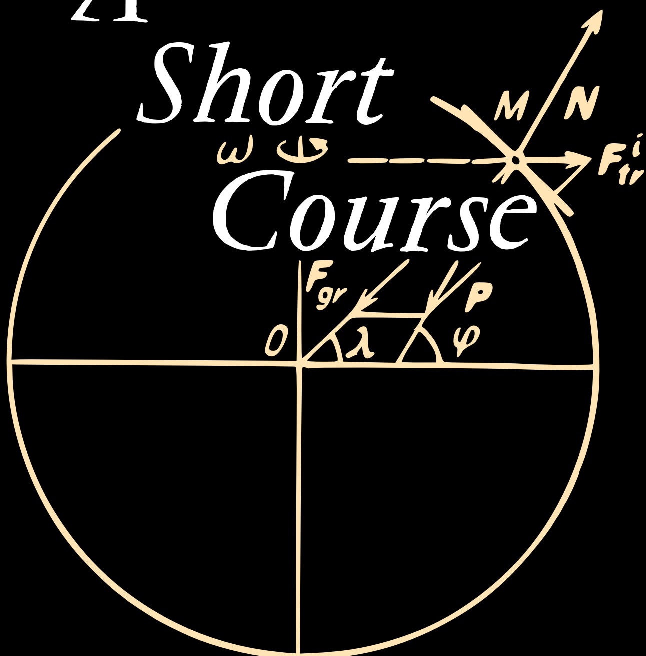
S. Targ

THEORETICAL MECHANICS

A Short Course



Mir Publishers Moscow

THEORETICAL MECHANICS

A SHORT COURSE

С. М. Тарг

КРАТКИЙ КУРС ТЕОРЕТИЧЕСКОЙ МЕХАНИКИ

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Preface to the English Edition

This *Short Course of Theoretical Mechanics* is designed for students of higher and secondary technical schools. It treats of the basic methods of theoretical mechanics and spheres of their application along with some topics which are of such importance todays that no course of mechanics, even a short one, can neglect them altogether.

In preparing the original Russian edition for translation the text has been substantially revised, with additions, changes and corrections in practically all the chapters.

Most of the additions are new sections containing supplementary information on the motion of a rigid body about a fixed point (the kinematic and dynamic Euler equations) and chapters setting forth the fundamentals of the method of generalized coordinates (the Lagrange equations), since the demands to the course of theoretical mechanics in training engineers of different specialities makes it necessary to devote some space to this subject even in a short course.

Also the book presents an essential minimum on the elementary theory of the gyroscope and such highly relevant topics as motion in gravitational fields (elliptical paths and space flights) and the motion of a body of variable mass (rocket motion); a new section discusses weightlessness.

The structure of this book is based on the profound conviction, born out by many years of experience, that the best way of presenting study material, especially when it is contained in a short course, is to proceed from the particular to the general. Accordingly, in this book, plane statics comes before three-dimensional statics, particle dynamics before system dynamics, rectilinear motion before curvilinear motion, etc. Such an arrangement helps the student to understand and digest the material better and faster and the teaching process itself is made more graphic and consistent.

Alongside with the geometrical and analytical methods of mechanics the book makes wide use of the vector method as one of the main generally accepted methods, which, furthermore, possesses a number of indisputable advantages. As a rule, however, only those vector operations are used which are similar to corresponding operations with scalar quantities and which do not require an acquaintance with many new concepts.

Considerable space—more than one-third of the book—is devoted to examples and worked problems. They were chosen with an eye to ensure a clear comprehension of the relevant mechanical phenomena and cover all the main types of problems solved by the methods described. There are 176 such examples (besides worked problems); their solutions contain instructions designed to assist the student in his independent work on the course. In this respect the book should prove useful to all students of engineering, notably those studying by correspondence or on their own.

The numeration of the equations of each of the four parts is separate; therefore, references to equations are only by their numbers. References to equations in other parts are given with the number of the respective section.

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Introduction

The progress of technology confronts the engineer with a wide variety of problems connected with structural design (buildings, bridges, canals, dams, etc.), the design, manufacture and operation of various machines, motors and means of locomotion, such as automobiles, steam engines, ships, aircraft, rockets, and spaceships. Despite the diversity of problems that arise, their solution, at least in part, is based on certain general principles common to all of them, namely, the laws governing the motion and equilibrium of material bodies.

The science which treats of the general laws of motion and equilibrium of material bodies and of the resulting mutual interactions is called *theoretical*, or *general*, *mechanics*. Theoretical mechanics constitutes one of the scientific bedrocks of modern engineering.

Mechanics, in the broad sense of the term, may be defined as the science that deals with the solution of all problems connected with the motion or equilibrium of material bodies and the resulting interactions between them. Theoretical mechanics treats of the *general laws* of motion and interaction of material bodies, i.e, laws which apply equally, for example, to the earth's motion around the sun or to the flight of a rocket or an artillery projectile. Other branches of mechanics cover a variety of general and specialised engineering disciplines treating of the design and calculation of specific structures, motors and other machines and mechanisms or their parts. All these disciplines are based on the laws and methods of theoretical mechanics.

By motion in mechanics we mean mechanical motion, i.e., any change in the relative positions of material bodies in space which occurs in the course of time. By mechanical interaction between bodies is meant such reciprocal action which changes or tends to change the state of motion or the shape of the bodies involved (deformation). The physical measure of such mechanical interaction is called *force*.

Theoretical mechanics is primarily concerned with the general laws of motion and equilibrium of material bodies under the action of the forces to which they are subjected.

According to the nature of the problems treated, mechanics is divided into *statics*, *kinematics*, and *dynamics*. Statics studies the forces and the conditions of equilibrium of material bodies subjected to the action of forces. Kinematics deals with the general geometrical properties of the motion of bodies. Finally, dynamics studies the laws of motion of material bodies under the action of forces.

According to the nature of the objects under study, theoretical mechanics is subdivided into: (a) mechanics of a particle, i.e., of a body whose dimensions can be neglected in studying its motion or equilibrium, and systems of particles; (b) mechanics of a rigid body, i.e., a body whose deformation can be neglected in studying its motion or equilibrium; (c) mechanics of bodies of variable mass (i.e., bodies whose mass changes with time, due to the change in the number of particles constituting the body); (d) mechanics of deformable bodies (the theories of plasticity and elasticity); (e) mechanics of liquids (fluid mechanics); (f) mechanics of gases (aerodynamics).

The general course of theoretical mechanics conventionally treats of the mechanics of particles and rigid bodies and the general laws of motion of systems of particles.

Theoretical mechanics is one of the natural sciences and is based on laws derived from experience that reflect a specific class of natural phenomena associated with the motion of material bodies. It not only provides the scientific foundations for many modern engineering domains; its laws and methods are also useful in studying and interpreting a wide range of important phenomena in the surrounding world, thereby contributing to the continued advance of natural science as a whole, as well as to the formation of a correct materialist world outlook.

The emergence and development of mechanics*) as a science are inseparable from the development of the productive forces of society and the level of industry and technology at each stage of this development.

The study of the so-called simple machines (the pulley, the winch, the lever, and the inclined plane) and the general study of equilibrium of bodies (statics) began in ancient times when the requirements of engineering were limited mainly to the needs of building construction. The fundamentals of statics are already found in the works of Archimedes (287-212 B.C.), one of the great scholars of antiquity.

Dynamics developed at a much later stage. The emergence and growth of bourgeois relations in Western and Central Europe in the 15th and 16th centuries spurred a rapid upsurge in handicrafts, commerce, navigation and methods of warfare (the appearance of firearms). This, coupled with important astronomical discoveries, contributed to the accumulation of a vast amount of experimental data, systematisation and analysis of which led to the discovery

*) The term "mechanics" first appears in the works of Aristotle (384-322 B.C.), the great philosopher of antiquity. It is derived from the Greek word μηχανη, which has the meaning of "structure", "machine", "device".

of the laws of dynamics in the 17th century. The credit of laying the foundations of dynamics goes to Galileo Galilei (1564-1642) and Sir Isaac Newton (1643-1727). The fundamental laws of so-called classical mechanics, now known as Newton's laws, were formulated by Newton in his *Mathematical Principles of Natural Philosophy* published in 1687. Newton's laws have since been confirmed by a tremendous amount of practical evidence in the course of the technological advance of human society. This permits us to regard as conclusive our concepts of mechanics based on Newton's laws; and the engineer can, therefore, confidently rely on them in his practical work.*)

In the 18th century, analytical methods, i.e., methods based on the application of differential and integral calculus, began to develop rapidly in mechanics. The methods of solving problems of particle and rigid body dynamics by the integration of differential equations were elaborated by the great mathematician and mechanic Leonhard Euler (1707-1783). Among the other major contributions to the progress of mechanics were the works of the outstanding French scientists Jean d'Alembert (1717-1783), who enunciated his famous principle for solving problems of dynamics, and Joseph Lagrange (1736-1813), who evolved the general analytical method of solving problems of dynamics on the basis of d'Alembert's principle and the principle of virtual work. Today analytical methods predominate in solving problems of dynamics.

Kinematics emerged as a special branch of mechanics only in the first half of the 19th century under pressure from the growing machine-building industry. Today kinematics is essential in studying the motion of machines and mechanisms.

In Russia, the study of mechanics was greatly influenced by the works of the great Russian scientist and thinker Mikhail Lomonosov (1711-1765) and of Leonhard Euler, who for many years lived and worked in St. Petersburg. Prominent among the galaxy of Russian scientists who contributed to the development of different divisions of theoretical mechanics were M. V. Ostrogradsky (1801-1861), the author of a number of important studies in analytical methods of problem solution in mechanics; P. L. Chebyshev (1821-1894), who started a new school in the study of the motion of mechanisms; S. V. Kovalevskaya (1850-1891), who solved one of the most difficult

*) Subsequent scientific developments revealed that at velocities approaching that of light the motion of bodies is governed by the mechanical laws of the theory of relativity, while the motion of "elementary" particles (electrons, positrons, etc.) is described by the laws of quantum mechanics. These discoveries, however, only served to define more accurately the spheres of application of classical mechanics and reaffirm the validity of its laws for the motion of all bodies other than subatomic particles at velocities not approaching the velocity of light, i.e., for motions with which engineering and celestial mechanics are primarily concerned.

problems of rigid body dynamics; A. M. Lyapunov (1857-1918), who elaborated new methods of studying the stability of motion; I. V. Meshchersky (1859-1935), who laid the foundations of the mechanics of bodies of variable mass; K. E. Tsiolkovsky (1857-1935), who made a number of fundamental discoveries in the theory of jet propulsion; and A. N. Krylov (1863-1945), who elaborated the theory of vessels and contributed much to the development of the theory of gyroscopic instruments.

Of tremendous importance to the further study of mechanics were the works of N. E. Zhukovsky (1847-1921), the "father of Russian aviation", and his closest pupil S. A. Chaplygin (1869-1942). Zhukovsky's special contribution was in the field of applying methods of mechanics to the solution of actual engineering problems. Zhukovsky's ideas have greatly influenced the teaching of theoretical mechanics in Soviet higher technical educational institutions.

Part 1

STATICS OF RIGID BODIES

Chapter 1

Basic Concepts and Principles

§ 1. The Subject of Statics

Statics is the branch of mechanics which studies the laws of composition of forces and the conditions of equilibrium of material bodies under the action of forces.

Equilibrium is the state of rest of a body relative to other material bodies. If the frame of reference relative to which a body is in equilibrium can be treated as fixed, the given body is said to be in absolute equilibrium, otherwise it is in relative equilibrium. In statics we shall study only absolute equilibrium. In actual engineering problems equilibrium relative to the earth or to bodies rigidly connected with the earth is treated as absolute equilibrium. The justification of this premise will be found in the course of dynamics, where the concept of absolute equilibrium will be defined more strictly. And there, too, we shall examine the concept of relative equilibrium.

Conditions of equilibrium depend on whether a given body is solid, liquid or gaseous. The equilibrium of liquids and gases is studied in the courses of hydrostatics and aerostatics respectively. General mechanics deals essentially with the equilibrium of solids.

All solid bodies change their shape to a certain extent when subjected to external forces. This is known as deformation. The amount of deformation depends on the material, shape and dimensions of the body and the acting forces. In order to ensure the necessary strength of engineering structures and elements, the material and dimensions of various parts are chosen in such a way that the deformation under specified loads would remain tolerably small*). This makes it possible, in studying the general conditions of equilibrium, to treat solid bodies as undeformable or absolutely rigid, ignoring the small deformations that actually occur. A *perfectly rigid body* is said to be one in which the distance between any pair of particles is always constant. In solving problems of statics, in this book we shall always consider bodies as perfectly rigid, and shall simply refer to them as rigid bodies. It will be shown at the end of § 3 that the laws of equilibrium of perfectly rigid bodies can be applied not only to solid bodies with relatively small deformation, but to any deformable bodies as well. Thus, the sphere of application of rigid body statics is extremely wide.

Deformation is of great importance in calculating the strength of engineering structures and machine parts. These questions are studied in the courses of strength of materials and theory of elasticity.

For a rigid body to be in equilibrium (at rest) when subjected to the action of a system of forces, the system must satisfy certain *conditions of equilibrium*. The determination of these conditions is one of the principal problems of statics. In order to find out the equilibrium conditions for various force systems and to solve other problems of mechanics one must know the principles of the composition, or addition, of forces acting on a rigid body, the principles of replacing one force system by another and, particularly, the principles of reducing a given force system to as simple a form as possible. Accordingly, statics of rigid bodies treats of two basic problems: (1) the composition of forces and reduction of force systems acting on rigid bodies to as simple a form as possible, and (2) the determination of the conditions for the equilibrium of force systems acting on rigid bodies.

The problems of statics may be solved either by geometrical constructions (the graphical method) or by mathematical calculus (the analytical method). The present course discusses both methods, but it should always be borne in mind that geometrical constructions are of special importance in solving problems of mechanics.

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*) For example, the material and the diameter of rods in various structures are so chosen that, under the working loads, they would extend (or contract) by no more than one-thousandth of their original length. Deformations of a similar order are tolerated in bending, torsion, etc.

§ 2. Force

The state of equilibrium or motion of a given body depends on its mechanical interactions with other bodies, i.e., on the loads, attractions or repulsions it experiences as a result of such interactions. *In mechanics, the quantitative measure of the mechanical interaction of material bodies is called force.*

Quantities employed in mechanics are either scalars (possessing magnitude alone) or vectors which besides magnitude are also characterised by direction in space.

Force is a vector quantity. Its action on a body is characterised by its (1) *magnitude*, (2) *direction*, and (3) *point of application.*

The magnitude of a force is expressed in terms of a standard force accepted as a unit. In mechanics, the fundamental units of force are the *newton* (N) and the *kilogram* (*force*) (kgf), with 1 kgf = 9,81 N (for more on units see § 101). Static forces are measured with dynamometers, which are described in the course of physics.

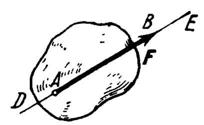

Fig. 1

The direction and the point of application of a force depend on the nature of the interaction between the given bodies and their respective positions. The force of gravity acting on a body, for example, is directed vertically downwards; the forces with which two smooth contacting spheres act on each other are normal to both their surfaces at the points of contact and are applied at those points, etc.

Force is represented graphically by a directed straight line segment with an arrowhead. The length of the line (AB in Fig. 1) denotes the magnitude of the force to some scale, the direction of the line shows the direction of the force, its initial point (point A in Fig. 1) usually indicating the point of application of the force, though sometimes it may be more convenient to depict a force as "pushing" a body with its tip (as in Fig. 4*c*). The line DE along which the force is directed is called the *line of action* of the force. We shall denote force, as all vector quantities, by a boldface-type letter ($\boldsymbol{F}$) or by overlined standard-type letters ($\overline{AB}$). The absolute value of a force is represented by the symbol $|\boldsymbol{F}|$ (two vertical lines "flanking" a vector) or simply by a standard-type letter (F). (In handwriting overlined letters are used.)

We shall call any set of forces acting on a rigid body a *force system.* We shall also use the following definitions:

1. A body not connected with other bodies and which from any given position can be displaced in any direction in space is called a *free body.*

2. If a force system acting on a free rigid body can be replaced by another force system without disturbing the body's initial condition of rest or motion, the two systems are said to be *equivalent*.
3. If a free rigid body *can* remain at rest under the action of a force system, that system is said to be *balanced* or *equivalent to zero*.
4. If a given force system is equivalent to a single force, that force is the *resultant* of the system. Thus, *a resultant is a single force capable of replacing the action of a system of forces on a rigid body.*

A force equal in magnitude, collinear with, and opposite in direction to the resultant is called an *equilibrant* force.

5. Forces acting on a rigid body can be divided into two groups: the external forces and the internal forces. *External forces* represent the action of other material bodies on the particles of a given body. *Internal forces* are those with which the particles of a given body act on each other.
6. A force applied to one point of a body is called a *concentrated force*. Forces acting on all the points of a given volume or given area of a body are called *distributed forces*.

A concentrated force is a purely notional concept, insofar as it is actually impossible to apply a force to a single point of a body. Forces treated in mechanics as concentrated are in fact the resultants of corresponding systems of distributed forces.

Thus, the force of gravity acting on a rigid body, as conventionally treated in mechanics, is the resultant of the gravitational forces acting on its particles. The line of action of this resultant force passes through the body's centre of gravity.*)

§ 3. Fundamental Principles

All theorems and equations in statics are deduced from a few fundamental principles, which are accepted without mathematical proof, and are known as the principles, or axioms, of statics. The principles of statics represent general formulations obtained as a result of a vast number of experiments with, and observations of, the equilibrium and motion of bodies and which, furthermore, have been consistently confirmed by actual experience. Some of these principles are corollaries of the fundamental laws of mechanics, which will be examined in the course of dynamics.

1st Principle. *A free rigid body subjected to the action of two forces can be in equilibrium if, and only if, the two forces are equal in magnitude* ($F_1 = F_2$), *collinear, and opposite in direction* (Fig. 2).

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*) The determination of the centre of gravity of bodies will be discussed in Chapter 9. Meanwhile it may be noted that if a homogeneous body has a centre of symmetry (e.g., a rectangular beam, a cylinder, a sphere, etc.) its centre of gravity is in the centre of symmetry.

The 1st principle defines the simplest balanced force system, since we know from experience that a free body subjected to the action of a single force cannot be in equilibrium.

2nd Principle. *The action of a given force system on a rigid body remains unchanged if another balanced force system is added to, or subtracted from, the original system.*

This principle establishes that two force systems differing from each other by a balanced system are equivalent.

Corollary of the 1st and 2nd Principles. *The point of application of a force acting on a rigid body can be transferred to any other point on the line of action of the force without altering its effect.*

Consider a rigid body with a force $\boldsymbol{F}$ applied at a point A (Fig. 3). Now take an arbitrary point B on the line of action of the force and

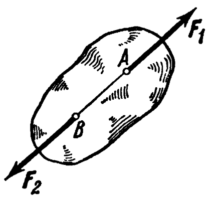

Fig. 2

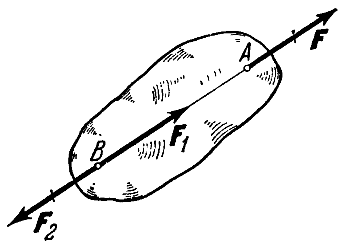

Fig. 3

apply to that point two equal and opposite forces $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ such that $\boldsymbol{F}_1 = \boldsymbol{F}$ and $\boldsymbol{F}_2 = -\boldsymbol{F}$. This operation does not affect the action of $\boldsymbol{F}$ on the body. From the 1st principle it follows that forces $\boldsymbol{F}$ and $\boldsymbol{F}_2$ also form a balanced system and thus cancel each other*), leaving force $\boldsymbol{F}_1$, equal to $\boldsymbol{F}$ in magnitude and direction, with the point of application shifted to point B.

Thus, the vector denoting force $\boldsymbol{F}$ can be regarded as applied at any point along the line of action (such a vector is called a *sliding vector*).

This principle holds good only for forces acting on perfectly rigid bodies. In engineering problems it can be used only when we determine the conditions of equilibrium of a structure without taking into account the internal stresses experienced by its parts.

For example, the rod AB in Fig. 4*a* will be in equilibrium if $F_1 = F_2$. It will remain in equilibrium if both forces are transferred to point C (Fig. 4*b*) or if force $\boldsymbol{F}_1$ is transferred to point B and force $\boldsymbol{F}_2$ to point A (Fig. 4*c*). The stresses in the rod, however, differ in each case. In the first case the rod extends under the action of

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*) We shall denote cancelled or transferred forces in diagrams by a dash across the respective vectors.

the applied force, in the second there are no internal stresses, and in the third it compresses.*⁾ Consequently, in determining the internal stresses the point of application of a force cannot be transferred along the line of action.

3rd Principle (the Parallelogram Law). *Two forces applied at one point of a body have as their resultant a force applied at the same point*

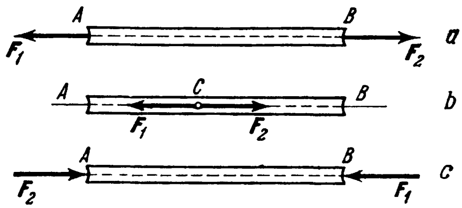

Fig. 4

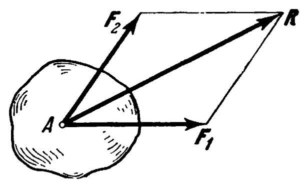

Fig. 5

and represented by the diagonal of a parallelogram constructed with the two given forces as its sides.

