frac{l_0}{2})\delta x_1.$$

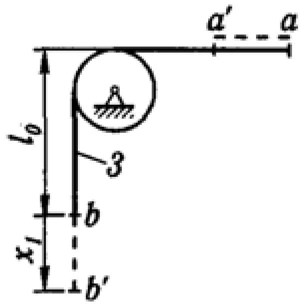


Fig. 126

Then, taking into account the data of the problem, we receive

$$\delta A_1 = \left[2{,}75mg - b\dot{x}_1 - \frac{M}{r} + \frac{2mg}{L}(x_1 + \frac{l_0}{2})\right]\delta x_1.$$

Hence,

$$Q_1 = 2{,}75mg - b\dot{x}_1 - \frac{M}{r} + \frac{2mg}{L}(x_1 + \frac{l_0}{2}).$$

Define virtual work of all the forces done in a virtual displacement δx_2. It should be born in mind that now $\delta x_1 = 0$. So we have

$$\delta A_2 = \delta A(\boldsymbol{P}) + \delta A(m_7\boldsymbol{g}) + \delta A(M).$$

Determine all of these members:

$$\delta A(\boldsymbol{P}) = P\delta x_2, \delta A(m_7\boldsymbol{g}) = -\frac{m_7 g}{2}\sin 30^\circ\delta x_2,\ \delta A(M) = -\frac{M}{r}\delta x_2.$$

Thus,

$$\delta A_2 = \left(P - \frac{mg}{4} - \frac{M}{r}\right)\delta x_2,$$

whence

$$Q_2 = P - \frac{mg}{4} - \frac{M}{r}.$$

Substituting all results in the Lagrange's equations, we receive the differential equations of motion of a system:

$$\frac{75}{8}m\ddot{x}_1 + \frac{11 - 2\sqrt{3}}{8}m\ddot{x}_2 = 2mg\frac{x_1}{L} + mg\left(2{,}75 + \frac{2l_0}{L}\right) - b\dot{x}_1 - \frac{M}{r};$$
$$\frac{99 - 4\sqrt{3}}{8}m\ddot{x}_2 + \frac{11 - 2\sqrt{3}}{8}m\ddot{x}_1 = -\frac{mg}{4} + P - \frac{M}{r}.$$

Expressing $\ddot{x}_2$ from the second equation and substituting in the first one, we obtain

$$\ddot{x}_1 + 2n\dot{x}_1 - cx_1 = a,$$

where

$$n = \frac{0{,}0538b}{m};\ c = \frac{0{,}215g}{L};$$

$$a = g(0{,}298 + 0{,}215 l_0/L) - 0{,}099 M/(rm) — 0{,}0088 P/m.$$

Let's define the solution of this linear non-uniform differential equation. Its characteristic equation is

$$z^2 + 2nz - c = 0.$$

Its equation roots are

$$z_{1,2} = -n \pm \sqrt{n^2 + c}.$$

Then the general solution of the differential equation has a form:

$$x_1 = e^{-nt}\left(C_1 e^{\sqrt{n^2 + c}t} + C_2 e^{-\sqrt{n^2 + c}t}\right) - \frac{a}{c}.$$

In order to define constants C_1 and C_2, differentiate this solution

$$\dot{x}_1 = e^{-nt}\left[\left(-n + \sqrt{n^2 + c}\right) C_1 e^{\sqrt{n^2 + c}t} - \left(n + \sqrt{n^2 + c}\right) C_2 e^{-\sqrt{n^2 + c}t}\right].$$

Using initial conditions: at $t = 0$, $x_1 = 0; \dot{x}_1 = 0$, we have

$$C_1 + C_2 - a/c = 0;$$

$$\left(-n + \sqrt{n^2 + c}\right) C_1 - \left(n + \sqrt{n^2 + c}\right) C_2 = 0,$$

whence

$$\left.\begin{aligned} C_1 &= \frac{a}{2c + \sqrt{n^2 + c}}\left(n + \sqrt{n^2 + c}\right); \\ C_2 &= \frac{a}{2c + \sqrt{n^2 + c}}\left(-n + \sqrt{n^2 + c}\right). \end{aligned}\right\}$$