Vector $\boldsymbol{R}$, which is the diagonal of the parallelogram with vectors $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ as its sides (Fig. 5), is called the geometrical sum of the vectors $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$:

$$\boldsymbol{R} = \boldsymbol{F}_1 + \boldsymbol{F}_2.$$

Hence, the 3rd principle can also be formulated as follows: *The resultant of two forces applied at the point of a body is the geometrical (vector) sum of those forces and is applied at that point.*

It is important to discriminate between concepts of a sum of forces and their resultant. Consider, say, a body to which two forces $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ are applied at points A and B (Fig. 6). In Fig. 6, force $\boldsymbol{Q}$ is the geometrical sum of forces $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ ($\boldsymbol{Q} = \boldsymbol{F}_1 + \boldsymbol{F}_2$) as the diagonal of the corresponding parallelogram. But $\boldsymbol{Q}$ is not the resultant of the two forces, for it will be readily observed that $\boldsymbol{Q}$ alone cannot replace the action of $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ on the body. Moreover, forces $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$, as will be shown later, have no resultant at all (§ 48, Problem 42).

4th Principle. *To any action of one material body on another there is always an equal and oppositely directed reaction.*

The law of action and reaction is one of the fundamental laws of mechanics. It follows from it that when a body A acts on a body B with a force $\boldsymbol{F}$, body B simultaneously acts on body A with a force $\boldsymbol{F}'$ equal in magnitude, collinear with, and opposite in sense to force

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*⁾ For the rod to be stretched (or compressed) with a force $\boldsymbol{F}_1$, the force should be applied at one end of the rod, with the other end supported rigidly or constrained by a force $\boldsymbol{F}_2 = -\boldsymbol{F}_1$, as in Fig. 4. The tensile (or compressional) stress is the same in both cases and is equal to $\boldsymbol{F}_1$, and not to $2\boldsymbol{F}_1$ as is sometimes erroneously supposed.

$\boldsymbol{F}$ ($\boldsymbol{F}' = -\boldsymbol{F}$) (Fig. 7). Forces $\boldsymbol{F}$ and $\boldsymbol{F}'$, however, do not form a balanced system as they are applied to different bodies.

Internal Forces. It follows from the 4th principle that any two particles of a rigid body act on each other with forces equal in magnitude and opposite in sense. Since in studying the general conditions of equilibrium a body can be treated as rigid, all internal forces (according to the 1st principle) form a balanced system, which (according to the 2nd principle) can be neglected. Hence, in studying the conditions of equilibrium of a body (structure) only the external forces acting on a given solid body or structure have to be taken into

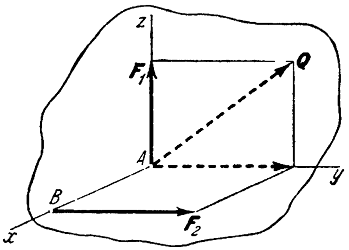


Fig. 6

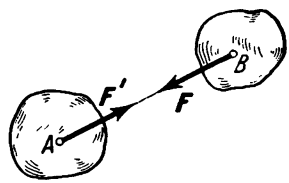


Fig. 7

account. The term "force" will henceforth be used in the sense of "external force".

5th Principle (Principle of Solidification). *If a freely deformable body subjected to the action of a force system is in equilibrium, the state of equilibrium will not be disturbed if the body solidifies (becomes rigid).*

The idea expressed in this principle is self-evident, for, obviously, the equilibrium of a chain will not be disturbed if its links are welded together, and a flexible string will remain in equilibrium if it turns into a bent rigid rod. Since the same force system acts on a reposing body before and after solidification, the 5th principle can also be formulated as follows: *If a deformable body is in equilibrium, the forces acting on it satisfy the conditions for the equilibrium of a rigid body.* For a deformable body, however, these conditions, though necessary, may not be sufficient.

For example, for a flexible string with two forces applied at its ends to be in equilibrium, the same conditions are necessary as for a rigid rod (the forces must be of equal magnitude and directed along the string in opposite directions). These conditions, though, are not sufficient. For the string to be in equilibrium the forces must be tensile, i.e., they must stretch the body as shown in Fig. 4*a*.

The principle of solidification is widely employed in engineering problems. It makes it possible to determine equilibrium conditions by treating a flexible body (a belt, cable, chain, etc.) or structure as a rigid body and to apply to it the methods of rigid-body statics.

If the equations obtained for the solution of a problem prove insufficient, additional equations must be derived which take into account either the conditions for the equilibrium of separate parts of the given structure or their deformation (problems requiring consideration of deformation are studied in the course of strength of materials).

§ 4. Constraints and Their Reactions

As has been defined above, a body not connected with other bodies and capable of displacement in any direction is called a *free body* (e.g., a balloon floating in the air). A body whose displacement in space is restricted by other bodies, either connected to or in contact with it, is called a *constrained body*. *We shall call a constraint anything that restricts the displacement of a given body in space.*

Examples of constrained bodies are a weight lying on a table or a door swinging on its hinges. The constraints in these cases are the surface of the table, which prevents the weight from falling, and the hinges, which prevent the door from sagging from its jamb.

A body acted upon by a force or forces whose displacement is restricted by a constraint acts on that constraint with a force which is customarily called the load or pressure acting on that constraint. At the same time, according to the 4th principle, the constraint reacts with a force of the same magnitude and opposite sense. *The force with which a constraint acts on a body, thereby restricting its displacement, is called the force of reaction of the constraint* (*force of constraint*), or simply *the reaction of the constraint.*

The procedure will be to call all forces which are not the reactions of constraints (e.g., gravitational forces) *applied* or *active forces.* Characteristic of active forces is that their magnitude and direction do not depend on the other forces acting on a given body. The difference between a force of constraint and an active force is that the magnitude of the former always depends on the active forces and is not therefore immediately apparent: if there are no applied forces acting on a body, the forces of constraint vanish. The reactions of constraints are determined by solving corresponding problems of statics. *The reaction of a constraint points away from the direction in which the given constraint prevents a body's displacement.* If a constraint prevents the displacement of a body in several directions, the sense of the reactions is not immediately apparent and has to be found by solving the problem in hand.

The correct determination of the direction of forces of constraint is of great importance in solving problems of statics. Let us therefore consider the direction of the forces of constraints (reactions) of some common types of constraints (more examples are given in § 25).

1. Smooth Plane (Surface) or Support. A smooth surface is one whose friction can be neglected in the first approximation. Such a surface prevents the displacement of a body perpendicular (normal) to both contacting surfaces at their point of contact (Fig. 8*a*)*). Therefore, *the reaction* N *of a smooth surface or support is directed*

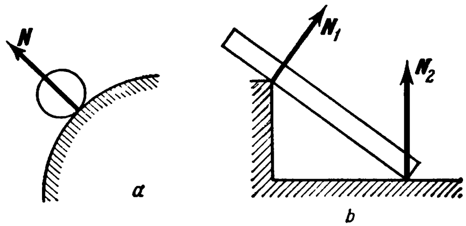


Fig. 8

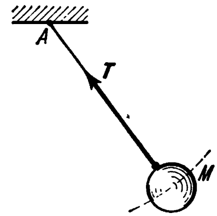


Fig. 9

normal to both contacting surfaces at their point of contact and is applied at that point. If one of the contacting surfaces is a point (Fig. 8*b*) the reaction is directed *normal to the other surface.*

2. String. A constraint provided by a flexible inextensible string (Fig. 9) prevents a body M from receding from the point of suspension

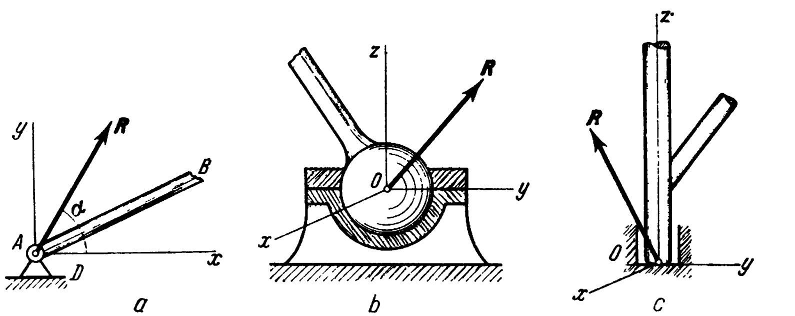


Fig. 10

of the string in the direction AM. *The reaction* T *of the string is thus directed along the string towards the point of suspension.*

3. Cylindrical Pin (Bearing). When two bodies are joined by means of a pin passing through holes in them, the connection is called a pin joint or hinge. The axial line of the pin is called the axis of the joint. Body AB in Fig. 10*a* is hinged to support D and can rotate freely about the axis of the joint (i.e., in the plane of the diagram); at the same time, point A cannot be displaced in any

*) In Figs. 8-11, the applied forces acting on the bodies are not shown. In the cases illustrated in Figs. 8 and 9, the reactions act in the indicated directions *regardless* of the applied forces and irrespective of whether the bodies are at rest or in motion.

direction perpendicular to the axis. Thus, *the reaction* $\boldsymbol{R}$ *of a pin can have any direction in the plane perpendicular to the axis of the joint* (plane Axy in Fig. 10*a*). In this case neither the magnitude R nor the direction (angle α) of force $\boldsymbol{R}$ are immediately apparent.

4. **Ball-and-Socket Joint and Step Bearing.** This type of constraint prevents displacement in any direction. Examples of such a constraint is a ball-pivot with which a camera is attached to a tripod (Fig. 10*b*) and a step bearing (Fig. 10*c*). *The reaction* $\boldsymbol{R}$ *of a ball-and-socket joint or a pivot can have any direction in space.* Neither its magnitude R nor its angles with the x, y, and z axes are immediately apparent.

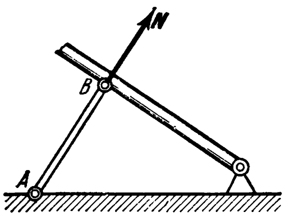


Fig. 11

5. **Rod.** Let a rod AB secured by hinges at its ends be the constraint of a certain structure (Fig. 11). The weight of the rod compared with the load it carries may be neglected. Then only two forces applied at A and B will act on the rod. If rod AB is in equilibrium, the forces, according to the 1st principle, must be collinear and directed along the axis of the rod (see Figs. 4*a* and 4*c*). Consequently, a rod subjected to forces applied at its tips, *where the weight of the rod as compared with the magnitude of the forces can be neglected*, can be only under tension or under compression. Hence, if in the structure such a rod AB is used as a constraint (Fig. 11) *the reaction* $\boldsymbol{N}$ *will be directed along its axis.*

§ 5. Axiom of Constraints

The equilibrium of constrained bodies is studied in statics on the basis of the following axiom: *Any constrained body can be treated as*

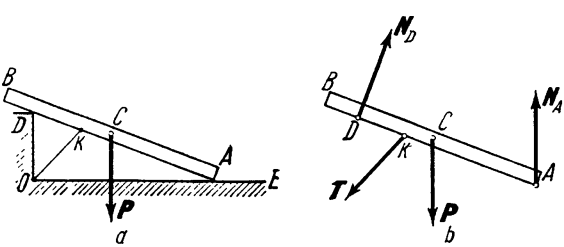


Fig. 12

a free body relieved from its constraints, provided the latter are represented by their reactions.

For example, the beam AB of weight P in Fig. 12*a*, for which surface OE, support D, and cable KO are the constraints, can be regarded as a free body (Fig. 12*b*) in equilibrium under the action of the given force $\boldsymbol{P}$ and the reactions $\boldsymbol{N}_A$, $\boldsymbol{N}_D$, and $\boldsymbol{T}$ of the constraints. The magnitudes of these reactions, which are unknown, can be determined from the conditions for the equilibrium of the forces acting on the now free body. This is the basic method of solving problems of statics.

The determination of the reactions of constraints is of practical importance because, from the 4th principle, if we know them, we shall know the loads acting on the constraints, i.e., the basic information necessary to calculate the strength of structural elements.

Chapter 2

Composition of Forces. Concurrent Force Systems

§ 6. Geometrical Method of Composition of Forces. Resultant of Concurrent Forces

Force being a vector quantity, the solution of many problems of mechanics involves operations in vector addition according to the laws of vector algebra. We shall commence our study of statics with the geometric method of composition of forces. The quantity which is the geometric sum of all the forces of a given system is called the *principal vector* of the system. As noted in § 3 (see Fig. 6), the concept of the geometric sum of forces should not be confused with that of the resultant; as we shall see later on, many force systems have no resultant at all, but the geometric sum (principal vector) can be calculated for *any* force system.

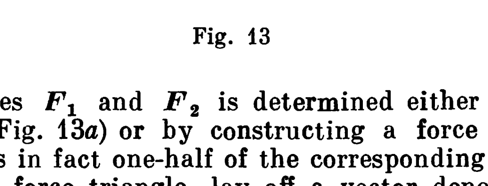


Fig. 13

(1) Composition of Two Forces. The geometric sum $\boldsymbol{R}$ of two concurrent forces $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ is determined either by the parallelogram rule (Fig. 13*a*) or by constructing a force triangle (Fig. 13*b*), which is in fact one-half of the corresponding parallelogram. To construct a force triangle, lay off a vector denoting one of the forces from an arbitrary point A_1, and from its tip lay off a vector denoting the second force. Joining the initial point of the first vector with the terminal point of the second vector, we get the vector that represents a force $\boldsymbol{R}$.

The magnitude of $\boldsymbol{R}$ is the side A_1C_1 of the triangle $A_1B_1C_1$, i.e.,

$$R^2 = F_1^2 + F_2^2 - 2F_1F_2\cos(180^\circ - \alpha),$$

where α is the angle between the two forces. Hence,

$$R = \sqrt{F_1^2 + F_2^2 + 2F_1F_2\cos\alpha}. \tag{1}$$

The angles β and γ which the force $\boldsymbol{R}$ makes with the component forces can be determined by the law of sines. As sin (180° — α) = = sin α, we have:

$$\frac{F_1}{\sin\gamma} = \frac{F_2}{\sin\beta} = \frac{R}{\sin\alpha}. \tag{2}$$

(2) Composition of Three Non-Coplanar Forces. The geometrical sum $\boldsymbol{R}$ of three non-coplanar forces $\boldsymbol{F}_1$, $\boldsymbol{F}_2$, $\boldsymbol{F}_3$ is represented by the

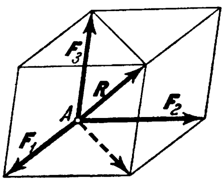


Fig. 14

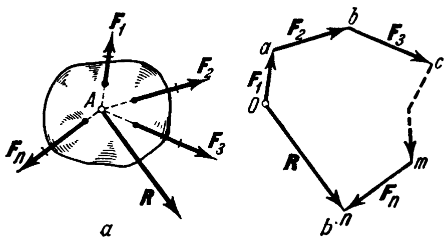


Fig. 15

diagonal of a parallelepiped with the given forces for its edges (the *parallelepiped rule*). This rule can be verified by successively applying the parallelogram rule (Fig. 14).

(3) Composition of a System of Forces. The geometrical sum (principal vector) of any force system can be determined either by successively compounding the forces of the system according to the parallelogram law, or by constructing a force polygon. The latter method is simpler and more convenient. In order to find the sum of the forces $\boldsymbol{F}_1$, $\boldsymbol{F}_2$, $\boldsymbol{F}_3$, . . ., $\boldsymbol{F}_n$ (Fig. 15*a*) lay off from an arbitrary point O (Fig. 15*b*) a vector $\overline{Oa}$ denoting force $\boldsymbol{F}_1$. Now from point a lay off a vector $\overline{ab}$ denoting force $\boldsymbol{F}_2$, from point b lay off a vector $\overline{bc}$ denoting force $\boldsymbol{F}_3$, and so on; from the tip m of the penultimate vector lay off vector $\overline{mn}$ denoting force $\boldsymbol{F}_n$. Vector $\overline{On} = \boldsymbol{R}$, laid off from the initial point of the first vector to the tip of the last vector, represents the geometrical sum, or the principal vector, of the component forces:

$$\boldsymbol{R} = \boldsymbol{F}_1 + \boldsymbol{F}_2 + \ldots + \boldsymbol{F}_n \text{ or } \boldsymbol{R} = \sum \boldsymbol{F}_k. \tag{3}$$

The magnitude and direction of $\boldsymbol{R}$ do not depend on the order in which the vectors are laid off. It will be noted that the construction carried out is in effect a consecutive application of the triangle rule.

The figure constructed in Fig. 15*b* is called a *force polygon* (or, generally speaking, a *vector polygon*). Thus, *the geometrical sum, or the principal vector, of a set of forces is represented by the closing side of a force polygon constructed with the given forces as its sides* (*the polygon law*). In constructing a vector polygon, care should be taken to arrange all the component vectors in one sense along the periphery of the polygon, with vector $\boldsymbol{R}$ being drawn in the opposite sense.

Resultant of Concurrent Forces. In studying statics we shall proceed from simple to more complex force systems. Let us commence, then, with systems of concurrent forces. *Forces whose lines of action intersect at one point are called concurrent* (see Fig. 15*a*). It follows from the first two principles of statics that a system of concurrent forces acting on a rigid body is equivalent to a system of forces applied at one point (point A in Fig. 15*a*).

Consecutively applying the parallelogram rule, we come to the conclusion that the *resultant of a system of concurrent forces is equal to the geometrical sum* (*principal vector*) *of those forces and* that it is *applied at the point of intersection of these forces.* Hence, if the forces $\boldsymbol{F}_1, \boldsymbol{F}_2, \ldots, \boldsymbol{F}_n$ intersect at point A (Fig. 15*a*), the resultant of the system is a force equal to the principal vector $\boldsymbol{R}$, obtained by constructing a force polygon, and applied at point A.

If the forces intersect beyond the diagram, the point through which their resultant passes can be determined by the graphical method described in § 30.

§ 7. Resolution of Forces

To resolve a force into two or more components means to replace it by a force system whose resultant is the original force. This problem is indeterminate and can be solved uniquely only if additional conditions are stated. Two cases are of particular interest:

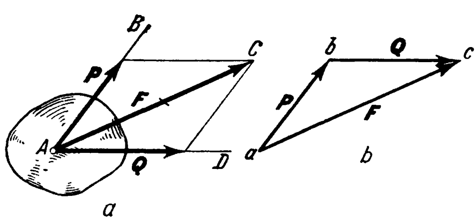


Fig. 16

1. Resolution of a Force Into Two Components of Given Direction. Consider the resolution of force $\boldsymbol{F}$ (Fig. 16) into two components parallel to the lines AB and AD (the force and the lines are coplanar). The task is to construct a parallelogram with force $\boldsymbol{F}$ as its diagonal and its sides respectively parallel to AB and AD. The problem is solved by drawing through the beginning and the tip of $\boldsymbol{F}$ lines parallel to AB and AD. Forces $\boldsymbol{P}$ and $\boldsymbol{Q}$ are the respective components, as $\boldsymbol{P} + \boldsymbol{Q} = \boldsymbol{F}$.

The resolution can also be carried out by applying the triangle rule (Fig. 16*b*). For this, force $\boldsymbol{F}$ is laid off from an arbitrary point a and through its initial and terminal points lines parallel to AB and AD are drawn to their point of intersection. Forces $\boldsymbol{P}$ and $\boldsymbol{Q}$ replace $\boldsymbol{F}$ if applied at point A or at any other point along the line of action of $\boldsymbol{F}$.

2. Resolution of a Force Into Three Components of Given Direction. If the given directions are not coplanar, the problem is defined and it reduces to the construction of a parallelepiped with the given force $\boldsymbol{F}$ as its diagonal and its sides parallel to the given directions (see Fig. 14).

The student is invited to consider the resolution of a given force $\boldsymbol{F}$ into two components $\boldsymbol{P}$ and $\boldsymbol{Q}$ coplanar with $\boldsymbol{F}$ if their magnitudes, P and Q, are given and if $P + Q \geqslant F$. The problem has two solutions.

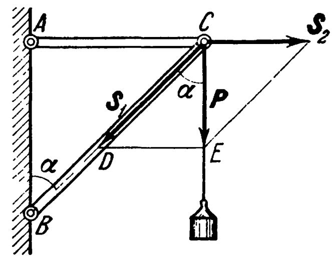


Fig. 17

Solution of Problems. The method of resolution of forces is useful in determining the pressure on constraints induced by the applied forces. Loads acting on rigid constraints are determined by resolving the given forces along the directions of the reactions of the constraints as, according to the 4th principle, a force acting on a constraint and its reaction have the same line of action. It follows, then, that this method can be applied only if the *directions* of the reactions of the respective constraints are immediately apparent.

Problem 1. A bracket consists of two members AC and BC attached to each other and to a wall by means of joints, with $\angle BAC = 90°$ and $\angle ABC = \alpha$ (Fig. 17). A load P is suspended from joint C. Neglecting the weight of the members, determine the force that contracts BC.

Solution. Force $\boldsymbol{P}$ acts on both members, and the reactions are directed along them. The unknown force is determined by applying $\boldsymbol{P}$ at point C and resolving it along AC and BC. Component $\boldsymbol{S}_1$ is the required force. From the triangle CDE we obtain:

$$\boldsymbol{S}_1 = \frac{\boldsymbol{P}}{\cos \alpha}.$$

From the same triangle we find that member AC is under a tension of

$$\boldsymbol{S}_2 = \boldsymbol{P} \tan \alpha.$$

The larger the angle α, the greater the load on both members, which increases rapidly as α approaches 90°. For example, at $P = 100$ kgf and $\alpha = 85°$, $S_1 \approx 1150$ kgf and $S_2 \approx 1140$ kgf. Thus, to lessen the load angle α should be made smaller.

We see from these results that a small applied force can cause very large stresses in structural elements (see also Problem 2). The reason for this is that forces are compounded and resolved according to the parallelogram rule: the diagonal of a parallelogram can be very much smaller than its sides. If, therefore, in solving a problem you find that the loads, or reactions, seem too big as compared with the applied forces, this does not necessarily mean that your solution is wrong.

Finally, we shall show why in similar problems the given force must be resolved into the components along the directions of the

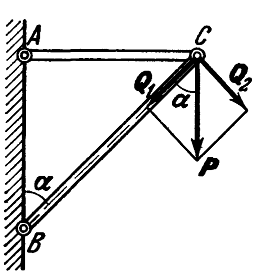


Fig. 18

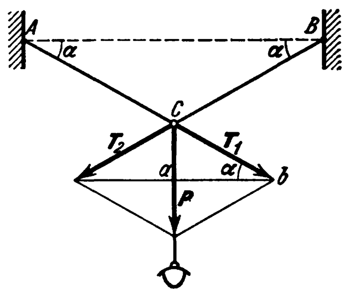


Fig. 19

constraints. In Problem 1 we have to determine the force acting on member BC. If we were to apply force P at C (Fig. 18) and resolve it into a component Q_1 along BC and a component Q_2 perpendicular to it, we should obtain:

$$Q_1 = P \cos \alpha, \quad Q_2 = P \sin \alpha.$$

Although force P was resolved according to the rule, component Q_1 is not the required force acting on BC because not all of force Q_2 acts on AC. Actually, force Q_2 acts on both members and, consequently, it increases the load acting on BC and adds to Q_1.

This example shows that if a force is not resolved along the directions of the constraints, the required result cannot be obtained.

Problem 2. A lamp of weight $P = 20$ kgf (Fig. 19) hangs from two cables AC and BC forming equal angles $\alpha = 5°$ with the horizontal. Determine the stresses in the cables.

Solution. Resolve force P applied at C into the components directed along the cables. The force parallelogram in this case is a rhombus, whose diagonals are mutually perpendicular and bisecting. From triangle aCb we obtain:

$$\frac{P}{2} = T_1 \sin\alpha$$

whence

$$T_1 = T_2 = \frac{P}{2\sin\alpha} \approx 115 \text{ kgf}.$$

The equation shows that the smaller the angle α, the greater the tension in the cables (at $\alpha = 1°$, for instance, $T \approx 573$ kgf). Should we attempt to stretch the cable absolutely horizontally it would break, for, at $\alpha \to 0$, T tends to infinity.

Problem 3. Neglecting the weight of rod AB and crank OB of the reciprocating gear in Fig. 20, determine the circumferential

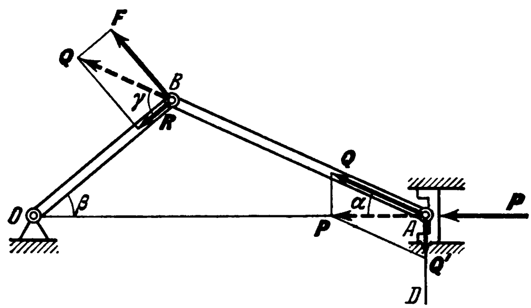


Fig. 20

force at B and the load on axle O of the crank caused by the action of force P applied to piston A if the known angles are α and β.

Solution. In order to determine the required forces we have to know the force Q with which the connecting rod AB acts on pin B. The magnitude of Q can be found by resolving force P along AB and AD (AD being the direction in which piston A acts on the slides). Thus we obtain:

$$Q = \frac{P}{\cos\alpha}.$$

Transferring force Q to point B and resolving it as shown in Fig. 20 into the circumferential force F and the load R on the axle, we obtain:

$$F = Q\sin\gamma, \quad R = Q\cos\gamma.$$

Angle γ is an external angle of triangle OBA and equals $\alpha + \beta$. Hence, we finally obtain:

$$F = P\frac{\sin(\alpha+\beta)}{\cos\alpha}, \quad R = P\frac{\cos(\alpha+\beta)}{\cos\alpha}.$$

As $\alpha + \beta \leqslant 180°$ and $\alpha < 90°$, force F is always greater than zero, i.e., it is always directed as shown in the diagram. Force R, however, is directed from B to O only as long as $\alpha + \beta < 90°$; at $\alpha + \beta > 90°$, $\boldsymbol{R}$ reverses its sense. At $\alpha + \beta = 90°$, $R = 0$.

This example shows that the method of force resolution can be employed even if the forces act on a body which is not in equilibrium. In such cases the load on a constraint is determined by resolving the applied force along the direction of the constraint and the direction of the displacement of the point at which the force is applied (point B in Fig. 20). The pressure on a constraint, as determined by this method, is called a *static load*, since in calculating it the masses, velocities and accelerations of moving bodies are not taken into account. In actual physical situations such calculations can be employed only if the velocities and accelerations are small. If the masses, velocities and accelerations of the moving bodies are taken into account, the determined forces are called *dynamic loads* and are calculated by the methods of dynamics (§ 169).

§ 8. Projection of a Force on an Axis and on a Plane

Let us now discuss analytical (mathematical) methods of solving problems of statics. These methods are based on the concept of the projection of a force on an axis. The same as for any other vector, *the projection of a force on an axis is an algebraic quantity equal to the length*

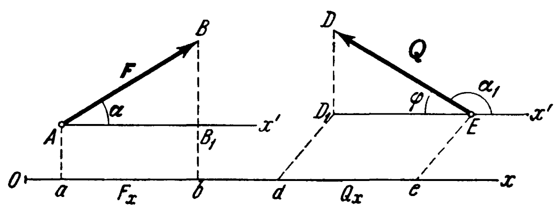


Fig. 21

of the line segment comprised between the projections of the initial and terminal points of the force taken with the appropriate sign: "plus" if the direction from the initial to the terminal point is the positive direction of the axis, and "minus" if it is the negative direction of the axis. It follows from this definition that the projections of a given force on any parallel axes of same sense are equal. This is useful in calculating the projection of a force on an axis not coplanar with that force. We shall denote the projection of a force $\boldsymbol{F}$ on an axis Ox by the symbol F_x. For the forces in Fig. 21 we have*):

$$F_x = AB_1 = ab, \quad Q_x = -ED_1 = -ed.$$

*) The positive direction of an axis shall be taken to mean the direction from point O (the origin) towards the letter x denoting the given axis. We shall use arrowheads in diagrams only to show the direction of vectors.

But it is apparent from the diagram that $AB_1 = F \cos \alpha$ and $ED_1 = Q \cos \varphi = -Q \cos \alpha_1$. Hence,

$$F_x = F \cos \alpha, \quad Q_x = -Q \cos \varphi = Q \cos \alpha_1, \tag{4}$$

i.e., *the projection of a force on an axis is equal to the product of the magnitude of the force and the cosine of the angle between the direction of the force and the positive direction of the axis.* The projection is positive if the angle between the direction of the force and the positive direction of the axis is acute, and negative if the angle is obtuse; *if the force is perpendicular to the axis, its projection on the axis is zero.*

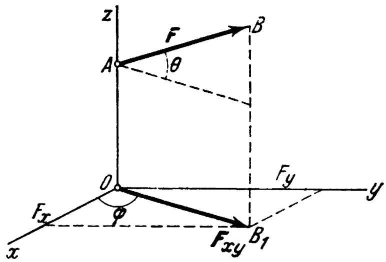


Fig. 22

The projection of a force $\boldsymbol{F}$ *on a plane Oxy is a vector* $\boldsymbol{F}_{xy} = \overline{OB_1}$ *comprised between the projections of the initial and terminal points of the force* $\boldsymbol{F}$ *on the plane* (Fig. 22). Thus, unlike the projection of a force on an axis, the projection of a force on a plane is a vector quantity characterised by both magnitude and direction in that plane. The magnitude of the projection is $F_{xy} = F \cos \theta$, where θ is the angle between the direction of force $\boldsymbol{F}$ and its projection $\boldsymbol{F}_{xy}$.

In some cases it may prove easier to find the projection of a force on an axis by first finding its projection on a plane through that axis and then to project the latter on the given axis. Thus, in the case shown in Fig. 22 we find that

$$\begin{aligned} F_x &= F_{xy} \cos \varphi = F \cos \theta \cos \varphi, \\ F_y &= F_{xy} \sin \varphi = F \cos \theta \sin \varphi. \end{aligned} \tag{5}$$

§ 9. Analytical Method of Defining a Force

For the analytical definition of a force we select a system of coordinate axes *Oxyz* as a frame of reference for defining the direction of our force in space. In mechanics right-hand coordinate systems are usually employed, i.e., systems in which a counterclockwise rotation about *Oz* carries *Ox* into *Oy* by the shortest way (Fig. 23). We can construct the vector denoting force $\boldsymbol{F}$, if we know the magnitude of the force F and the angles α, β, γ it makes with the coordinate axes. The quantities F, α, β, γ define the given force $\boldsymbol{F}$. The point A at which the force is applied must be defined additionally by its coordinates x, y, z.

For the solution of problems of statics it is often more convenient to define a force by its projections. Let us show that force $\boldsymbol{F}$ is completely defined if its projections F_x, F_y, F_z on the axes of a cartesian coordinate system are known. From formula (4) we have

$$F_x = F\cos\alpha, \quad F_y = F\cos\beta, \quad F_z = F\cos\gamma.$$

Squaring these equations and adding them, we obtain $F_x^2 + F_y^2 + F_z^2 = F^2$ since $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$, whence

$$F = \sqrt{F_x^2 + F_y^2 + F_z^2}, \quad \cos\alpha = \frac{F_x}{F}, \quad \cos\beta = \frac{F_y}{F}, \quad \cos\gamma = \frac{F_z}{F}. \tag{6}$$

Eqs. (6) give the magnitude of a force and the angles it makes with the coordinate axes in terms of its projections on the given axes, i.e., they define the force. It should be noted that in the first equation the sign before the radical is always positive as the formula gives only the *magnitude* of the force.

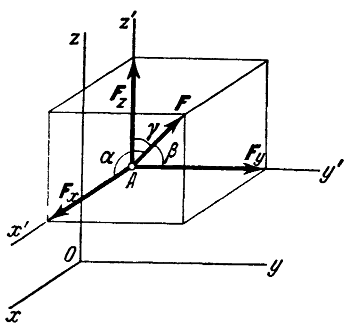


Fig. 23

If force $\boldsymbol{F}$ is resolved parallel to the axes, its rectangular components $\boldsymbol{F}_x$, $\boldsymbol{F}_y$, $\boldsymbol{F}_z$ are equal in magnitude to the projections of the force on the respective axes (see Fig. 23). It follows then that the vector of a force can be constructed geometrically according to the parallelepiped rule if its rectangular components or projections on the axes of a coordinate system are known.

If a set of given forces is coplanar, each force can be defined by its projections on two coordinate axes Ox and Oy. Then Eqs. (5) take the form:

$$F = \sqrt{F_x^2 + F_y^2}, \quad \cos\alpha = \frac{F_x}{F}, \quad \cos\beta = \frac{F_y}{F}. \tag{7}$$

The force can be constructed graphically according to its x and y components or projections by the parallelogram rule.

§ 10. Analytical Method of Composition of Forces

Operations with vectors can be expressed in terms of operations with their projections by the following geometrical theorem: *The projection of the vector of a sum on an axis is equal to the algebraic sum of the projections of the component vectors on the same axis.* As force is a vector, it follows that if, for example, $\boldsymbol{R} = \boldsymbol{F}_1 + \boldsymbol{F}_2 + \boldsymbol{F}_3 + \boldsymbol{F}_4$ (Fig. 24), then

$$R_x = F_{1x} + F_{2x} + F_{3x} + F_{4x},$$

where

$$F_{1x}=ab,\ F_{2x}=bc,\ F_{3x}=cd,\ F_{4x}=-de,\ R_x=ae.$$

From this theorem we obtain that for any force system $\boldsymbol{F}_1, \boldsymbol{F}_2, \ldots, \boldsymbol{F}_n$ whose sum (principal vector) is $\boldsymbol{R}$, where $\boldsymbol{R} = \sum \boldsymbol{F}_k$, we obtain

$$R_x=\sum F_{kx},\quad R_y=\sum F_{ky},\quad R_z=\sum F_{kz}. \tag{8}$$

Knowing R_x, R_y, and R_z, from Eqs. (6) we have

$$\left.\begin{aligned} &R=\sqrt{R_x^2+R_y^2+R_z^2};\\ &\cos\alpha=\frac{R_x}{R},\quad \cos\beta=\frac{R_y}{R},\quad \cos\gamma=\frac{R_z}{R}. \end{aligned}\right\} \tag{9}$$

Eqs. (8) and (9) provide the analytical solution of the problem of the composition of forces.

For coplanar forces the respective equations are

$$\left.\begin{aligned} &R_x=\sum F_{kx},\quad R_y=\sum F_{ky};\\ &R=\sqrt{R_x^2+R_y^2},\\ &\cos\alpha=\frac{R_x}{R},\quad \cos\beta=\frac{R_y}{R}. \end{aligned}\right\} \tag{10}$$

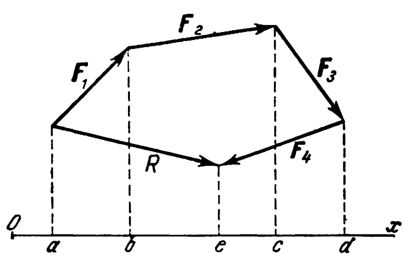


Fig. 24

If the forces are defined by their magnitudes and their angles with the coordinate axes, it is first necessary to determine their projections on the coordinate axes.

Problem 4. Determine the sum of three forces $\boldsymbol{P}$, $\boldsymbol{Q}$, $\boldsymbol{F}$ whose respective projections are:

$$\begin{aligned} &P_x=6\text{ N},\quad P_y=3\text{ N},\quad P_z=12\text{ N};\\ &Q_x=3\text{ N},\quad Q_y=-7\text{ N},\quad Q_z=1\text{ N};\\ &F_x=5\text{ N},\quad F_y=2\text{ N},\quad F_z=-8\text{ N}. \end{aligned}$$

Solution. From Eqs. (8) we find $R_x = 6+3+5 = 14$ N, $R_y = 3-7+2 = -2$ N, $R_z = 12+1-8 = 5$ N. Substituting these quantities in Eqs. (9), we obtain:

$$R=\sqrt{14^2+(-2)^2+5^2}=15\text{ N},\qquad \cos\alpha=\frac{14}{15},$$

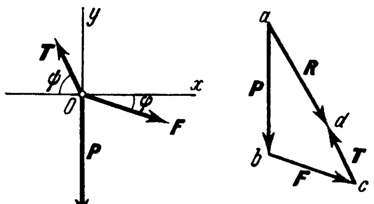
$$\cos\beta=-\frac{2}{15},\quad \cos\gamma=\frac{1}{3},$$

and finally

$$R = 15\text{ N},\ \alpha = 21°,\ \beta = 97°40',\quad \gamma = 70°30'.$$

Problem 5. Determine the resultant of the three coplanar forces in Fig. 25*a* if $F = 17.32$ kgf, $T = 10$ kgf, $P = 24$ kgf, $\varphi = 30°$, $\psi = 60°$.

Solution. Compute the projections of the given forces: $F_x = F\cos\varphi = 15$ kgf, $T_x = -T\cos\psi = -5$ kgf, $P_x = 0$; $F_y = -F\sin\varphi = -8.66$ kgf, $T_y = T\sin\psi = 8.66$ kgf, $P_y = -P = -24$ kgf. Then, by Eqs. (10), $R_x = 15 - 5 = 10$ kgf, $R_y = -8.66 + 8.66 - 24 = -24$ kgf, whence



a *b*

Fig. 25

$$R = \sqrt{10^2 + (-24)^2} = 26 \text{ kgf},$$

$$\cos\alpha = \frac{5}{13}, \quad \cos\beta = -\frac{12}{13},$$

and finally:

$$R = 26 \text{ kgf}, \alpha = 67°20', \beta = 157°20'.$$

To solve the problem graphically, choose a scale (e.g., 1 cm corresponds to 10 kgf) and construct a force polygon with forces $\boldsymbol{P}$, $\boldsymbol{F}$, and $\boldsymbol{T}$ as its sides (Fig. 25*b*). Side $\overline{ad}$ represents to scale the direction and magnitude of the resultant $\boldsymbol{R}$. If on measuring we find that $ad \approx 2.5$ cm, then $R \approx 25$ kgf with an error of 4% to the exact solution.

§ 11. Equilibrium of a System of Concurrent Forces

It follows from the laws of mechanics that a rigid body subjected to the action of an external set of mutually balanced forces can either be at rest or in motion. We shall call this kind of motion "motion under no forces", "inertial" or "coasting" motion, of which uniform rectilinear translatory motion is an example.

From this we derive two important conclusions: (1) Forces acting on bodies at rest and on bodies in "inertial" motion equally satisfy the conditions of equilibrium treated of in statics (see Problem 6). (2) The equilibrium of forces acting on a free rigid body is a necessary but insufficient condition for the equilibrium (rest) of the body. The body will remain at rest only if it was at rest before the moment when the balanced forces were applied.

For a system of concurrent forces acting on a body to be in equilibrium it is necessary and sufficient for the resultant of the forces to be zero. The conditions which the forces themselves must satisfy can be expressed either in graphical or in analytical form.

(1) Graphical Condition of Equilibrium. Since the resultant $\boldsymbol{R}$ of a system of concurrent forces is defined as the closing side of a force polygon constructed with the given forces, it follows that $\boldsymbol{R}$ can be zero only if the terminal point of the last force of the polygon coincides with the initial point of the first force, i.e., if the polygon is closed.

Thus, *for a system of concurrent forces to be in equilibrium it is necessary and sufficient for the force polygon drawn with these forces to be closed.*

(2) Analytical Conditions of Equilibrium. Analytically the resultant of a system of concurrent forces is determined by the formula

$$R=\sqrt{R_x^2+R_y^2+R_z^2}.$$

As the expression under the radical is a sum of positive components, R can be zero only if simultaneously $R_x=0$, $R_y=0$, $R_z=0$, which follows from Eqs. (8), i.e., when the forces acting on the body satisfy the equations

$$\sum F_{kx}=0, \quad \sum F_{ky}=0, \quad \sum F_{kz}=0. \tag{11}$$

Eqs. (11) express the conditions of equilibrium in analytical form: *The necessary and sufficient condition for the equilibrium of a three-dimensional system of concurrent forces is that the sums of the projections of all the forces on each of three coordinate axes must separately vanish.*

If all the concurrent forces acting on a body lie in one plane, they form a *coplanar system of concurrent forces.* Obviously, for such a force system only two equations are required to express the conditions of equilibrium:

$$\sum F_{kx}=0, \quad \sum F_{ky}=0. \tag{12}$$

Eqs. (11) and (12) also express the *necessary* conditions (or equations) of equilibrium of a free rigid body subjected to the action of concurrent forces.

(3) The Theorem of Three Forces. The following theorem will often be found useful in solving problems of statics: *If a free rigid body remains in equilibrium under the action of three nonparallel coplanar forces, the lines of action of those forces intersect at one point.*

To prove the theorem, first draw two of the forces acting on the body, say $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$. As the theorem states that the forces are not parallel and lie in the same plane, their lines of action intersect at some point A (Fig. 26). Now attach forces $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ to point A and replace them by their resultant $\boldsymbol{R}$. Two forces will be acting on the body: $\boldsymbol{R}$ and $\boldsymbol{F}_3$, which is applied at some point B of the body. If the body is to be in equilibrium, then, according to the 1st principle, forces $\boldsymbol{R}$ and $\boldsymbol{F}_3$ must be directed along the same line, i.e., along AB. Consequently, force $\boldsymbol{F}_3$ also passes through A, and the theorem is proved.

It should be noted that the reverse is not true, i.e., if the action lines of three forces intersect at one point, the body on which they are acting is not necessarily in equilibrium. Thus, the theorem expresses a *necessary*, *but not sufficient*, condition for the equilibrium of a free rigid body acted upon by three forces.

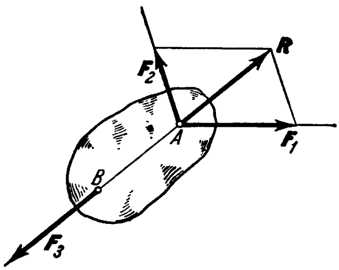


Fig. 26

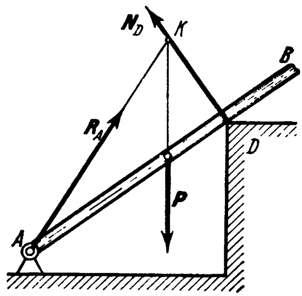


Fig. 27

Example. Consider a beam AB (Fig. 27) hinged at A and reclining on a ledge at D. By replacing the constraints with their reactions we can treat the beam as a free body in equilibrium under the action of three forces $\boldsymbol{P}$, $\boldsymbol{N}_D$ and $\boldsymbol{R}_A$, whose lines of action, according to the theorem just proved, must intersect at one point. But the lines of action of forces $\boldsymbol{P}$ and $\boldsymbol{N}_D$ are known, and they intersect at K. Consequently, the reaction $\boldsymbol{R}_A$ of the hinge applied at A must also pass through K, i.e., it is directed along AK. In this case the theorem of three forces has helped us to determine the unknown direction of the reaction of hinge A.