Thus, we have equation of motion of the system describing change of the first generalized coordinate. To receive the second equation of motion, we find

$$\ddot{x}_2 = \frac{8}{99 - 4\sqrt{3}}\left(-\frac{g}{4} + \frac{P}{m} - \frac{M}{m \cdot r}\right) - \frac{11 - 2\sqrt{3}}{99 - 4\sqrt{3}}\ddot{x}_1, \text{ or } \ddot{x}_2 = k - 0{,}0818\ddot{x}_1,$$

where $k = 0{,}0869\left[\frac{1}{m}\left(P - \frac{M}{r}\right) - \frac{g}{4}\right]$.

Integrating, we obtain

$$\dot{x}_2 = kt — 0{,}0818\dot{x}_1 + C_3; \; x_2 = kt^2/2 - 0{,}0818 x_1 + C_3 t + C_4.$$

Using initial conditions: at $t = 0$, $x_1 = 0$; $x_2 = 0$; $\dot{x}_1 = 0$; $\dot{x}_2 = \dot{x}_{20}$, we find $C_3 = \dot{x}_{20}$; $C_4 = 0$.

Finally we have

$$x_2 = \frac{kt^2}{2} - 0{,}0818\left[e^{-nt}\left(C_1 e^{\sqrt{n^2 + c}t} + C_2 e^{-\sqrt{n^2 + c}t}\right) - \frac{a}{c}\right] + \dot{x}_{20} t.$$

This is the second equation of motion of a system.