§ 12. Problems Statically Determinate and Statically Indeterminate

In problems where the equilibrium of constrained rigid bodies is considered, the reactions of the constraints are unknown quantities. Their number depends on the number and type of the constraints. A problem of statics can be solved only if the number of unknown reactions is not greater than the number of equilibrium equations in which they are present. Such problems are called *statically determinate*, and the corresponding systems of bodies are called *statically determinate systems*.

Problems in which the number of unknown reactions of the constraints is greater than the number of equilibrium equations in which they are present are called *statically indeterminate*, and the corresponding systems of bodies are called *statically indeterminate systems*.

An example of a statically indeterminate system is a weight hanging from three strings lying in one plane (Fig. 28). There are three unknown quantities in this problem (the tensions T_1, T_2, T_3 of the strings), but only the two equations (12) for the equilibrium of a coplanar system of concurrent forces. Other examples of statically indeterminate systems are given in § 25.

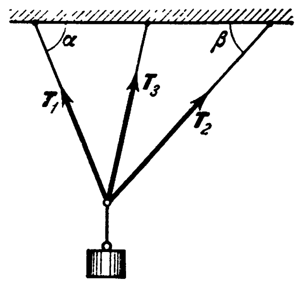


Fig. 28

It can be seen that the static indeterminateness of a problem is a result of the presence of too many constraints. In the present case two strings are sufficient to keep the weight in equilibrium at any values of angles α and β (see Problem 2, Fig. 19), and the third string is redundant.

We shall be concerned only with statically determinate problems, i.e., problems in which the number of reactions is the same as the number of equilibrium equations involving them. For the solution of statically indeterminate problems the assumption of the rigidity of the bodies under consideration must be given up and their deformations taken into account. Problems of this kind are solved in the courses of strength of materials and statics of structures.

§ 13. Solution of Problems of Statics

The problems solved by the methods of statics fall into one of two types: (1) Problems in which some or all of the forces acting on the body are known and where one has to determine in what position, or under what distribution of the forces acting on it, the body will be in equilibrium (Problems 6 and 7). (2) Problems in which the body is known to be in equilibrium (or in inertial motion) and one has to determine all or some of the forces acting on it (Problems 8, 9, 10, and others). The reactions of constraints are unknown quantities in all problems of statics.

In practical applications, problems of statics are used to determine the conditions of equilibrium of structures (when they are not rigidly anchored by constraints) and the loads on the supports or stresses in different parts of a structure in equilibrium. As any structure is an association of several connected bodies, the solution of any problem must begin with *the isolation of the specific body whose equilibrium should be examined in order to obtain the unknown quantities.*

The solution consists of the following steps:

1. Choose the Body Whose Equilibrium Should Be Examined. For the problem to lend itself to solution, the given and required forces, or their equivalents, should all be applied to the body whose equilibrium is being examined (for instance, if the problem is to determine a load acting on a support, we can examine the equilibrium of the body experiencing the reaction of the support, which is equal in magnitude to the required load).

If the given forces act on one body and the required on another, it may be necessary to examine the equilibrium of each body separately, or even of some intermediate bodies as well.

2. Isolate the Body From Its Constraints and Draw the Given Forces and the Reactions of the Removed Constraints. Such a drawing is called a free-body diagram and is drawn separately, as in Fig. 12*b**⁾. In drawing the reactions, the points brought up in § 4 in connection with the reactions of constraints should be taken into account.

3. State the Conditions of Equilibrium. The statement of these conditions depends on the force system acting on the free body and the method of solution (graphical or analytical). Special cases of stating the equilibrium conditions for different force systems will be examined in the respective chapters of this course.

4. Determine the Unknown Quantities, Verify the Answer and Analyse the Results. In solving a problem it is important to have a carefully drawn diagram, which helps to choose the correct method of solution and prevents errors in stating the conditions of equilibrium. All computations should be carried out in strict order.

The computations should, as a rule, be written out in general (algebraic) form. This provides formulas for determining the unknown quantities which can then be used to analyse the results. Solution in general form also makes it possible to catch mistakes by checking the dimensions (the dimensions of the terms in each side of an equation should be the same). If the problem is solved in general form, the numerical values should be substituted in the final equations.

In this section we shall discuss equilibrium problems involving concurrent forces. They can be solved by either the graphical or the analytical method.

(a) The graphical method is suitable when the total number of given and required forces acting on a body is three. If the body is in equilibrium the force triangle must be closed (the construction

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*⁾ When sufficient experience is gained, the student may mentally isolate the body he is examining and draw the given forces and the reactions of the constraints acting on the body (and on it alone) on the general diagram (as in Fig. 31). However, if the equilibrium of two or more structural elements has to be examined, it is best to draw a free-body diagram.

should start with the known force). By solving the triangle we obtain the unknown quantities.

(b) The analytical method can be applied for any number of forces. Before writing the conditions of equilibrium—which for a coplanar system of concurrent forces will be in the form of the two Eqs. (12), and for a three-dimensional system the three Eqs. (11)—the coordinate axes must be chosen. The choice is arbitrary, but the equations can be simplified by taking one of the axes perpendicular to an unknown force. The beginner is advised to commence his solution by determining the projections of all the forces on each of the coordinate axes and tabulating the information (see Problems 6, 10, and 11).

Other suggestions are offered in the sample problems below.

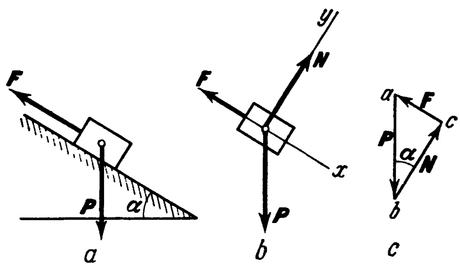

Fig. 29

Problem 6. A load of weight P lies on a plane inclined at α degrees to the horizontal (Fig. 29*a*). Determine the magnitude of the force F parallel to the plane which should be applied to the load to keep it in equilibrium, and the pressure Q exerted by the load on the plane.

Solution. The required forces act on different bodies: $\boldsymbol{F}$ on the load, and $\boldsymbol{Q}$ on the plane. To solve the problem we shall determine instead of $\boldsymbol{Q}$ the reaction $\boldsymbol{N}$ of the plane, which is equal to $\boldsymbol{Q}$ in magnitude and opposite in sense. In this case the given force $\boldsymbol{P}$ and the required forces $\boldsymbol{F}$ and $\boldsymbol{N}$ all act on the load, i.e., on one body. Now consider the equilibrium of the load as a free body (Fig. 29*b*, where $\boldsymbol{P}$ and $\boldsymbol{F}$ are the applied forces and $\boldsymbol{N}$ is the reaction of the constraint, or the plane). The required forces can be determined by employing either the graphical or the analytical method of stating the equilibrium conditions of the body.

Graphical Method. If the body is in equilibrium, the force triangle with $\boldsymbol{P}$, $\boldsymbol{F}$, and $\boldsymbol{N}$ as its sides must be closed. Start the construction with the given force: from an arbitrary point *a* lay off to scale force $\boldsymbol{P}$ (Fig. 29*c*). Through its initial and terminal points draw straight lines parallel to the directions of the forces $\boldsymbol{F}$ and $\boldsymbol{N}$. The intersection of the lines gives us the third vertex *c* of the closed force triangle *abc*, whose sides *bc* and *ca* denote the required forces in the chosen scale. The direction of the forces is determined by the arrow rule: as the resultant is zero, no two arrowheads can meet in any vertex of the triangle.

The magnitudes of the required forces can also be computed from the triangle *abc*, in which case the diagram need not be drawn to scale. Observing that $\angle bca = 90°$, and $\angle abc = \alpha$, we have

$$F = P \sin \alpha, \quad N = P \cos \alpha.$$

Analytical Method. Since the force system is coplanar, only two coordinate axes are needed. To simplify the computation, take axis Ox perpendicular to the unknown force N. Compute the projections of all the forces on each of the coordinate axes and tabulate the information*):

F_k	P	F	N
F_{kx}	$P \sin \alpha$	$-F$	0
F_{ky}	$-P \cos \alpha$	0	N

From Eqs. (12) we obtain:

$$P \sin \alpha - F = 0,$$
$$-P \cos \alpha + N = 0.$$

Thanks to our choice of coordinate axes, each equation contains only one unknown quantity. Solving the equations, we obtain:

$$F = P \sin \alpha, \quad N = P \cos \alpha.$$

The force exerted by the load on the plane is equal in magnitude to the calculated force $N = P \cos \alpha$ and opposite in sense.

It will be noted that the force F needed to hold the load on the inclined plane is less than its weight P. Thus, an inclined plane represents a simple machine which makes it possible to balance a large force with a smaller one.

As was shown in the beginning of § 11, these results hold equally good for a body at rest or in motion "under no forces". It follows, then, that in order to push the load with uniform velocity up a smooth surface the same force $F = P \sin \alpha$ has to be applied which is needed to keep it in equilibrium; similarly, the same force $F = P \sin \alpha$ has to be applied to brake the load if we wish it to slide with uniform velocity down the plane. For either motion to take place the load must receive an initial velocity, otherwise the force $F = P \sin \alpha$ acting on it will keep it at rest. The force exerted on the plane will in all cases be $P \cos \alpha$.

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*) The table should be filled by vertical columns: first compute the x and y projections of P, then of F, etc. The use of tables reduces the possibility of mistakes in the equations. The beginner will find tables especially useful until he acquires the necessary experience in operating with force projections.

A general conclusion can be drawn from the solution of the above problem: *In problems of statics solved by the equations of equilibrium, the forces exerted by a body on its constraints should be replaced by the reactions of the constraints acting on the body, which are equal in magnitude and opposite in sense to the applied forces.* In solving problems by the method of force resolution (§ 7) the forces exerted by the constraints are determined directly.

Problem 7. The rod AB in Fig. 30a is hinged to a fixed support by a pin A. Attached to the rod at B is a load $P = 10$ kgf and

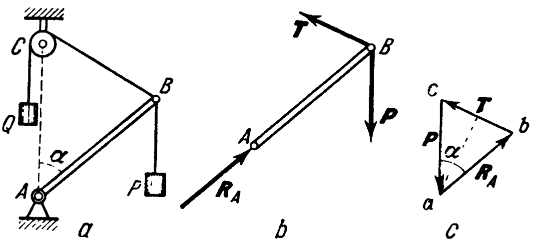

Fig. 30

a string passing over a pulley C with a load $Q = 14.1$ kgf tied to the other end of the string. The axes of the pulley C and the pin A lie on the same vertical and $AC = AB$. Neglecting the weight of the rod and the diameter of the pulley, determine the angle α at which the system will be in equilibrium and the stress in the rod AB.

Solution. Consider the conditions for the equilibrium of rod AB, to which all the given and required forces are applied. Removing the constraints and treating the rod as a free body (Fig. 30b), draw the forces acting on it: the weight of the load $\boldsymbol{P}$, the tension $\boldsymbol{T}$ in the string, and the reaction $\boldsymbol{R}_A$ of the pin, which is directed along AB, since in the present case the rod can only be in tension or in compression (see § 4). If the friction of the rope on the pulley is neglected, the tension in the string can be regarded as uniform throughout its length, whence $T = Q$.

For the graphical method of solution, construct a closed force triangle abc with forces $\boldsymbol{P}, \boldsymbol{T}, \boldsymbol{R}_A$ as its sides (Fig. 30c) starting with force $\boldsymbol{P}$. As triangles abc and ABC are similar, we have $ca = ab$ and $\angle cab = \alpha$. Hence, as $T = Q = 2P \sin \frac{\alpha}{2}$,

$$R_A = P \text{ and } \sin\frac{\alpha}{2} = \frac{Q}{2P}.$$

It follows from these results that at $\alpha < 180°$ equilibrium is possible only if $Q < 2P$ and that the rod will be compressed with a force equal to P at any values of Q and α.

The case of $\alpha = 180°$ should be examined separately. It is apparent that in this case equilibrium is possible at any values of P and Q. If $P > Q$ the rod will be stretched with a force $P - Q$; if $Q > P$ the rod will be compressed with a force $Q - P$.

Substituting the numerical values, we obtain $R_A = 10$ kgf, $\alpha = 90°$ (the rod is horizontal).

Note that force $\boldsymbol{Q}$ (the weight of the load) was not directly included in the equilibrium condition (in the force triangle), as it is applied to the load and not to the rod AB whose equilibrium was considered.

Further on in this book we shall not draw free-body diagrams, but in picturing the forces acting on a given body it should always be visualised as free, as in Figs. 29*b*, 30*b*, and 12*b*.

Problem 8. A crane held in position by a journal bearing A and a thrust bearing B carries a load P (Fig. 31). Neglecting the weight of the structure, determine the reactions R_A and R_B of the constraints if the jib is of length l and $AB = h$.

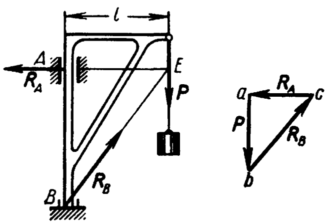

Fig. 31

Solution. Consider the equilibrium of the crane as a whole under the action of the given and required forces. Mentally remove the constraints A and B, considering the crane as a free body, and draw the given force $\boldsymbol{P}$ and the reaction $\boldsymbol{R}_A$ of the journal bearing perpendicular to AB. The reaction $\boldsymbol{R}_B$ of the thrust bearing can have any direction in the plane of the diagram. But the crane is in equilibrium under the action of three forces and consequently their lines of action must intersect in one point. This is point E, where the lines of action of $\boldsymbol{P}$ and $\boldsymbol{R}_A$ cross. Hence, the reaction $\boldsymbol{R}_B$ is directed along BE.

To solve the problem by the graphical method draw a closed triangle abc with forces $\boldsymbol{P}$, $\boldsymbol{R}_A$, $\boldsymbol{R}_B$ as its sides, starting with the given force $\boldsymbol{P}$. From the similarity of triangles abc and ABE we obtain:

$$\frac{R_A}{P} = \frac{l}{h}, \quad \frac{R_B}{P} = \frac{\sqrt{h^2 + l^2}}{h},$$

whence

$$R_A = \frac{l}{h} P, \quad R_B = \sqrt{1 + \frac{l^2}{h^2}}\, P.$$

From the triangle abc we see that the directions of the reactions $\boldsymbol{R}_A$ and $\boldsymbol{R}_B$ were drawn correctly. The loads acting on the journal bearing A and the thrust bearing B are respectively equal in magnitude to R_A and R_B but opposite in sense. The greater the ratio $l : h$, the greater the load acting on the constraints.

This problem is an example of the application of the theorem of three forces.

Note the following conclusion arising from it: *If the statement of a problem gives the linear dimensions of structural elements, it is more convenient to solve the force triangle by the rule of similarity; if the angles are given* (Problem 6), *the formulas of trigonometry should be used.*

Problem 9. A horizontal force $\boldsymbol{P}$ is applied to hinge A of the toggle-press in Fig. 32*a*. Neglecting the weight of the rods and piston, determine the force exerted by the piston on body M when the given angles are α and β.

Solution. First consider the equilibrium of the hinge A to which the given force $\boldsymbol{P}$ is applied. Regarding the hinge as a free body, we find that also acting on it are the reactions $\boldsymbol{R}_1$ and $\boldsymbol{R}_2$ of the rods directed along them. Construct a force triangle (Fig. 32*b*). Its angles are $\varphi = 90° - \alpha$, $\psi = 90° - \beta$, $\gamma = \alpha + \beta$. By the law of sines we have:

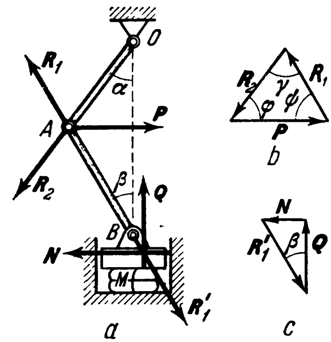

Fig. 32

$$\frac{R_1}{\sin\psi} = \frac{P}{\sin\gamma}, \quad R_1 = \frac{P\cos\alpha}{\sin(\alpha+\beta)}.$$

Now consider the equilibrium of the piston, regarding it as a free body. Acting on it are three forces: $\boldsymbol{R}_1' = -\boldsymbol{R}_1$ exerted by rod AB, the reaction $\boldsymbol{N}$ of the wall, and the reaction $\boldsymbol{Q}$ of the pressed body. The three forces are in equilibrium, consequently they are concurrent. Constructing a triangle with the forces as its sides (Fig. 32*c*), we find:

$$Q = R_1' \cos\beta.$$

Substituting for $\boldsymbol{R}_1'$ its equivalent $\boldsymbol{R}_1$, we finally obtain:

$$Q = \frac{P\cos\alpha\cos\beta}{\sin(\alpha+\beta)} = \frac{P}{\tan\alpha + \tan\beta}.$$

The force with which the piston compresses the body M is equal to Q in magnitude and opposite in sense.

From the last formula we see that with a constant applied force P the pressure Q increases as the angles α and β diminish.

If the rods OA and AB are of equal length, then $\alpha = \beta$ and $Q = 0.5P\cot\alpha$.

The following conclusion can be drawn from this solution: *In some problems the given force or forces are applied to one body and the required force or forces act on another; in such cases the equilibrium of the first* *body is considered and the force with which it acts on the other body is found; then the second body is examined and the required quantities are obtained.*

Problem 10. Rods AB and BC of the bracket in Fig. 33a are joined together and attached to the wall by hinges. Over a pulley attached to the bracket at B passes a string, one end of which is fastened to the wall while the other supports a load of weight Q. Neglecting the weight of the rods and the diameter of the pulley, determine the reactions of the rods if angles α and β are given.

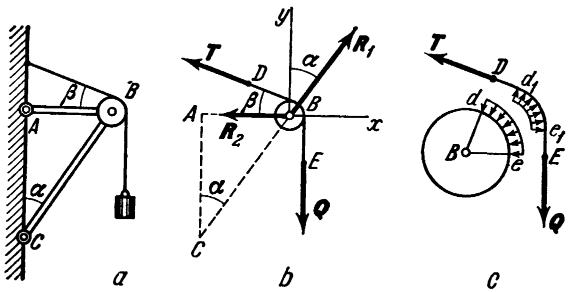

Fig. 33

Solution. Consider the equilibrium of the pulley with the section DE of the string which is in contact with it *). Isolate the body and draw the reactions of the constraints (Fig. 33b). Acting on the pulley and the segment of the string passing over it are four external forces: the tension $\boldsymbol{Q}$ in the right-hand part of the string, the tension $\boldsymbol{T}$ in the left-hand part of the string, which is equal to Q in magnitude ($T = Q$), and the reactions $\boldsymbol{R}_1$ and $\boldsymbol{R}_2$ of the rods directed along the rods. Neglecting the diameter of the pulley, the forces can be treated as concurrent. As there are more than three forces the analytical method of solution is more convenient. Draw the coordinate axes as shown in the diagram and compute the x and y projections of all the forces:

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*) In such cases it is best to treat the pulley as one body together with the section of the string with which it is in contact. The unknown reciprocal actions of the string and the pulley distributed along the arc de constitute a system of balanced internal forces and do not enter into the equilibrium conditions (see § 3, corollary of the 4th principle). Should we treat the pulley separately (Fig. 33c, in a larger scale) we would have to consider the forces exerted by the string on the pulley along the arc de, the resultant of which would have to be found by additionally examining the conditions of equilibrium of section DE of the string (applying the principle of solidification). This would make the calculations much more involved.

F_k	Q	T	R_1	R_2
F_{kx}	0	$-T\cos\beta$	$R_1\sin\alpha$	$-R_2$
F_{ky}	$-Q$	$T\sin\beta$	$R_1\cos\alpha$	0

Now apply the formulas of equilibrium (12) and write the corresponding equations, substituting for T the equal quantity Q:

$$-Q\cos\beta + R_1\sin\alpha - R_2 = 0;$$
$$-Q + Q\sin\beta + R_1\cos\alpha = 0.$$

From the second equation we find:

$$R_1 = \frac{1-\sin\beta}{\cos\alpha}Q.$$

Substituting this value of R_1 into the first equation and transposing, we obtain:

$$R_2 = Q\,\frac{\sin\alpha - \cos(\alpha-\beta)}{\cos\alpha}.$$

It follows from the expression for R_1 that at any acute angles α and β, $R_1 > 0$. This means that the reaction $\boldsymbol{R}_1$ is always directed as shown in the diagram. The force which the pulley exerts on the rod is oppositely directed (rod BC is under thrust). For $\boldsymbol{R}_2$ we obtain a different result. Let us assume that angles α and β are always acute. Since

$$\sin\alpha - \cos(\alpha-\beta) = \sin\alpha - \sin(90° - \alpha + \beta),$$

the difference is positive if $\alpha > (90° - \alpha + \beta)$ or if $2\alpha > 90° + \beta$. Hence, at $\alpha > \left(45° + \frac{\beta}{2}\right)$, $R_2 > 0$, i.e., the reaction $\boldsymbol{R}_2$ is directed as shown in the diagram. But if $\alpha < \left(45° + \frac{\beta}{2}\right)$, $R_2 < 0$, and the reaction $\boldsymbol{R}_2$ is of opposite sense and is directed from A to B. In the first case rod AB is in tension, in the second it is in compression. At $\alpha = 45° + \frac{\beta}{2}$, $R_2 = 0$.

The following conclusions are important: (1) If a system includes pulleys with strings passing over them, in writing the equations of equilibrium a pulley and the section of the string with which it is in contact should be treated as a single body. If the friction of the cable on the pulley or the friction of the pulley axle can be neglected, the tension in both portions of the string is equal in magnitude and directed away from the pulley—otherwise the string would slip in the direction of the greater tension or the pulley would turn (see also Problem 13).

(2) If, in drawing the reactions of constraints, any reaction is pointed in the wrong direction, this will show up immediately in the force polygon of a graphical solution (the arrowhead rule); in an analytical solution the sign of the respective reaction will be negative.

Whenever possible, however, the forces should be directed correctly. In Problem 8, for instance, the direction of the reaction of bearing A can be determined by the following consideration: if the bearing is removed force $\boldsymbol{P}$ will tend to overturn the crane to the right, consequently, force $\boldsymbol{R}_A$, which replaces the action of the bearing, should be directed to the left in order to keep the crane in equilibrium.

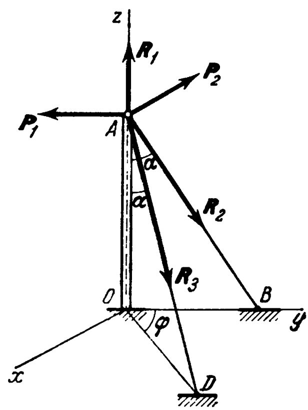


Fig. 34

Problem 11. The vertical pole OA in Fig. 34 is anchored down by guy wires AB and AD which make equal angles $\alpha = 30°$ with the pole; the angle between the planes AOB and AOD is $\varphi = 60°$. Two horizontal wires parallel to the axes Ox and Oy are attached to the pole, and the tension in each of them is $P = 100$ kgf. Neglecting the weight of all the elements, determine the vertical load acting on the pole and the tensions in the guy wires.

Solution. Consider the equilibrium of point A to which the guys and horizontal wires are attached. Acting on it are the reactions $\boldsymbol{P}_1$ and $\boldsymbol{P}_2$ of the horizontal wires ($P_1 = P_2 = P$), the reactions $\boldsymbol{R}_2$ and $\boldsymbol{R}_3$ of the guys, and the reaction $\boldsymbol{R}_1$ of the pole. The force system is three-dimensional, and the analytical method of solution is most suitable. Draw a coordinate system as shown in the diagram, compute the projections of all the forces on each of the axes, and tabulate the information (the x and y projections of $\boldsymbol{R}_3$ are calculated as explained at the end of § 8):

$\boldsymbol{F}_k$	$\boldsymbol{P}_1$	$\boldsymbol{P}_2$	$\boldsymbol{R}_1$	$\boldsymbol{R}_2$	$\boldsymbol{R}_3$
F_{kx}	0	$-P$	0	0	$R_3 \sin\alpha \sin\varphi$
F_{ky}	$-P$	0	0	$R_2 \sin\alpha$	$R_3 \sin\alpha \cos\varphi$
F_{kz}	0	0	R_1	$-R_2 \cos\alpha$	$-R_3 \cos\alpha$

From the equilibrium Eqs. (11) we have:

$$-P + R_3 \sin\alpha \sin\varphi = 0,$$
$$-P + R_2 \sin\alpha + R_3 \sin\alpha \cos\varphi = 0,$$
$$R_1 - R_2 \cos\alpha - R_3 \cos\alpha = 0,$$

solving which we obtain:

$$R_3 = \frac{P}{\sin\alpha \sin\varphi}, \quad R_2 = P\frac{1-\cot\varphi}{\sin\alpha},$$
$$R_1 = P\left(1 + \tan\frac{\varphi}{2}\right)\cot\alpha.$$

The results show that at $\varphi < 45°$, $R_2 < 0$, and the reaction $\boldsymbol{R}_2$ is of opposite sense than shown in the diagram. As a wire cannot be in compression, it follows that the guy AD should be anchored in such a way that angle φ would be greater than 45°. Substituting the quantities in the equations, we obtain: $R_3 = 231$ kgf, $R_2 = 85$ kgf, $R_1 = 273$ kgf.

§ 14. Moment of Force About an Axis (or a Point)

We know from experience that a force acting on a body tends either to displace it in some direction or to rotate it about a point. The tendency of a force to turn a body about a point or an axis is called the *moment* of that force.

Consider a force $\boldsymbol{F}$ applied at a point A of a rigid body (Fig. 35) which tends to rotate the body about a point O. The perpendicular distance h from O to the line of action of $\boldsymbol{F}$ is called the *moment arm* of force $\boldsymbol{F}$ about the centre O. As the point of application of the force can be transferred arbitrarily along its line of action, it is apparent that the rotational action of any force depends only on (1) the magnitude of the force F and the length of its moment arm h; (2) the position of the plane OAB of rotation through the centre O and the force $\boldsymbol{F}$; and (3) the sense of the rotation in that plane.

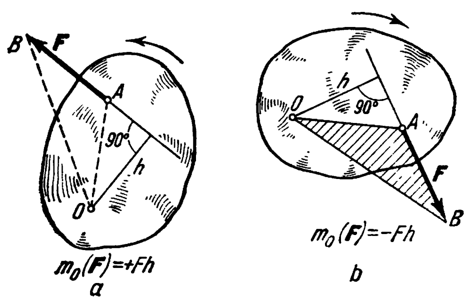


Fig. 35

For the present we shall limit ourselves to coplanar force systems, in which the plane of rotation is the same for all the forces and does not have to be specifically defined. The sense of rotation is denoted by (+) or (—), assuming it to be positive in some particular direction.

Thus, we may formulate the concept of moment of a force as a measure of the tendency of the force to turn the body on which it acts: *The moment of a force* $\boldsymbol{F}$ *about a centre O is defined as the product of the force magnitude and the length of the moment arm taken with appropriate sign.*

We shall denote the moment of a force $\boldsymbol{F}$ about a centre O by the symbol $m_O(\boldsymbol{F})$. Thus,

$$m_O(\boldsymbol{F}) = \pm Fh. \tag{13}$$

We shall call a moment positive if the applied force tends to rotate the body counterclockwise, and negative if it tends to rotate the body clockwise. Thus, the sign of the moment of the force $\boldsymbol{F}$ about O is (+) in Fig. 35*a*, and (—) in Fig. 35*b*. If the arm is measured in metres, the moment of the force is measured in newton-metres (N·m) or in kilogram-metres (kgf·m).

Note the following properties of the moment of a force:

(1) The moment of a force does not change if the point of application of the force is transferred along its line of action.

(2) The moment of a force about a centre O can be zero only if the force is zero or if its line of action passes through O (i.e., the moment arm is zero).

(3) The magnitude of the moment of a force is represented by twice the area of the triangle OAB (Fig. 35*b*):

$$m_O(\boldsymbol{F}) = \pm 2 \text{ areas of } \triangle OAB. \tag{14}$$

This follows from the fact that

$$\text{area of } \triangle OAB = \frac{1}{2} AB \cdot h = \frac{1}{2} F \cdot h.$$

§ 15. Varignon's Theorem[*] of the Moment of a Resultant

The moment of the resultant of a coplanar system of concurrent forces about any centre is equal to the algebraic sum of the moments of the component forces about that centre.

Consider a coplanar system of concurrent forces $\boldsymbol{F}_1, \boldsymbol{F}_2, \ldots, \boldsymbol{F}_n$ intersecting at a point A (Fig. 36). Take an arbitrary point O and draw axis Ox perpendicular to OA; we select the positive direction of axis Ox such that the sense of the projections of all the forces on the axis is the same as the sense of their respective moments about the centre O.

To prove the theorem, determine the respective expressions of the moments $m_O(\boldsymbol{F}_1)$, $m_O(\boldsymbol{F}_2)$, ... From Eq. (14), $m_O(\boldsymbol{F}_1) =$

[*] Pierre Varignon (1654-1722), a celebrated French mathematician and mechanic, who outlined the fundamentals of statics in his book *Projet d'une nouvelle mécanique* (1686).

$= +2$ areas of $\triangle OAB_1$. But it is apparent from the figure that 2 areas of $\triangle OAB_1 = OA \cdot Ob = OA \cdot F_{1x}$, where F_{1x} is the projection of force $\boldsymbol{F}_1$ on axis Ox. Hence,

$$m_O(\boldsymbol{F}_1) = OA \cdot F_{1x} \qquad (15)$$

The moments of the other forces are calculated similarly. Equation (15) is valid also when force $\boldsymbol{F}$ passes below OA. In this case the moment will be negative, because the projection F_x will be negative.

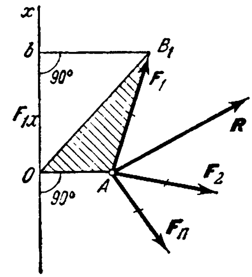


Fig. 36

Denote the resultant of the forces $\boldsymbol{F}_1, \boldsymbol{F}_2, \ldots, \boldsymbol{F}_n$ as $\boldsymbol{R}$, where $\boldsymbol{R} = \sum \boldsymbol{F}_k$. From the theorem of the projection of a sum of forces on an axis we have $R_x = \sum F_{kx}$. Multiplying through by OA we obtain:

$$OA \cdot R_x = \sum (OA \cdot F_{kx})$$

or, by Eq. (15),

$$m_O(\boldsymbol{R}) = \sum m_O(\boldsymbol{F}_k). \qquad (16)$$

Eq. (16) is the mathematical expression of Varignon's theorem.

§ 16*. Equations of Moments of Concurrent Forces

The analytical conditions of the equilibrium of concurrent forces can be expressed in terms of either their projections or their moments. Let us demonstrate that the necessary and sufficient conditions for

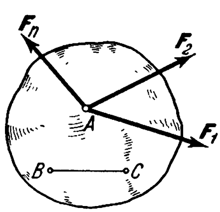


Fig. 37

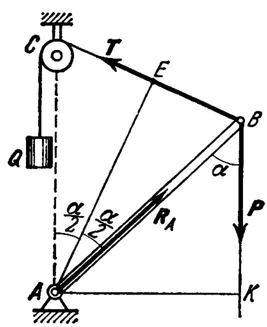


Fig. 38

the equilibrium of a coplanar system of concurrent forces are:

$$\sum m_B(\boldsymbol{F}_k) = 0, \quad \sum m_C(\boldsymbol{F}_k) = 0, \qquad (17)$$

where B and C are arbitrary points not collinear with the point A where the forces intersect (Fig. 37).

The necessity of these conditions is apparent since, for example, if $\sum m_B(\boldsymbol{F}_k) \neq 0$ then, by Eq. (16), $m_B(\boldsymbol{R}) \neq 0$, whence $\boldsymbol{R} \neq 0$, and equilibrium is impossible.

Let us prove the sufficiency of these conditions. If the conditions (17) are fulfilled, then according to Varignon's theorem, $m_B(\boldsymbol{R}) = 0$ and $m_C(\boldsymbol{R}) = 0$, which is possible only if $\boldsymbol{R} = 0$ or if the line of action of $\boldsymbol{R}$ passes through both B and C. The latter condition being impossible, as the resultant of the concurrent forces must pass through A (Fig. 37), which is stipulated as not collinear with B and C. Thus, Eqs. (17) can be valid only if the resultant $\boldsymbol{R} = 0$, i.e., if the force system is in equilibrium.

It is apparent that satisfaction of only one of the conditions (17) is insufficient for equilibrium.

In solving problems with Eqs. (17) the equations can be made to contain a single unknown quantity each by taking the centres of moments on the lines of action of the unknown forces.

Problem 12. Solve Problem 7 using the equations of moments.

Solution. Introducing the symbol $a = AB = AC$, take points A and C as the centres of the moments (Fig. 38). Drawing perpendiculars AE and AK from A to the lines of action of forces $\boldsymbol{T}$ and $\boldsymbol{P}$, we obtain: $AE = a\cos\frac{\alpha}{2}$, $AK = a\sin\alpha$, whence $m_A(\boldsymbol{T}) = Ta\cos\frac{\alpha}{2}$, $m_A(\boldsymbol{P}) = -Pa\sin\alpha$. Furthermore, $m_A(\boldsymbol{R}_A) = 0$. The moments of the forces about C are computed similarly. From the equilibrium Eqs. (17) we obtain:

$$\sum m_A(\boldsymbol{F}_k) \equiv Ta\cos\frac{\alpha}{2} - Pa\sin\alpha = 0, \qquad \text{(a)}$$

$$\sum m_C(\boldsymbol{F}_k) \equiv R_A a\sin\alpha - Pa\sin\alpha = 0. \qquad \text{(b)}$$

As $T = Q$, we obtain from (a)

$$\left(Q - 2P\sin\frac{\alpha}{2}\right)\cos\frac{\alpha}{2} = 0,$$

whence angle α, which defines the position of equilibrium, has two values:

$$\alpha = 180^\circ \text{ and } \sin\frac{\alpha}{2} = \frac{Q}{2P}.$$

From equation (b) we find that at $\alpha \neq 180^\circ$, $R_A = P$.

Problem 13. In Problem 10 determine the reaction R_2 by means of the equations of moments.

Solution. Taking the moments about C (see Fig. 33*b*) and assuming $CB = a$, we obtain:

$$\sum m_C(\boldsymbol{F}_k) \equiv Ta\cos(\alpha - \beta) + R_2 a\cos\alpha - Qa\sin\alpha = 0.$$

Whence, as $T = Q$, we immediately find:

$$R_2 = Q \frac{\sin\alpha - \cos(\alpha - \beta)}{\cos\alpha}.$$

The reaction R_1 can be found by taking the moments about A. The equations of moments can thus also be used to verify answers obtained by other methods.

Note that the validity of the equality $T = Q$ can be verified by writing an equation of moments about the centre of the pulley (this equation, as will be proved in § 24, also holds good for non-concurrent forces). In this case we obtain $Tr - Qr = 0$, where r is the radius of the pulley, or $T = Q$.

Chapter 3
Parallel Forces and Force Couples in a Plane

§ 17. Composition and Resolution of Parallel Forces

Let us find the resultant of two parallel forces acting on a rigid body. Two cases are possible: (1) the forces are of same sense, and (2) the forces are of opposite sense.

(1) Composition of Two Forces of Same Sense. Consider a rigid body on which two parallel forces $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ are acting (Fig. 39). By applying the 1st and 2nd principles of statics we can replace the given system of parallel forces with an equivalent system of concurrent forces $\boldsymbol{Q}_1$ and $\boldsymbol{Q}_2$. For this, apply two balanced forces $\boldsymbol{P}_1$ and $\boldsymbol{P}_2$ ($\boldsymbol{P}_1 = -\boldsymbol{P}_2$) directed along AB at points A and B, compound them with forces $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ according to the parallelogram law, transfer the resultants $\boldsymbol{Q}_1$ and $\boldsymbol{Q}_2$ to the point O where their lines of action intersect, and resolve them into their initial components. As a result we have applied at point O two balanced forces $\boldsymbol{P}_1$ and $\boldsymbol{P}_2$ which can be neglected, and two forces $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ directed along the same line. Now transfer the latter two forces to C and replace them by their resultant $\boldsymbol{R}$ of magnitude

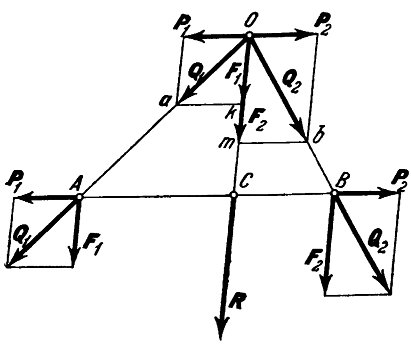


Fig. 39

$$R = F_1 + F_2. \tag{18}$$

Thus, force $\boldsymbol{R}$ is the resultant of the parallel forces $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ applied at points A and B. To determine the position of C consider the triangles OAC, Oak, and OCB, Omb. From the similarity of the respective triangles it follows that:

$$\frac{AC}{OC}=\frac{P_1}{F_1} \text{ and } \frac{BC}{OC}=\frac{P_2}{F_2}$$

or $AC \cdot F_1 = BC \cdot F_2$, as $P_1 = P_2$.

From the property of proportions, and taking into account that $BC + AC = AB$ and $F_1 + F_2 = R$, we obtain:

$$\frac{BC}{F_1}=\frac{AC}{F_2}=\frac{AB}{R}. \tag{19}$$

Thus, *the resultant of two parallel forces of the same sense acting on a rigid body is equal to the sum of their magnitudes, parallel to them, and is of same sense*; *the line of action of the resultant is between the points of application of the component forces, its distances from the points being inversely proportional to the magnitudes of the forces.*

(2) Composition of Two Forces of Opposite Sense. Consider the concrete case of $F_1 > F_2$ (Fig. 40). Take a point C on the extension of BA and apply two balanced forces $\boldsymbol{R}$ and $\boldsymbol{R}'$ parallel to the given forces $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$. The magnitudes of $\boldsymbol{R}$ and $\boldsymbol{R}'$ and the location of C are chosen so as to satisfy the equations:

$$R=F_1-F_2; \tag{20}$$

$$\frac{BC}{F_1}=\frac{AC}{F_2}=\frac{AB}{R}. \tag{21}$$

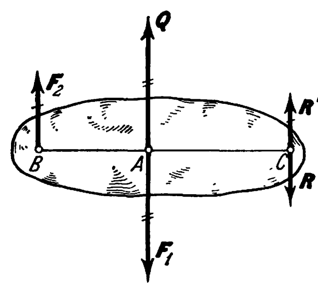


Fig. 40

Compounding forces $\boldsymbol{F}_2$ and $\boldsymbol{R}'$, we find from Eqs. (18) and (19) that their resultant $\boldsymbol{Q}$ is equal in magnitude to F_2+R', i.e., it is equal to force $\boldsymbol{F}_1$ and is applied at point A. Forces $\boldsymbol{F}_1$ and $\boldsymbol{Q}$ are balanced and can be discarded. As a result, the given forces $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ are replaced by a single force $\boldsymbol{R}$, their resultant. The magnitude and point of application of the resultant is determined by Eqs. (20) and (21). Thus, *the resultant of two parallel forces of opposite sense acting on a rigid body is equal in magnitude to the difference between their magnitudes, parallel to them, and has the same sense as the greater force*; *the line of action of the resultant lies on the extension of the line segment connecting the points of application of the component forces, its distances from the points being inversely proportional to the forces.*

If several parallel forces act on a body, their resultant, if any, can be found by consecutively applying the rule of composition of two forces, or by a method which will be examined in Chapter 4.

(3) Resolution of Forces. The above formulas can be used to solve problems on the resolution of a given force into two forces parallel to it and of same or opposite sense. The problem becomes determinate when additional conditions are given (e.g., the lines of action of both required forces or the magnitude and line of action of one of them).

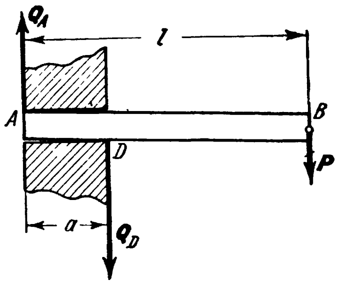


Fig. 41

Problem 14. A beam AB of length $l = 2.5$ m passes through a wall of thickness $a = 0.5$ m (Fig. 41). Suspended from end B of the beam is a load of weight $P = 3$ tonf. Neglecting the weight of the beam, determine the forces acting on the wall, assuming them applied at points A and D (the beam is slightly biased).

Solution. Resolve force $\boldsymbol{P}$ into forces $\boldsymbol{Q}_D$ and $\boldsymbol{Q}_A$ along the reactions of the constraints D and A. As force $\boldsymbol{P}$ does not lie between the required forces, they must be of opposite sense, and $\boldsymbol{Q}_D$, as being closer to $\boldsymbol{P}$, is greater than $\boldsymbol{Q}_A$ and of the same sense as $\boldsymbol{P}$. From the equations

$$\frac{Q_D}{l} = \frac{P}{a} \quad \text{and} \quad P = Q_D - Q_A$$

we obtain

$$Q_D = \frac{l}{a} P = 15 \text{ tonf}, \quad Q_A = 12 \text{ tonf}.$$

The results can be verified by the proportion:

$$\frac{Q_A}{l-a} = \frac{P}{a}.$$

§ 18. A Force Couple. Moment of a Couple

A *force couple* is a system of two parallel forces of same magnitude and opposite sense acting on a rigid body (Fig. 42). Clearly, a force system constituting a couple is not in equilibrium (see the 1st principle). Furthermore, unlike previously examined systems, a couple has no resultant. For if the couple $(\boldsymbol{F}, \boldsymbol{F}')$ had a resultant $\boldsymbol{Q} \neq 0$, there would also have to be a force $\boldsymbol{Q}_1 = -\boldsymbol{Q}$ capable of balancing it, i.e., the system of forces $\boldsymbol{F}$, $\boldsymbol{F}'$, $\boldsymbol{Q}$ would be in equilibrium. But as will be shown later on, for any system of forces to be in equilibrium their geometrical sum must be zero; in the present case this would require that $\boldsymbol{F} + \boldsymbol{F}' + \boldsymbol{Q}_1 = 0$, which is impossible since $\boldsymbol{F} + \boldsymbol{F}' = 0$ but $\boldsymbol{Q}_1 \neq 0$. Thus, a couple cannot be replaced or balanced by a single force. For this reason the properties of the couple as a special mode of mechanical interaction between bodies are the subject of a special study.