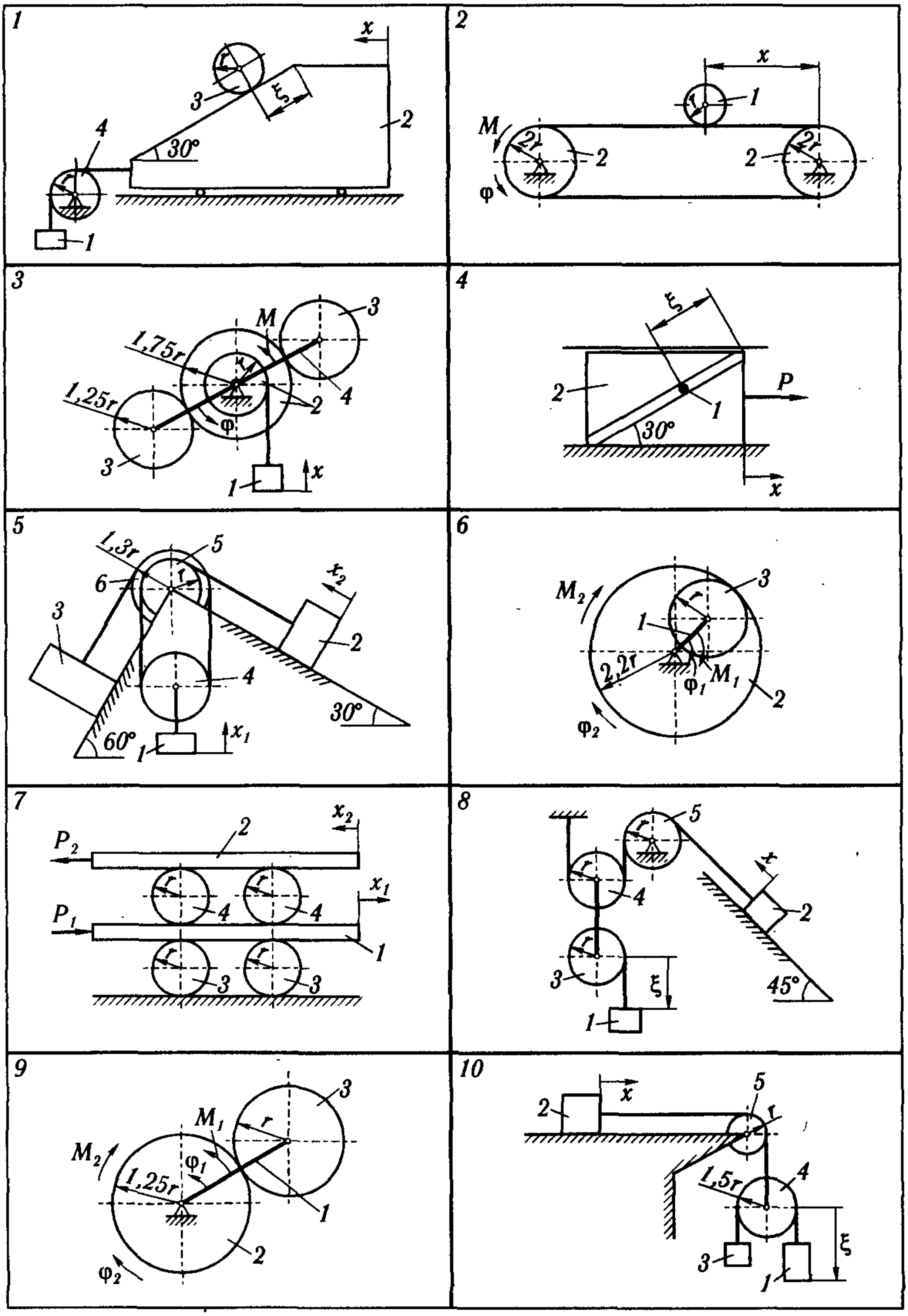
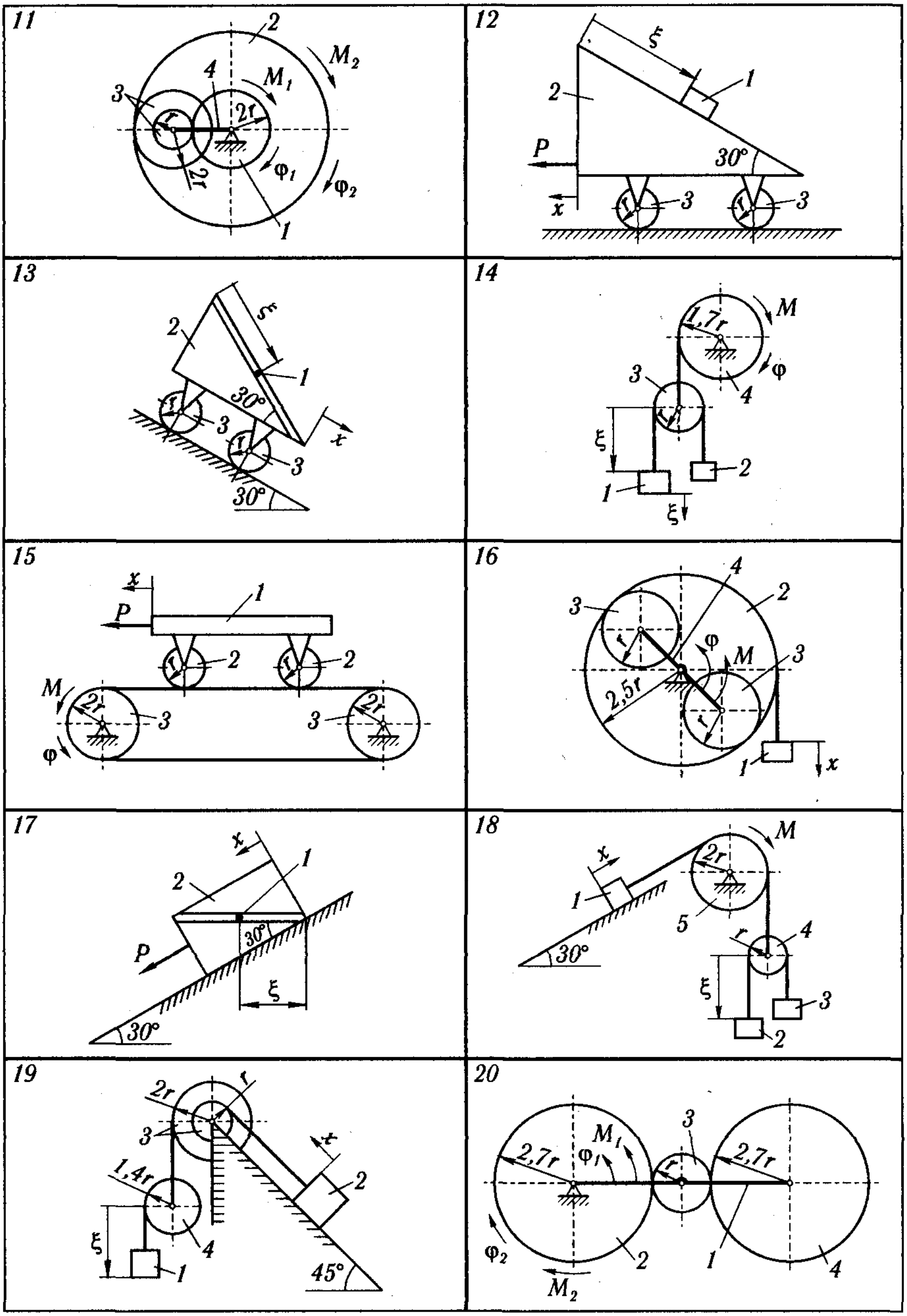