The plane through the lines of action of both forces of a couple is called the *plane of action of the couple*. The perpendicular distance d between the lines of action of the forces is called the *arm of the couple*. The action of a couple on a rigid body is a tendency to turn it; it depends on: (1) the magnitude F of the forces of the couple and the perpendicular distance d between them; (2) the location of the plane of action of the couple; and (3) the sense of rotation in that plane. A couple is characterised by its *moment*.

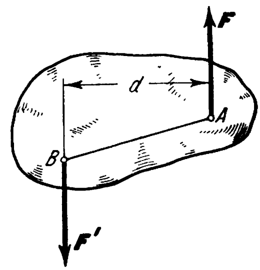


Fig. 42

In this chapter we shall discuss the properties of couples of coplanar forces. For this case the following definition can be given in analogy with that of the moment of a force (§ 14): *The moment of a couple is defined as a quantity equal to the product of the magnitude of one of the forces of the couple and the perpendicular distance between the forces, taken with the appropriate sign**). Denoting the moment of a couple by the symbol m or M, we have:

$$m = \pm Fd. \tag{22}$$

The moment of a couple (as that of a force) *is said to be positive if the action of the couple tends to turn a body counterclockwise, and negative if clockwise.* The moment of a couple is measured in the same units as the moment of a force. It is apparent from Fig. 42 that the moment of a couple is equal to the moment of one of its forces about the point of application of the other, i.e.,

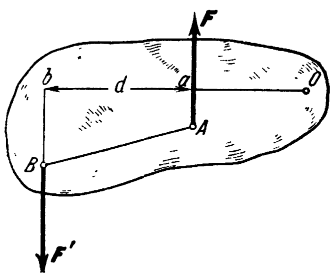


Fig. 43

$$m = m_B(\boldsymbol{F}) = m_A(\boldsymbol{F}'). \tag{23}$$

Let us prove the following **theorem of the moments of the forces of a couple:** *The algebraic sum of the moments of the forces of a couple about any point in its plane of action is independent of the location of that point and is equal to the moment of the couple.* For, taking an arbitrary point O in the plane of a couple (Fig. 43), we find: $m_O(\boldsymbol{F}) =$

*) This concept should not be confused with the moment of a force. The concept of moment of a force presumes a point about which the moment is taken. The moment of a couple is defined only by its forces and the perpendicular distance between them: it is not associated with any point in the plane. The theory of couples was elaborated by the eminent French mathematician and geometrician Louis Poinsot (1777-1859).

$= -F \cdot Oa$, $m_O(F') = F' \cdot Ob$. Adding the two equations and noting that $F' = F$ and $Ob - Oa = d$, where d is the couple arm, we obtain:

$$m_O(F) + m_O(F') = m. \tag{24}$$

This theorem will be found useful in computing the moments of couples about any centre.

§ 19. Equivalent Couples

Before stating the conditions necessary for two couples to be equivalent let us prove the following theorem: *A couple acting on a rigid body can be replaced by any other couple of the same moment lying in the same plane without altering the external effect on that body.*

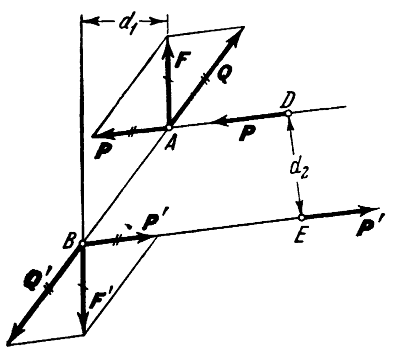


Fig. 44

Consider a couple (F, F') acting on a rigid body, and let the arm of this couple be d_1. Through arbitrary points D and E in the plane where the two forces act draw two parallel lines intersecting the lines of action of the forces at points A and B (Fig. 44) and apply the forces at those points. Before the forces F and F' could be applied at any point on their lines of action. Denote the distance between the lines AD and BE by d_2. Resolve F along AB and AD into forces Q and P, and F' along AB and BE into forces Q' and P'. Obviously $P = -P'$ and $Q = -Q'$. Forces Q and Q' are balanced and can be discarded. The couple (F, F') is replaced by the couple (P, P') with different magnitudes of the forces and arm, which forces can, obviously, be applied at any points D and E on their lines of action. Since points D and E and the directions of AD and BE are arbitrary, the location of the couple (P, P') in the plane is also arbitrary (the situation in which P and P' are parallel to F can be attained by performing the mentioned transformations with the couple twice).

Let us now show that the moments of the two couples (P, P') and (F, F') are equal. As force F is the resultant of forces P and Q, from Varignon's theorem

$$m_B(F) = m_B(P) + m_B(Q).$$

But $m_B(\boldsymbol{F}) = Fd_1$, $m_B(\boldsymbol{P}) = Pd_2$, and $m_B(\boldsymbol{Q}) = 0$; consequently, $Fd_1 = Pd_2$, i.e., the moments of the two couples are equal, which proves the theorem.

The following properties of a couple follow from this theorem:

(1) A couple can be transferred anywhere in its plane of action;

(2) It is possible to change the magnitudes of the forces of a couple or the perpendicular distance between them arbitrarily without changing its moment.

It follows from these properties that two coplanar couples of equal moment are equivalent as, by performing the above operations, i.e., by changing the arm and displacing in the plane of action, they can be transformed into one another. From the above theorems it is also apparent that the action of a couple on a rigid body is really characterised by its moment.

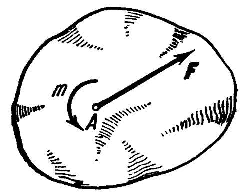


Fig. 45

Hence, a couple in a given plane is completely defined by its moment; the magnitudes of the forces, the distance between them, and their location in the plane of action are immaterial. That is why in engineering problems a couple is often denoted by a semicircular arrow indicating the direction of the rotation, without drawing the forces of the couple (Fig. 45, for example, shows a force $\boldsymbol{F}$ and a couple of moment m acting on the body).

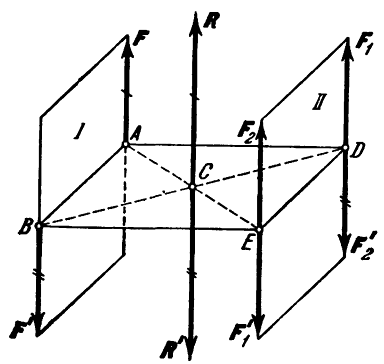


Fig. 46

Let us now prove another theorem: *The external effect of a couple on a rigid body remains the same if the couple is transferred from a given plane into any other parallel plane.*

Consider a couple $(\boldsymbol{F}, \boldsymbol{F}')$ lying in plane I (Fig. 46). Now take a plane II parallel to the given plane I, and in it a line DE equal and parallel to AB. At points D and E apply two pairs of balanced forces such that $\boldsymbol{F}_1 = \boldsymbol{F}_2 = \boldsymbol{F}$ and $\boldsymbol{F}_1' = \boldsymbol{F}_2' = \boldsymbol{F}'$. Note that $ABED$ is a parallelogram, the diagonals of which bisect at their point of intersection C. Compound now the equal parallel forces $\boldsymbol{F}$ and $\boldsymbol{F}_2$ and replace them by their resultant $\boldsymbol{R} = 2\boldsymbol{F}$ applied at the middle of AE, i.e., at C. The resultant of forces $\boldsymbol{F}'$ and $\boldsymbol{F}_2'$ is $\boldsymbol{R}' = 2\boldsymbol{F}' = -\boldsymbol{R}$ applied at the middle of BD, i.e., also at C, and forces $\boldsymbol{R}$ and $\boldsymbol{R}'$ therefore cancel each other. As a result the couple $(\boldsymbol{F}, \boldsymbol{F}')$ is replaced by a similar couple $(\boldsymbol{F}_1, \boldsymbol{F}_1')$ in plane II.

It follows from the above theorems that two couples of equal moment lying in parallel planes are equivalent.

Attention is drawn to the following analogy: A force acting on a rigid body is defined by its magnitude, line of action, and sense; its point of application can be transferred arbitrarily along the line of action. A couple acting on a rigid body is defined by the magnitude of its moment, its plane of action, and the sense of rotation; a couple can be situated anywhere in its plane of action.

The motion of a rigid body subjected to a couple is discussed in dynamics. It will be proved in theorems of dynamics that any couple acting on a free rigid body tends to turn that body about its centre of gravity (see p. 381). If a body has a fixed axis of rotation, the couple, regardless of its location in the plane perpendicular to this axis, will turn the body around the given axis with the same rotational effort (moment), which follows from Eq. 24.

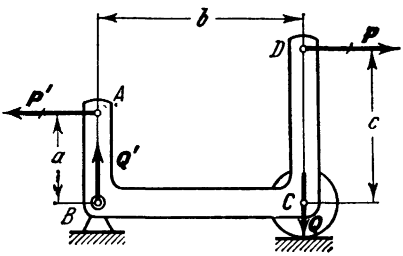


Fig. 47

Problem 15. The lever $ABCD$ in Fig. 47 is in equilibrium under the action of two parallel forces $\boldsymbol{P}$ and $\boldsymbol{P}'$ making a couple. Determine the load on the supports if $AB = a = 15$ cm, $BC = b = 30$ cm, $CD = c = 20$ cm, and $P = P' = 30$ kgf.

Solution. Replace couple $(\boldsymbol{P}, \boldsymbol{P}')$ with an equivalent couple $(\boldsymbol{Q}, \boldsymbol{Q}')$ whose two forces are directed along the reactions of the constraints. The moments of the two couples are equal, i.e., $P(c-a) = Qb$, consequently the loads on the constraints are:

$$Q = Q' = \frac{c-a}{b} P = 5 \text{ kgf}$$

and are directed as shown in the diagram.

§ 20. Composition of Coplanar Couples. Conditions for the Equilibrium of Couples

Let us prove the following theorem of the composition of couples: *A system of coplanar couples is equivalent to a single couple lying in the same plane the moment of which equals the algebraic sum of the moments of the component couples.* Let three couples of moments m_1, m_2, and m_3 be acting on a body (Fig. 48). By the theorem of equivalent couples they can be replaced by couples $(\boldsymbol{P}_1, \boldsymbol{P}_1')$, $(\boldsymbol{P}_2, \boldsymbol{P}_2')$, and ($P_3$, P_3') with a common arm d and the same moments:

$$P_1 d = m_1, \quad -P_2 d = m_2, \quad P_3 d = m_3.$$

Compounding the forces applied at A and B respectively, we obtain a force $\boldsymbol{R}$ at B and a force $\boldsymbol{R}'$ at A the magnitudes of which are:

$$R = R' = P_1 - P_2 + P_3.$$

As a result the set of couples is replaced by a single couple ($\boldsymbol{R}$, $\boldsymbol{R}'$) with a moment

$$M = Rd = P_1 d + (-P_2 d) + $$
$$+ P_3 d = m_1 + m_2 + m_3.$$

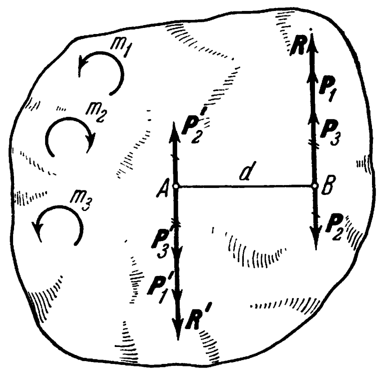


Fig. 48

The theorem is proved for three couples, but apparently the same result can be obtained for any number of couples, and a set of n couples of moments m_1, m_2, . . ., m_n can be replaced by a single couple with a moment

$$M = \sum m_k. \qquad (25)$$

It follows from this theorem that for a coplanar system of couples to be in equilibrium it is necessary and sufficient for the algebraic sum of their moments to be zero:

$$\sum m_k = 0. \qquad (26)$$

Problem 16. A couple of moment m_1 acts on gear *1* of radius r_1 in Fig. 49*a*. Determine the moment m_2 of the couple which should be applied to gear *2* of radius r_2 in order to keep the system in equilibrium.

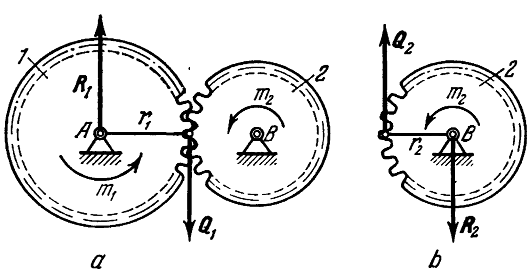


Fig. 49

Solution. Consider first the conditions for the equilibrium of gear *1*. Acting on it is the couple of moment m_1 which can be balanced only by the action of another couple, in this case the couple $(\boldsymbol{Q}_1, \boldsymbol{R}_1)$, where $\boldsymbol{Q}_1$ is the component of the force exerted on the tooth by gear *2* perpendicular to the radius and $\boldsymbol{R}_1$ is the component of the reaction of the axle A, also perpendicular to the radius (the force acting on the tooth and the reaction of axle A have another component along the radius which mutually balance and do not enter the equilibrium conditions). From Eq. (26) we have $m_1 + (-Q_1 r_1) = 0$, or $Q_1 = m_1/r_1$.

Consider now the conditions for the equilibrium of gear *2*. By the 4th principle we know that gear *1* acts on gear *2* with a force $\boldsymbol{Q}_2 = -\boldsymbol{Q}_1$ (Fig. 49*b*), which together with the component of the reaction of axle B perpendicular to the radius makes a couple $(\boldsymbol{Q}_2, \boldsymbol{R}_2)$ of moment $-Q_2 r_2$. This couple must be balanced by a couple of moment m_2 acting on gear *2*; from Eq. (26) we have $m_2 + (-Q_2 r_2) = 0$. Hence, as $Q_2 = Q_1$,

$$m_2 = \frac{r_2}{r_1} m_1.$$

It will be noticed that the couples of moments m_1 and m_2 do not satisfy the equilibrium condition (26), which could be expected, as the couples are applied to *different* bodies.

The force $\boldsymbol{Q}_1$ (or $\boldsymbol{Q}_2$) obtained in the course of the solution is called the *circumferential force* acting on the gear. The circumferential force is thus equal to the moment of the acting couple divided by the radius of the gear:

$$Q_1 = \frac{m_1}{r_1} = \frac{m_2}{r_2}.$$

Chapter 4
General Case of Forces in a Plane

§ 21. Theorem of Translation of a Force

The resultant of a concurrent force system can be found directly with the help of the parallelogram rule. We solved the problem for the case of two parallel forces by replacing them with an equivalent system of concurrent forces (see Fig. 39). Obviously, the problem can be solved for any force system if we find a method of reducing it to a system of concurrent forces. Such a method is given by the following theorem: *A force acting on a rigid body can be moved parallel to its line of action to any point of the body, if we add a couple of a moment equal to the moment of the force about the point to which it is translated.*

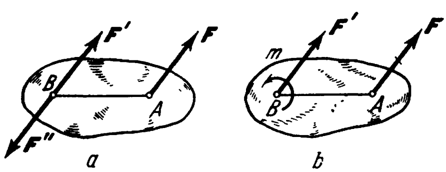


Fig. 50

Consider a force $\boldsymbol{F}$ applied to a rigid body at a point A (Fig.50a). The action of the force will not change if two balanced forces $\boldsymbol{F}' = \boldsymbol{F}$ and $\boldsymbol{F}'' = -\boldsymbol{F}$ are applied at any point B of the body. The resulting three-force system consists of a force $\boldsymbol{F}'$, equal to $\boldsymbol{F}$ and applied at B, and a couple $(\boldsymbol{F}, \boldsymbol{F}'')$ of moment

$$m = m_B(\boldsymbol{F}). \tag{27}$$

This equation follows from Eq. (23). The theorem is thus proved. The result can also be denoted as in Fig. 50b, with force $\boldsymbol{F}$ neglected. Here now are two examples on the application of this theorem.

Example 1. In order to maintain a homogeneous bar AB of weight P and length $2a$ in Fig. 51a in equilibrium, it is obviously necessary to apply to its middle C a force Q directed vertically up and equal to P in magnitude. According to the theorem just proved, force Q can be replaced by a force Q' applied at the end A of the bar and a couple of moment $m = Qa$. If the moment arm of this

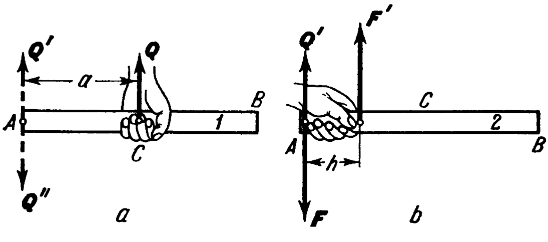


Fig. 51

couple is reduced to h (Fig. 51b), the component forces have to be increased so that $Fh = Qa$. Consequently, to hold the bar at the end A we have to add a couple (F, F') to force Q'. This result, which follows from the theorem, is immediately "felt" when you shift your grip from the middle of the bar (Fig. 51a) to the end (Fig. 51b).

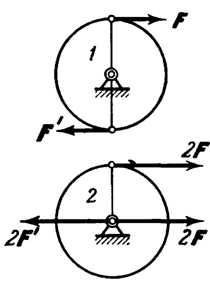


Fig. 52

Example 2. Forces F and $F' = -F$ are applied to the ends of two threads wrapped in opposite directions around a drum *1* of radius r (Fig. 52); a force $2F$ is applied to the end of a single thread wrapped on drum *2* of same radius r. Let us analyse the difference between the actions of these forces.

Acting on drum *1* is only the couple (F, F') of moment $2Fr$ which revolves the drum. The force acting on drum *2* can be replaced by a force-couple system consisting of a force $2F'' = 2F$ applied at the axle of the drum and a couple ($2F$, $2F'$). Thus we find that acting on the drum are: (1) a couple of the same moment as the couple acting on drum *1*, namely $2Fr$, which rotates the drum; and (2) a force $2F''$ exerting pressure on the drum axle. In other words, both drums revolve similarly, but the axle of drum *2* carries a load $2F$, which drum *1* does not.

§ 22. Reduction of a Coplanar Force System to a Given Centre

Let a set of coplanar forces F_1, F_2, . . ., F_n be acting on a rigid body and let O be any point coplanar with them which we shall call the *centre of reduction*. By the theorem proved in § 21, we can transfer all the forces to O as in Fig. 53a. As a result we have acting on the body a system of forces

$$\boldsymbol{F}'_1=\boldsymbol{F}_1,\quad \boldsymbol{F}'_2=\boldsymbol{F}_2,\ \ldots,\ \boldsymbol{F}'_n=\boldsymbol{F}_n \tag{28}$$

applied at O and a system of couples of moments (by Eq. 27):

$$m_1 = m_O(\boldsymbol{F}_1),\quad m_2 = m_O(\boldsymbol{F}_2),\ \ldots,\ m_n = m_O(\boldsymbol{F}_n). \tag{28'}$$

The forces applied at O can be replaced with their resultant $\boldsymbol{R} = \sum \boldsymbol{F}'_k$ acting at the same point, or, by Eqs. (28),

$$\boldsymbol{R} = \sum \boldsymbol{F}_k. \tag{29}$$

Similarly, by the theorem of composition of couples, we can replace all the couples with a coplanar resultant with a moment $M_O = \sum m_k$; or, by Eqs. (28'),

$$M_O = \sum m_O(\boldsymbol{F}_k). \tag{30}$$

The quantity $\boldsymbol{R}$, which is the geometrical sum of all the forces of the given system, it will be recalled, is called the *principal vector*

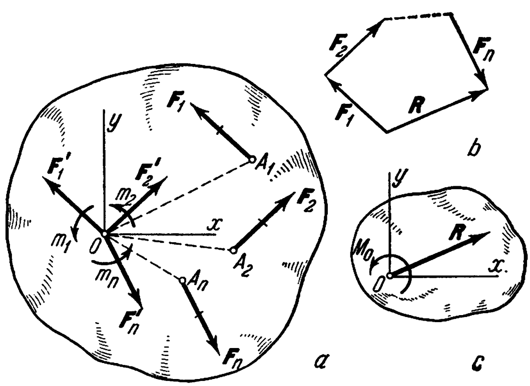


Fig. 53

of the system; we shall call the quantity M_O, which is the sum of the moments of all the forces of the system about O, the *principal moment of the system about O*. Thus we have proved the following theorem: *Any system of coplanar forces acting on a rigid body can be reduced to an arbitrary centre O in such a way that it is replaced by a single force $\boldsymbol{R}$ equal to the principal vector of the system and applied at the centre of reduction O and a single couple of moment M_O equal to the principal moment of the system about O* (Fig. 53c).

It should be noted that $\boldsymbol{R}$ is not the resultant of the force system, as it replaces the system only together with a couple.

It is apparent from the theorem that two force systems with equal principal vectors and principal moments are statically equivalent. Hence, *in order to define a coplanar force system it is sufficient to define its principal vector* $\boldsymbol{R}$ *and principal moment* M_O *about a centre O.*

The magnitude of $\boldsymbol{R}$ can be determined either graphically by constructing a force polygon (see Fig. 53*b*) or analytically by Eqs. (10) in § 10; obviously, the magnitude of $\boldsymbol{R}$ does not depend on the location of O. The magnitude of M_O is determined by Eq. (30). In the general case the value of M_O may change if the position of O changes due to the change in the moments of the component forces. Therefore, in defining the principal moment it is necessary to state the point with reference to which it is taken.