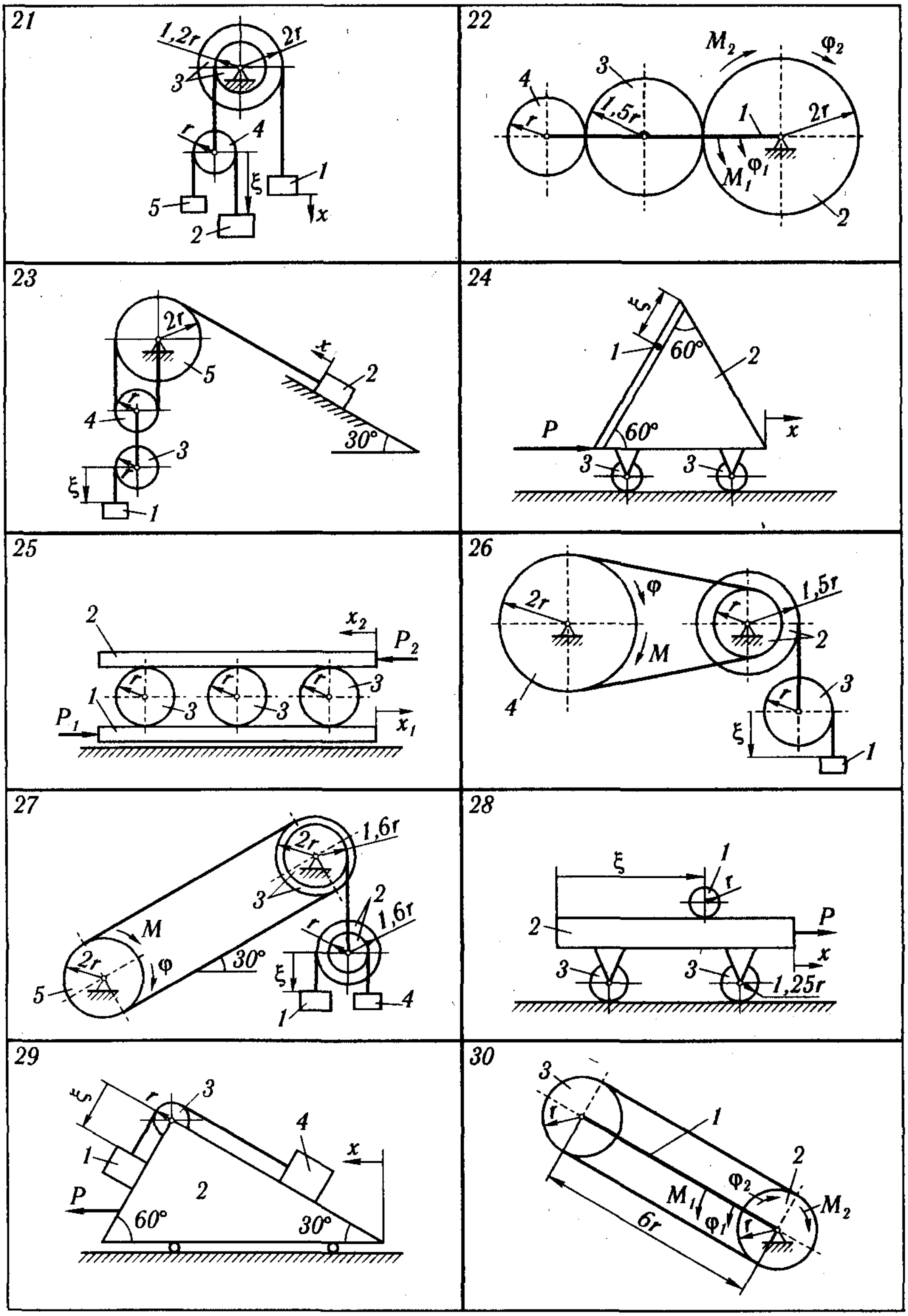

1
x
3
ξ
4
30°
2
r
1
2
x
1
r
M
2r
2
φ
3
M
1,75r
1,25r
r
φ
2
4
3
1
x
4
ξ
2
P
1
30°
x
5
1,3r
5
x2
6
r
3
2
4
30°
60°
x1
1
6
M2
3
r
1
2,2r
M1
φ1
2
φ2
7
x2
P2
2
r
x1
4
P1
1
3
8
5
r
x
4
2
3
ξ
45°
1
9
3
M1
r
M2
φ1
1,25r
1
2
φ2
10
2
x
5
r
1,5r
4
3
ξ
1

Fig. 127



11
2
M2
3
4
M1
r
2r
2r
φ1
φ2
1
12
ξ
1
2
P
30°
x
r
3
r
3
13
2
ξ
1
30°
r
3
x
r
3
30°
14
M
1,7r
φ
3
4
ξ
r
2
1
ξ
15
x
1
P
r
2
r
2
M
2r
3
3
2r
φ
16
4
2
3
r
φ
M
3
2,5r
r
1
x
17
x
1
2
30°
P
ξ
30°
18
M
x
2r
1
5
30°
r
4
ξ
3
2
19
2r
r
3
x
2
1,4r
ξ
4
1
45°
20
3
M1
2,7r
φ1
r
2,7r
φ2
2
1
4
M2

Fig. 128



21
1,2r
2r
3
r
4
ξ
1
5
x
2
22
M2
φ2
3
4
r
1,5r
1
2r
φ1
M1
2
23
2r
x
2
5
r
4
30°
3
ξ
r
1
24
ξ
60°
1
2
60°
P
x
3
3
25
2
x2
P2
r
r
r
3
1
3
3
P1
x1
26
φ
1,5r
2r
r
2
M
r
3
4
ξ
1
27
1,6r
2r
3
2
M
1,6r
r
2r
φ
30°
ξ
5
1
4
28
ξ
1
r
2
P
x
3
3
1,25r
29
ξ
r
3
4
x
1
2
P
60°
30°
30
3
1
r
2
φ2
M1
M2
6r
φ1
r