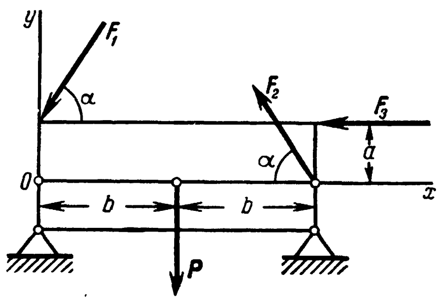


Fig. 54

Problem 17. Reduce the force system $\boldsymbol{P}$, $\boldsymbol{F}_1$, $\boldsymbol{F}_2$, $\boldsymbol{F}_3$ in Fig. 54 to the centre O if $P = 30$ kgf, $F_1 = F_2 = F_3 = F = 20$ kgf, $a = 0.3$ m, $b = 0.5$ m, and $\alpha = 60°$.

Solution. The task is to find the principal vector $\boldsymbol{R}$ of the given force system, which we shall determine from its projections R_x, R_y, and their principal moment M_O about O. Drawing the axes Oxy as shown, calculate the projections of each force on them and their moments about O (see table).

$\boldsymbol{F}_k$	$\boldsymbol{P}$	$\boldsymbol{F}_1$	$\boldsymbol{F}_2$	$\boldsymbol{F}_3$
F_{kx}	0	$-F\cos\alpha$	$-F\cos\alpha$	$-F$
F_{ky}	$-P$	$-F\sin\alpha$	$F\sin\alpha$	0
$m_O(\boldsymbol{F}_k)$	$-bP$	$aF\cos\alpha$	$2bF\sin\alpha$	aF

Substituting the problem conditions, we obtain:

$$R_x \equiv \sum F_{kx} = -40 \text{ kgf}, \quad R_y \equiv \sum F_{ky} = -30 \text{ kgf},$$

$$M_O \equiv \sum m_O(F_k) = 11.3 \text{ kgf-m}.$$

Thus, in reducing the given force system to the centre O it is substituted by force $\boldsymbol{R}$ with projections $R_x = -40$ kgf, $R_y = -30$ kgf ($R = 50$ kgf), applied at point O and by a couple with a moment $M_O = 11.3$ kgf-m.

§ 23. Reduction of a Coplanar Force System to the Simplest Possible Form

The theorem proved in § 22 makes it possible to reduce a given coplanar force system to the simplest possible form. The result will depend on the values of the principal vector $\boldsymbol{R}$ and the principal moment M_O of the system:

(1) If $\boldsymbol{R} = 0$ and $M_O = 0$, the system is in equilibrium. This case of equilibrium will be further examined in the following section.

(2) If $\boldsymbol{R} = 0$ and $M_O \neq 0$, the system can be reduced to a couple of moment $M_O = \sum m_O(\boldsymbol{F}_k)$. In this case the magnitude of M_O does not depend on the location of the centre O, otherwise we would find that the same system could be replaced by non-equivalent couples, which is impossible.

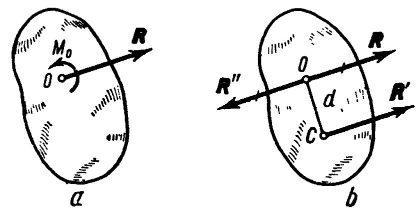


Fig. 55

(3) If $\boldsymbol{R} \neq 0$, the system can be reduced to a resultant force. Two cases are possible:

(a) $\boldsymbol{R} \neq 0$, $M_O = 0$. In this case the system can immediately be replaced by a single force, i.e., the resultant $\boldsymbol{R}$ going through O;

(b) $\boldsymbol{R} \neq 0$, $M_O = 0$ (Fig. 55*a*). In this case the couple of moment M_O can be represented by two forces $\boldsymbol{R}'$ and $\boldsymbol{R}''$, such that $R' = R$ and $R'' = -R$ (Fig. 55*b*). If $d = OC$, the arm of the couple must be

$$Rd = |M_O|. \tag{31}$$

Discarding the mutually balanced forces $\boldsymbol{R}$ and $\boldsymbol{R}''$, we find that the whole force system can be replaced by the resultant $\boldsymbol{R}' = \boldsymbol{R}$ passing through point C. The position of C is determined by two conditions: (1) The distance $OC = d$ ($OC \perp \boldsymbol{R}$) must satisfy equation (31); (2) The sense of the moment of force $\boldsymbol{R}'$ about O, i.e., the sense of $m_O(\boldsymbol{R}')$, is the same as the sense of M_O. An example of the calculations involved is presented in Problem 20.

These cases show that if a coplanar force system is not in equilibrium, it can be reduced either to a resultant (when $\boldsymbol{R} \neq 0$) or to a couple (when $\boldsymbol{R} = 0$).

Problem 18. Reduce to the simplest possible form the system of forces $\boldsymbol{P}_1$, $\boldsymbol{P}_2$, $\boldsymbol{P}_3$ acting on the beam AB in Fig. 56, and determine the loads on the supports A and B if $P_1 = P_2 = P_3 = P$.

Solution. The force polygon $\boldsymbol{P}_1$, $\boldsymbol{P}_2$, $\boldsymbol{P}_3$ is closed, consequently $\boldsymbol{R} = 0$. The sum of the moments of all the forces about any point (for instance, point C) is $-Pa$. Hence, the given force system can be reduced to a couple of moment $m = -Pa$. Placing this couple as shown in the diagram by the dashed vectors, we conclude that the forces $\boldsymbol{P}_1$, $\boldsymbol{P}_2$, and $\boldsymbol{P}_3$ act on the supports with forces $\boldsymbol{Q}_1$ and $\boldsymbol{Q}_2$ of magnitude $\frac{Pa}{b}$.

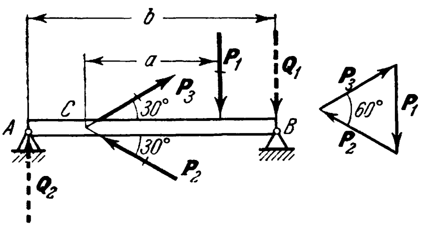


Fig. 56

Problem 19. Reduce to the simplest possible form the system of forces $\boldsymbol{F}_1$, $\boldsymbol{F}_2$, $\boldsymbol{F}_3$, acting on the truss AB in Fig. 57 and determine the loads on the supports A and B if $F_1 = F_2 = F_3 = F$.

Solution. Noting that forces $\boldsymbol{F}_2$ and $\boldsymbol{F}_3$ form a couple, transfer them as shown in the diagram by the dashed vectors. Forces $\boldsymbol{F}_1$ and $\boldsymbol{F}'_3$ balance, and the whole system is reduced to a resultant $\boldsymbol{R} = \boldsymbol{F}'_2$.

We see that the action of forces $\boldsymbol{F}_1$, $\boldsymbol{F}_2$, $\boldsymbol{F}_3$ is reduced to a vertical load acting on support A. Support B is under no load.

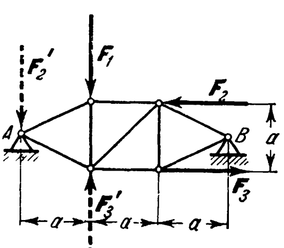


Fig. 57

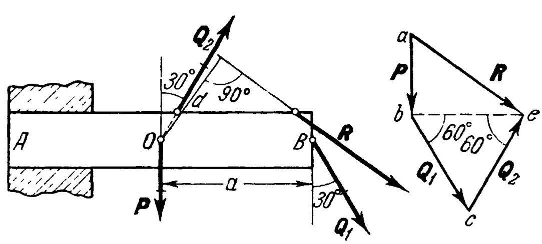


Fig. 58

Problem 20. Determine the resultant of the forces acting on the beam in Fig. 58 if $P = 3$ tonf, $Q_1 = Q_2 = Q = 4$ tonf, and $OB = a = 0.8$ m.

Solution. Constructing a force polygon with forces $\boldsymbol{P}$, $\boldsymbol{Q}_1$, and $\boldsymbol{Q}_2$, we find that the magnitude of force $\boldsymbol{R}$ (the principal vector of the system) is 5 tonf, for, in the polygon, $be = 2Q \cos 60° = 4$ tonf, and $ab = 3$ tonf. Now, taking point O, where forces $\boldsymbol{P}$ and $\boldsymbol{Q}_2$ intersect, as the centre of the moments, compute the principal moment: $M_O = m_O(\boldsymbol{Q}_1) = -aQ \cos 30° = -1.6\sqrt{3}$

tonf-m, and from Eq. (31) we obtain:

$$d=\frac{|M_O|}{R}=0\ 32\sqrt{3}\approx 0.55 \text{ m}.$$

By drawing from O a perpendicular to the direction of $\boldsymbol{R}$ and laying off a segment d on it, we obtain the line of action of the resultant. As $M_O<0$, the resultant lies to the right of O (the moment of $\boldsymbol{R}$ about O is negative).

§ 24. Conditions for the Equilibrium of a Coplanar Force System. The Case of Parallel Forces

For any given coplanar force system to be in equilibrium it is necessary and sufficient for the following two conditions to be satisfied simultaneously:

$$\boldsymbol{R}=0, \quad M_O=0, \tag{32}$$

where O is any point in a given plane, as at $\boldsymbol{R}=0$ the magnitude of M_O does not depend on the location of O [see § 23, item (2)].

The conditions (32) are necessary, for if one of them is not satisfied the force system acting on a body is reduced either to a resultant (when $\boldsymbol{R}\neq 0$) or to a couple (when $M_O\neq 0$) and consequently is not balanced. At the same time, conditions (32) are sufficient, for at $\boldsymbol{R}=0$ the system can be reduced only to a couple of moment M_O, but $M_O=0$, hence the system is in equilibrium.

Let us determine from Eqs. (32) the analytical conditions of equilibrium. They can be expressed in three different forms, as follows.

1. **The Basic Equations of Equilibrium.** The magnitudes of $\boldsymbol{R}$ and M_O are determined by the equations

$$R=\sqrt{R_x^2+R_y^2}, \quad M_O=\sum m_O(\boldsymbol{F}_k)$$

where $R_x=\sum F_{kx}$ and $R_y=\sum F_{ky}$. But R can be zero only if both $R_x=0$ and $R_y=0$. Hence, Eqs. (32) will be satisfied if

$$\sum F_{kx}=0, \quad \sum F_{ky}=0, \quad \sum m_O(\boldsymbol{F}_k)=0. \tag{33}$$

Eqs. (33) express the following analytical conditions of equilibrium: *The necessary and sufficient conditions for the equilibrium of any coplanar force system are that the sums of the projections of all the forces on each of the two coordinate axes and the sum of the moments of all the forces about any point in the plane must separately vanish.* Eqs. (33) also express the necessary conditions for the equilibrium of a free rigid body subjected to the action of a coplanar set of forces. In the mechanical sense the first two equations express the necessary conditions for a body to have no translation parallel to the coordinate axes, and the third equation expresses the condition for it to have no rotation in the plane Oxy.

2*. The Second Form of the Equations of Equilibrium: *The necessary and sufficient conditions for the equilibrium of any coplanar force system are that the sums of the moments of all the forces about any two points A and B and the sum of the projections of all the forces on any axis Ox not perpendicular to AB must separately vanish*:

$$\sum m_A(\boldsymbol{F}_k)=0, \quad \sum m_B(\boldsymbol{F}_k)=0, \quad \sum F_{kx}=0. \tag{34}$$

The necessity of these conditions is obvious, for if any one of them is not satisfied, then either $\boldsymbol{R}\neq 0$ or $M_A\neq 0$ ($M_B\neq 0$) and the

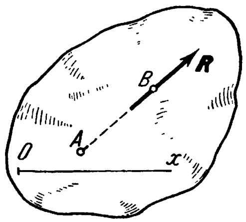


Fig. 59

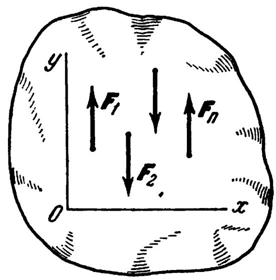


Fig. 60

forces will not be in equilibrium. Let us now prove that these conditions are sufficient. If for a given force system only the first two of Eqs. (34) are satisfied, then $M_A = 0$ and $M_B = 0$. By § 23, such a force system may not be in equilibrium as it may have a resultant $\boldsymbol{R}$ passing through the points A and B*) (Fig. 59). But from the third equation we must have $R_x = \sum F_{kx} = 0$. As Ox is not perpendicular to AB, the latter condition can be satisfied only if the resultant $\boldsymbol{R}$ is zero, i.e., if the system is in equilibrium.

3*. The Third Form of the Equations of Equilibrium (the Equations of Three Moments): *The necessary and sufficient conditions for the equilibrium of any coplanar force system are that the sums of the moments of all the forces about any three non-collinear points A, B, C must separately vanish*:

$$\sum m_A(\boldsymbol{F}_k)=0, \quad \sum m_B(\boldsymbol{F}_k)=0, \quad \sum m_C(\boldsymbol{F}_k)=0. \tag{35}$$

The necessity of these conditions, as in the previous form, is obvious. Their sufficiency follows from the consideration that if, with all the three equations satisfied, the system would not be in equilibrium, it could be reduced to a single resultant passing through

*) This follows from item (3), case (a), in § 23.

points A, B, and C, which is impossible as they are not collinear. Hence, if Eqs. (35) are satisfied, the system is in equilibrium.

In all the cases considered we have three conditions for the equilibrium of a coplanar force system. The Eqs. (33) are defined as the basic equations of equilibrium because they impose no restrictions on the choice of the coordinate axes or the centres of moments.

If acting on a body besides a coplanar force system $\boldsymbol{F}_1, \boldsymbol{F}_2, \ldots, \boldsymbol{F}_n$ is a system of couples of moments $m_1, m_2, \ldots, m_s$ in the same plane, the couples do not enter the equilibrium equations of the force components, as the sum of the components of the forces of a couple parallel to any axis is obviously zero. In the equations of moments, though, the moments of these couples should be added algebraically to the moments of the forces, as the sum of the moments of the two forces of a couple about any centre equals the moment of that couple [see Eq. (24), § 18].Thus, for example, the conditions of equilibrium (33) for a set of forces and couples acting on a body take the form:

$$\sum F_{kx}=0, \quad \sum F_{ky}=0, \quad \sum m_O(\boldsymbol{F}_k)+\sum m_i=0. \tag{36}$$

Eqs. (34) and (35) will change similarly.

Equilibrium of a Coplanar System of Parallel Forces. If all the forces acting on a body are parallel, we can take axis x of a coordinate system perpendicular to them and axis y parallel to them (Fig. 60). Then the x projections of all the forces will be zero, and the first one of Eqs. (33) becomes an identity $0 = 0$. Hence, for parallel forces we have two equations of equilibrium:

$$\sum F_{ky}=0, \quad \sum m_O(\boldsymbol{F}_k)=0, \tag{37}$$

where the y axis is parallel to the forces.

Another form of the conditions for the equilibrium of parallel forces derived from Eqs. (34) is:

$$\sum m_A(\boldsymbol{F}_k) = 0, \quad \sum m_B(\boldsymbol{F}_k) = 0. \tag{38}$$

The points A and B should not lie on a straight line parallel to the given forces.

§ 25. Solution of Problems

All the general rules of problem solution outlined in § 13 should be followed in solving the problems of this part of the course.

Attention is again drawn to the importance, in proceeding with the solution of a problem, of clearly visualising the specific body whose equilibrium is being considered. The next step is to isolate it and consider it as a free body, drawing all the given forces and the reactions of the rejected constraints.

Then write the equilibrium equations, choosing those which lead to the simplest system (in which each equation has only one unknown quantity).

The simplest possible equations can be obtained by the following procedure, provided, of course, that the actual computations do not become more involved: (a) *for the equations of the force projections, take one of the coordinate axes perpendicular to one of the unknown forces*; (b) *for the moment equations, take the moments with respect to the point where the greatest number of unknown forces intersect.*

In computing moments it may prove useful to resolve a force into two components and apply Varignon's theorem to find the moment of a force as the sum of the moments of its components.

Many problems in statics are solved to determine the reactions of the supports and connections of beams, trusses, etc. In addition to the constraints described in § 4, the following three types are widely used in engineering:

(1) Pin and Roller Support (Fig. 61, support A). The reaction N_A of such a support is normal to the surface on which the rollers rest on.

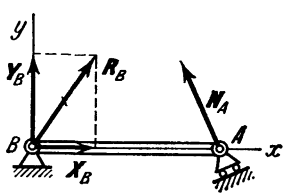


Fig. 61

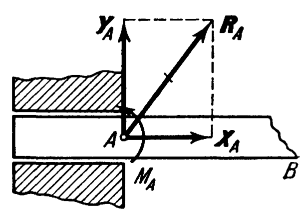


Fig. 62

(2) Fixed Pin (Fig. 61, support B). The reaction $\boldsymbol{R}_B$ of such a support is through the pin axis and can have any direction in the plane of the diagram. In solving problems we shall denote reaction $\boldsymbol{R}_B$ by its X_B and Y_B components in the direction of the axes of coordinates. If having solved the problem we find X_B and Y_B, we shall define the reaction $\boldsymbol{R}_B$ whose absolute value is

$$R_B=\sqrt{X_B^2+Y_B^2}.$$

Constraints of the type in Fig. 61 are employed to avoid additional stresses in beam AB due to temperature changes and bending.

Note that if support A of the beam in Fig. 61 were also fixed, the beam would be statically indeterminate for any coplanar force system, as three equilibrium equations (33) would include four unknown reactions X_A, Y_A, X_B, Y_B (see § 12).

(3) **Fixed Support (Rigid Clamp or Embedding)** (Fig. 62). In this case the action of the constraining surfaces on the embedded portion of the beam is that of a system of distributed forces of reaction. By reducing the forces of reaction to a common centre A, we can replace them with an immediately unknown force $\boldsymbol{R}_A$ attached at A and a couple M_A of immediately unknown moment. Force $\boldsymbol{R}_A$ can in turn be denoted by its rectangular components $\boldsymbol{X}_A$ and $\boldsymbol{Y}_A$. Thus, to determine the reactions of a fixed support we must find three unknown quantities X_A, Y_A, and M_A. If another support is added at point B, the beam will be statically indeterminate.

The direction of reactions in other types of constraints was examined in § 4.

Problem 21. The travelling crane in Fig. 63 weighs $P = 4$ tonf, its centre of gravity lies on DE, it lifts a load of weight $Q = 1$ tonf, the length of the jib (the distance of the load from DE) is $b = 3.5$ m, and the distance between the wheels is $AB = 2a = 2.5$ m. Determine the forces with which the wheels A and B act on the rails.

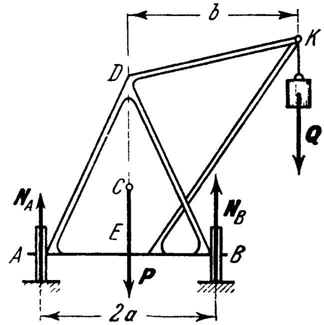


Fig. 63

Solution. Consider the equilibrium of the crane-and-load system taken as a free body: the active forces are $\boldsymbol{P}$ and $\boldsymbol{Q}$, the unknown forces are the reactions $\boldsymbol{N}_A$ and $\boldsymbol{N}_B$ of the removed constraints. Taking A as the centre of the moments of all the forces and projecting the parallel forces on a vertical axis, we obtain, by the equilibrium equations (37):

$$-Pa + N_B \cdot 2a - Q(a + b) = 0,$$
$$N_A + N_b - P - Q = 0,$$

solving which we find:

$$N_A = \frac{P}{2} - \frac{Q}{2}\left(\frac{b}{a} - 1\right) = 1.1 \text{ tonf},$$
$$N_B = \frac{P}{2} + \frac{Q}{2}\left(\frac{b}{a} + 1\right) = 3.9 \text{ tonf}.$$

To verify the solution, write the equation of the moments about B:

$$-N_A 2a + Pa - Q(b - a) = 0.$$

Substituting the value of N_A, we find that the equation is satisfied. The pressures exerted by the wheels on the rails are respectively equal to N_A and N_B in magnitude and directed vertically down.

From the solution we see that at

$$Q = \frac{a}{b-a} P = 2.22 \text{ tonf}$$

the reaction N_A is zero and the left wheel no longer presses on the rail. If the load Q is further increased the crane will topple over. The maximum load Q at which equilibrium is maintained is determined by the equation $\sum m_B(\boldsymbol{F}_k) = 0$.

Problem 22. One end of a uniform beam AB weighing P kgf (Fig. 64) leans at A against a corner formed by a smooth horizontal surface and a block D, and at B on a smooth plane inclined α degrees

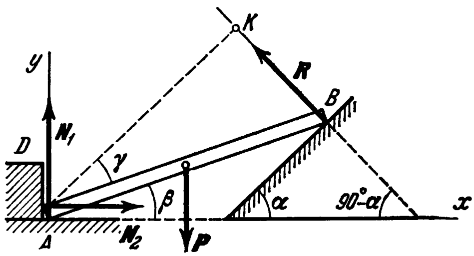


Fig. 64

to the horizontal. The beam's inclination to the horizontal is equal to β. Determine the pressure of the beam on its three constraints.