Fig. 129

Tab. 5

Variant number (Fig. 127-129)	Masses of bodies					Radii of gyration		Forces P	Moments M	Coefficients of		Generalized coordinates		Initial conditions				Supplementary data
	1	2	3	4	5	i_{2y}	i_{3y}			friction	viscousity	q_1	q_2	q_{10}	q_{20}	$\dot{q}_{10}$	$\dot{q}_{20}$	
1	$2m$	$6m$	m	m	–	–	–	–	–	–	–	x	ξ	0	0	0	0	
2	m	$3m$	–	–	–	–	–	–	M	–	–	φ	x	0	x_0	0	0	Neglect mass of the belt
3	m	$3m$	$2m$	–	–	$r\sqrt{2}$	–	–	M	–	–	φ	x	0	0	0	0	Moment M is applied at cage
4	m	$4m$	–	–	–	–	–	–	–	0	b	x	ξ	0	0	$\dot{x}_0$	0	Body *1* is a particle
5	m	$2m$	$4m$	$2m$	$2m$	–	–	–	–	f	–	x_1	x_2	0	0	0	0	Pulleys 5 and 6 are freely shafted on a common axle, their masses being equal
6	m	$2m$	$3m$	–	–	$2r$	–	–	M_1; M_2	–	–	φ_1	φ_2	0	0	0	0	Moment M_1 is applied at cage
7	$3m$	$3m$	m	m	–	–	–	P_1; P_2	–	–	–	x_1	x_2	0	0	0	0	
8	m	$2m$	$2m$	$2m$	$2m$	–	–	–	–	f	–	x	ξ	0	0	0	$\dot{\xi}_0$	
9	m	$2m$	$3m$	–	–	–	–	–	M_1; M_2	–	–	φ_1	φ_2	0	0	0	0	Moment M_1 is applied at cage
10	$2m$	$2m$	m	$2m$	m	–	–	–	–	f	–	x	ξ	0	0	$\dot{x}_0$	0	
11	m	$3m$	$2m$	m	–	$4r$	$r\sqrt{2}$	–	M_1; M_2	–	–	φ_1	φ_2	0	0	0	0	
12	$2m$	$5m$	m	–	–	–	–	P	–	f	–	x	ξ	0	ξ_0	0	0	
13	m	$3m$	$2m$	–	–	–	–	–	–	–	b	x	ξ	0	0	0	$\dot{\xi}_0$	Body *1* is a particle
14	$2m$	m	m	$2m$	–	–	–	–	M	–	–	φ	ξ	0	0	0	$\dot{\xi}_0$	
15	$3m$	m	$2m$	–	–	–	–	P	M	–	–	φ	x	0	0	0	0	Neglect mass of the belt

Continuation of the tab.5

Variant number (Fig. 127-129)	Masses of bodies					Radii of gyration		Forces P	Moments M	Coefficients of		Generalized coordinates		Initial conditions				Supplementary data
	1	2	3	4	5	i_{2y}	i_{3y}			friction	viscousity	q_1	q_2	q_{10}	q_{20}	$\dot{q}_{10}$	$\dot{q}_{20}$	
16	$2m$	$3m$	$2m$	m	–	$2r$	–	–	M	–	–	φ	x	0	0	0	$\dot{x}_0$	Moment M is applied at cage
17	m	$3m$	–	–	–	–	–	P	–	0	b	x	ξ	0	ξ_0	0	0	Body *1* is a particle
18	$2m$	$2m$	m	m	$3m$	–	–	–	M	f	–	x	ξ	0	0	0	0	
19	$2m$	$2m$	$3m$	m	–	–	$r\sqrt{2}$	–	–	f	–	x	ξ	0	0	$\dot{x}_0$	0	
20	$2m$	$3m$	m	$3m$	–	–	–	–	$M_1;\ M_2$	–	–	φ_1	φ_2	0	0	0	0	Moment M_1 is applied at cage
21	$2m$	$2m$	$3m$	$2m$	m	–	$r\sqrt{2}$	–	–	–	–	x	ξ	0	0	$\dot{x}_0$	0	
22	m	$3m$	$2m$	m	–	–	–	–	$M_1;\ M_2$	–	–	φ_1	φ_2	0	0	0	0	The same
23	$2m$	m	m	m	$3m$	–	–		–	f	–	x	ξ	0	0	0	$\dot{\xi}_0$	
24	m	$3m$	m	–	–	–	–	P	–	–	b	x	ξ	0	ξ_0	0	0	Body *1* is a particle
25	$2m$	$2m$	m	–	–	–	–	$P_1;\ P_2$	–	f	–	x_1	x_2	0	0	0	0	
26	m	$3m$	$2m$	$3m$	–	–	r	–	M	–	–	φ	ξ	0	0	0	$\dot{\xi}_0$	
27	$2m$	$2m$	$3m$	m	$2m$	$r\sqrt{2}$	$r\sqrt{3}$	–	M	–	–	φ	ξ	0	0	0	0	
28	m	$3m$	m	–	–	–	–	P	–	–	–	x	ξ	0	0	0	$\dot{\xi}_0$	
29	$2m$	$4m$	m	m	–	–	–	P	–	f	–	x	ξ	0	0	$\dot{x}_0$	0	
30	$3m$	$2m$	$2m$	–	–	–	–	–	$M_1;\ M_2$	–	–	φ_1	φ_2	0	0	$\dot{\varphi}_{10}$	0	Moment M_1 is applied at cage