Solution. Consider the equilibrium of the beam as a free body. Acting on it are the given force $\boldsymbol{P}$ applied at the middle of the beam and the reactions $\boldsymbol{R}$, $\boldsymbol{N}_1$, and $\boldsymbol{N}_2$ of the constraints directed normal to the respective surfaces. Draw the coordinate axes as in Fig. 64 and write the equilibrium equations (33), taking the moments about A, where two of the unknown forces intersect. First compute the projections of all the forces on the coordinate axes and their moments about A and tabulate the results*). The symbols in the table are $AB = 2a$ and $\angle KAB = \gamma$ (AK is the moment arm of force $\boldsymbol{R}$ about point A).

$\boldsymbol{F}_k$	$\boldsymbol{N}_1$	$\boldsymbol{N}_2$	$\boldsymbol{P}$	$\boldsymbol{R}$
F_{kx}	0	N_2	0	$-R\sin\alpha$
F_{ky}	N_1	0	$-P$	$R\cos\alpha$
$m_A(\boldsymbol{F}_k)$	0	0	$-Pa\cos\beta$	$R\,2a\cos\gamma$

*) On tabulating data for problem solutions see footnote on p. 50.

Now write the equilibrium equations:

$$N_2 - R\sin\alpha = 0,$$
$$N_1 - P + R\cos\alpha = 0,$$
$$-Pa\cos\beta + 2Ra\cos\gamma = 0.$$

From the last equation we find:

$$R = \frac{P\cos\beta}{2\cos\gamma}.$$

As AK is parallel to the inclined plane, $\angle KAx = \alpha$, whence $\gamma = \alpha - \beta$ and finally

$$R = \frac{P\cos\beta}{2\cos(\alpha-\beta)}.$$

Solving the first two equations, we obtain:

$$N_1 = P\left[1 - \frac{\cos\alpha\cos\beta}{2\cos(\alpha-\beta)}\right], \quad N_2 = P\frac{\sin\alpha\cos\beta}{2\cos(\alpha-\beta)}.$$

The forces exerted on the surfaces are equal in magnitude to the respective reactions and opposite in sense.

The values of N_1 and N_2 can be verified by solving the equations of moments about the points of intersection of $\boldsymbol{R}$ and $\boldsymbol{N}_2$ and $\boldsymbol{R}$ and $\boldsymbol{N}_1$.

From this solution we can draw the following conclusion: If, in order to determine the projections or moments of any force or forces, we need to know a quantity (e.g., the length of a line or size of an angle) not given in the statement of the problem, we should denote that quantity by a symbol and include it in the equilibrium equations. If the introduced quantity is not cancelled out in the course of the computations, it should be expressed in terms of the given quantities.

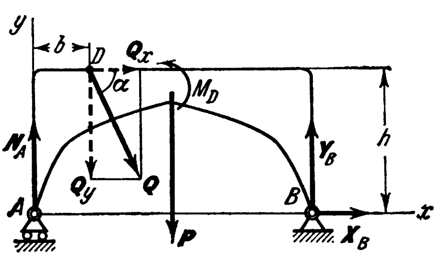


Fig. 65

Problem 23. Acting on a symmetrical arch of weight $P = 8$ tonf (Fig. 65) is a set of forces reduced to a force $Q = 4$ tonf applied at D and a couple with a moment $M_D = 12$ tonf-m. The dimensions of the arch are $a = 10$ m, $b = 2$ m, $h = 3$ m, and $\alpha = 60°$. Determine the reactions of the pin B and the roller A.

Solution. Consider the equilibrium of the arch as a free body. Acting on it are the given forces $\boldsymbol{N}$ and $\boldsymbol{Q}$ and a couple of moment M_D, and also the reactions of the supports $\boldsymbol{N}_A$, $\boldsymbol{X}_B$, $\boldsymbol{Y}_B$ (the reaction of the pin being denoted by its rectangular components as in Fig. 61). In this problem it is more convenient to use Eqs. (34), taking the moments about A and B and the force projections on axis Ax, and each equation will contain one unknown force. Compute the moments and the force projections and tabulate the information as shown. In computing the moments of force Q, resolve it into rectangular components Q_x and Q_y and apply Varignon's theorem.

F_k	N_A	P	X_B	Y_B	Q	M_D
F_{kx}	0	0	X_B	0	$Q\cos\alpha$	0
$m_A(F_k)$	0	$-P\frac{a}{2}$	0	Y_Ba	$-\lvert Q_x\rvert h-\lvert Q_y\rvert b$	M_D
$m_B(F_k)$	$-N_Aa$	$P\frac{a}{2}$	0	0	$-\lvert Q_x\rvert h+\lvert Q_y\rvert (a-b)$	M_D

Writing the equilibrium equations and taking into account that $\lvert Q_x \rvert = Q\cos\alpha$, and $\lvert Q_y \rvert = Q\sin\alpha$, we obtain:

$$X_B+Q\cos\alpha=0, \qquad \text{(a)}$$

$$Y_Ba-P\frac{a}{2}-hQ\cos\alpha-bQ\sin\alpha+M_D=0, \qquad \text{(b)}$$

$$-N_Aa+P\frac{a}{2}-hQ\cos\alpha+(a-b)Q\sin\alpha+M_D=0. \qquad \text{(c)}$$

Solving the equations, we find

$$X_B = -Q\cos\alpha = -2 \text{ tonf},$$

$$Y_B=\frac{P}{2}+Q\frac{b\sin\alpha+h\cos\alpha}{a}-\frac{M_D}{a}\approx 4.09 \text{ tonf},$$

$$N_A=\frac{P}{2}+Q\frac{(a-b)\sin\alpha-h\cos\alpha}{a}+\frac{M_D}{a}\approx 7.37 \text{ tonf}.$$

The value of X_B is negative, which means that the sense of the x component of the reaction at B is opposite to that shown in the diagram, which could have been foreseen. The total reaction at B can be found from the geometrical sum of the rectangular components X_B and Y_B, its magnitude being

$$R_B=\sqrt{X_B^2+Y_B^2}\approx 4.55 \text{ tonf}.$$

If the sense of the couple acting on the arch were opposite to that indicated in Fig. 65, we would have $M_D = -12$ tonf-m. In this case $Y_B = 6.49$ tonf, $N_A = 4.97$ tonf, while X_B would remain the same.

To check the solution write the equation for the projections on axis Ay:

$$N_A + Y_B - P - Q \sin \alpha = 0. \qquad \text{(d)}$$

Substituting the obtained values of N_A and Y_B, we find that they satisfy the equation (substitution should be carried out in both the general form, to verify the equations, and in the numerical solution to verify the computations).

It should be borne in mind that this method of verification may not reveal errors due to mistakes in computing the projections or the moments of the forces perpendicular to Ay. Therefore, that portion of the computations has to be further verified or an additional equation of moments about, say, D can be written.

Note also the following. In writing Eqs. (34) the projection axis should not be taken perpendicular to AB, or in our case not along Ay. If, nevertheless, we were to write a third equation of projections on Ay, we would have obtained a system of equations (b), (c), (d) with only two unknown quantities N_A and Y_B (one of the equations would be a corollary of the other two) and we would be unable to determine the reaction X_B.

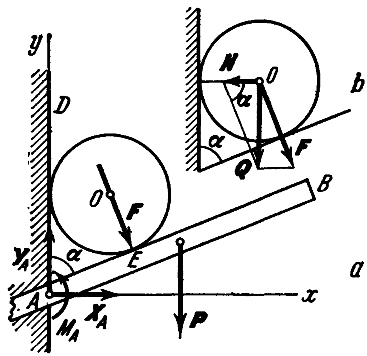


Fig. 66

Problem 24. The homogeneous beam in Fig. 66*a* is embedded in a wall at an angle $\alpha = 60°$ to it. The length of the portion AB is $b = 0.8$ m and its weight is $P = 100$ kgf. Lying inside the angle DAB is a cylinder of weight $Q = 180$ kgf, touching the beam at E, and $AE = a = 0.3$ m. Determine the reaction of the wall.

Solution. Consider the equilibrium of the beam as a free body. Acting on it are force $\boldsymbol{P}$ applied halfway between A and B, force $\boldsymbol{F}$ applied perpendicularly to the beam at E (but not force $\boldsymbol{Q}$, which is applied to the cylinder, not to the beam!), and the reactions of the embedding, indicated by the rectangular components $\boldsymbol{X}_A$ and $\boldsymbol{Y}_A$ and a couple of moment M_A (see Fig. 62).

In order to write the equations of equilibrium (33) let us compute the projections of all the forces on each of the coordinate axes and their moments about A (see table):

F_k	X_A	Y_A	F	P	M_A
F_{kx}	X_A	0	$F\cos\alpha$	0	0
F_{ky}	0	Y_A	$-F\sin\alpha$	$-P$	0
$m_A(F_k)$	0	0	$-Fa$	$-P\frac{b}{2}\sin\alpha$	M_A

To determine F we resolve force Q, which is applied at the centre of the cylinder, into components F and N respectively perpendicular to the beam and the wall (Fig. 66b). From the parallelogram we obtain

$$F=\frac{Q}{\sin\alpha}.$$

Writing the equations of equilibrium and substituting the value of F, we have:

$$X_A+Q\cot\alpha=0,\quad Y_A-Q-P=0,$$

$$M_A-Q\frac{a}{\sin\alpha}-P\frac{b}{2}\sin\alpha=0.$$

Solving these equations we find:

$$X_A = -Q\cot\alpha = -103.8 \text{ kgf},$$

$$Y_A = P+Q = 280 \text{ kgf},$$

$$M_A=Q\frac{a}{\sin\alpha}+P\frac{b}{2}\sin\alpha=96.9 \text{ kgf-m}.$$

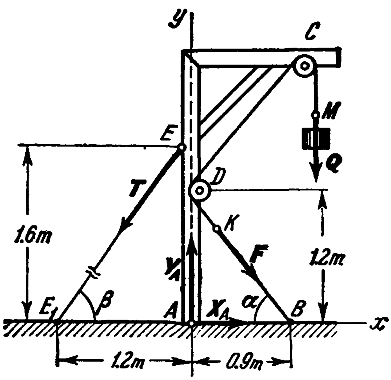


Fig. 67

The reaction of the wall consists of force $R_A=\sqrt{X_A^2+Y_A^2}$ and a couple with a moment M_A.

The solution of this problem again underlines the fundamental point: the equilibrium equations include only the forces *acting directly* on the body whose equilibrium is being considered.

Problem 25. A string supporting a weight $Q = 240$ kgf passes over two pulleys C and D as shown in Fig. 67. The other end of the string is secured at B, and the frame is kept in equilibrium by a guy-rope EE_1. Neglecting the weight of the frame and the friction in the pulleys, determine the tension in the guy-rope and the reactions at A, if the constraint at A is a smooth pivot allowing the frame to turn about its axis. The distance from the pulley C to pole is 1 m. Other dimensions are as shown in the diagram.

Solution. Neglecting the constraints, consider the whole system of the frame and the portion $KDCM$ of the string as a single free rigid body (see Problem 10). Acting on it are the following external forces: $\mathbf{Q}$ and $\mathbf{F}$, the tensions in the sections of the string at M and K, and the reactions $\mathbf{T}$, $\mathbf{X}_A$, and $\mathbf{Y}_A$ of the constraints. The internal forces cancel each other and are not shown in the diagram. As the friction of the pulleys is neglected, the tension in the cable is uniform throughout its whole length and $F = Q$.

Introducing angles α and β, let us compute the projections of all the forces on the coordinate axes and their moments about A.

$\mathbf{F}_k$	$\mathbf{Q}$	$\mathbf{F}$	$\mathbf{T}$	$\mathbf{X}_A$	$\mathbf{Y}_A$
F_{kx}	0	$F\cos\alpha$	$-T\cos\beta$	X_A	0
F_{ky}	$-Q$	$-F\sin\alpha$	$-T\sin\beta$	0	Y_A
$m_A(\mathbf{F}_k)$	$-1.0\,Q$	$-0.9\,F\sin\alpha$	$1.2T\sin\beta$	0	0

From the right-angle triangles AEE_1 and ADB we find that $EE_1 = 2.0$ m and $DB = 1.5$ m, whence $\sin\alpha = \sin\beta = 0.8$, $\cos\alpha = \cos\beta = 0.6$, and $\alpha = \beta$. Substituting for the trigonometric functions their values and assuming $F = Q$, the equations of equilibrium give:

$$0.6Q - 0.6T + X_A = 0,$$
$$-Q - 0.8Q - 0.8T + Y_A = 0,$$
$$-1.0Q - 0.72Q + 0.96T = 0,$$

solving which, we find:

$$T = \frac{43}{24}Q = 430 \text{ kgf}, \quad X_A = \frac{19}{40}Q = 114 \text{ kgf},$$
$$Y_A = \frac{97}{30}Q = 776 \text{ kgf}.$$

Attention is drawn to the following conclusions: (1) in writing equations of equilibrium, any system of bodies which remains fixed when the constraints are removed can be regarded as a rigid body; (2) the internal forces acting on the parts of a system (in this case the tension of the string DC acting on pulleys C and D) are not included in the equilibrium equations as they cancel each other.

§ 26. Equilibrium of Systems of Bodies

In many cases the static solution of engineering structures is reduced to an investigation of the conditions for the equilibrium of systems of connected bodies. We shall call the constraints connecting the parts of a given structure *internal*, as opposed to *external* constraints which connect a given structure with other bodies (e.g., the supports of a bridge).

If a structure remains rigid after the external constraints (supports) are removed, the problems of statics are solved for it as for a rigid body. Such examples were considered in Problems 23 and 25 (see Figs. 65 and 67).

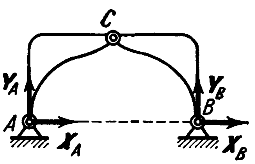


Fig. 68

However, an engineering structure may not necessarily remain rigid when the external constraints are removed. An example of such a structure is the three-pin arch in Fig. 68. If supports A and B are removed the arch is no longer rigid, for its parts can turn about pin C.

According to the principle of solidification, for a system of forces acting on such a structure to be in equilibrium it must satisfy the conditions of equilibrium for a rigid body. It was pointed out, though, that these conditions, while necessary, were not sufficient, and therefore not all the unknown quantities could be determined from them. In order to solve such a problem it is necessary to examine additionally the equilibrium of one or several parts on the given structure.

For example, for the forces acting on the three-pin arch in Fig. 68 we have three equations with four unknown quantities, X_A, Y_A, X_B, Y_B. By investigating the conditions for the equilibrium of the left- or right-hand members of the arch we obtain three more equations with two more unknown quantities, X_C and Y_C (not shown in Fig. 68). Solving the system of six equations we can determine all six unknown quantities (see Problem 26).

Another method of solving such problems is to divide a structure into separate bodies and write the equilibrium equations for each as for a free body (see Problem 27). The reactions of the internal constraints will constitute pairs of forces equal in magnitude and opposite in sense. For a structure of n bodies, each of which is subjected to the action of a coplanar force system, we thus have $3n$ equations from which we may determine $3n$ unknown quantities (in other force systems the number of equations is, of course, different). If the number of unknown quantities is greater than the number of equations, the problem is statically indeterminate.

Problem 26. A bracket consists of a horizontal member AD (Fig. 69) of weight $P_1 = 15$ kgf hinged to the wall and an inclined member CB of weight $P_2 = 12$ kgf that is also hinged to the member AD and to the wall (all dimensions are shown in the diagram). Suspended from the horizontal member at D is a load of weight $Q = 30$ kgf. Determine the reactions at A and C, considering AD and CB to be homogeneous.

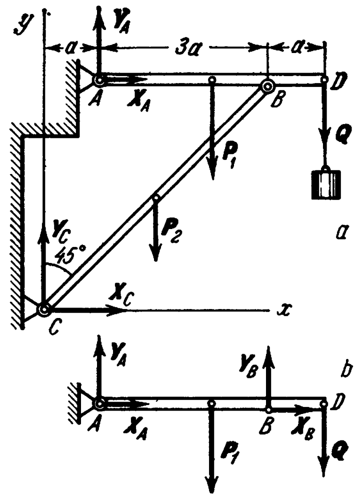


Fig. 69

Solution. Rejecting the external constraints, consider the bracket as a whole as a free body. We find that acting on it are the given forces $\boldsymbol{P}_1$, $\boldsymbol{P}_2$, $\boldsymbol{Q}$ and the reactions of the constraints $\boldsymbol{X}_A$, $\boldsymbol{Y}_A$, $\boldsymbol{X}_C$, $\boldsymbol{Y}_C$. But with its constraints removed, the bracket is no longer a rigid body, because the members can turn about pin B. On the other hand, by the principle of solidification, if it is in equilibrium the forces acting on it must satisfy the conditions of static equilibrium. We may therefore write the corresponding equations:

$$\sum F_{kx} \equiv X_A + X_C = 0,$$
$$\sum F_{ky} \equiv Y_A + Y_C - P_1 - P_2 - Q = 0,$$
$$\sum m_A(\boldsymbol{F}_k) \equiv X_C \cdot 4a - Y_C a - P_2 a - P_1 \cdot 2a - Q \cdot 4a = 0.$$

We find that the three equations contain four unknown quantities X_A, Y_A, X_C, Y_C. Let us therefore investigate additionally the equilibrium conditions of member AD (Fig. 69b). Acting on it are forces $\boldsymbol{P}_1$ and $\boldsymbol{Q}$ and the reactions $\boldsymbol{X}_A$, $\boldsymbol{Y}_A$, $\boldsymbol{X}_B$, and $\boldsymbol{Y}_B$. If we write the required fourth equation for the moments of these forces about B we shall avoid introducing two more unknown quantities, X_B and Y_B. We have:

$$\sum m_B(\boldsymbol{F}_k) \equiv -Y_A \cdot 3a + P_1 a - Qa = 0.$$

Solving the system of four equations (starting with the last one) we find:

$$Y_A = \frac{1}{3}(P_1 - Q) = -5 \text{ kgf}, \quad Y_C = \frac{2}{3} P_1 + P_2 + \frac{4}{3} Q = 62 \text{ kgf},$$
$$X_C = \frac{2}{3} P_1 + \frac{1}{2} P_2 + \frac{4}{3} Q = 56 \text{ kgf}, \quad X_A = -X_C = -56 \text{ kgf}.$$

We see that the sense of forces $\boldsymbol{Y}_A$ and $\boldsymbol{X}_A$ is opposite to that shown in the diagram. The reactions at B can be determined from equations for the x and y projections of the forces acting on AD; they are: $X_B = -X_A$, $Y_B = P_1 + Q - Y_A = 50$ kgf.

It should be noted that in solving a system of simultaneous equations the values in every successive equation should be substituted with the sign obtained in the preceding equation. In the present case, for example, the value substituted for Y_A in the last equations is —5 kgf, not 5 kgf. This also suggests that it would be wrong, on finding that $Y_A = -5$ kgf, to alter the sense of Y_A in the diagram and consider $Y_A = 5$ kgf, for this could lead to wrong solutions of subsequent equilibrium equations.

We see that in solving problems of statics there is no need to investigate *all* the conditions for the equilibrium of a given body. *If a problem does not require the determination of the reactions of a constraint, the equations should, if possible, not include those unknown reactions.* That is just what we did in the above problem in examining the equilibrium of AD when we wrote only one equation of the moments about B.

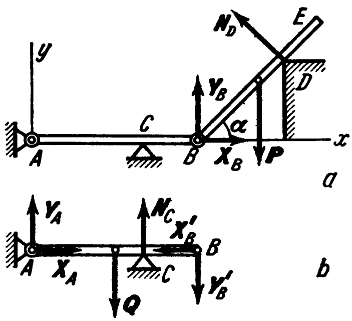


Fig. 70

Problem 27. The horizontal beam AB in Fig. 70a of weight $Q = 20$ kgf is attached to the wall by a pin at A and rests on a support at C. Beam BE of weight $P = 40$ kgf is hinged to B and rests against D as shown. Determine the reactions of the supports if $CB = \frac{1}{3} AB$, $DE = \frac{1}{3} BE$, and $\angle \alpha = 45°$ considering the beam and the block homogeneous

Solution. Let us consider the members of the system separately and investigate the equilibrium of BE and AB. Acting on BE, which we regard as a free body (Fig. 70a), are force P and the reactions of the supports N_D, X_B, Y_B. Assuming $BE = a$, we write the equations (33):

$$\sum F_{kx} \equiv X_B - N_D \sin\alpha = 0,$$
$$\sum F_{ky} \equiv Y_B - P + N_D \cos\alpha = 0,$$
$$\sum m_B(F_k) \equiv N_D \frac{2}{3} a - P \frac{a}{2} \cos\alpha = 0,$$

solving which we obtain:

$$N_D = \frac{3}{4} P \cos\alpha = 21.2 \text{ kgf},$$
$$X_B = \frac{3}{8} P \sin 2\alpha = 15 \text{ kgf},$$
$$Y_B = P\left(1 - \frac{3}{4}\cos^2\alpha\right) = 25 \text{ kgf}.$$

Considering beam AB as a free body, we have acting on it force $\boldsymbol{Q}$, the reactions of the external supports $\boldsymbol{N}_C$, $\boldsymbol{X}_A$, $\boldsymbol{Y}_A$, and the action $\boldsymbol{X}'_B$ and $\boldsymbol{Y}'_B$ of BE transmitted by pin B (Fig. 70*b*). According to the 4th principle, forces $\boldsymbol{X}'_B$ and $\boldsymbol{Y}'_B$ must be directed opposite to $\boldsymbol{X}_B$ and $\boldsymbol{Y}_B$, but in magnitude $X'_B = X_B$ and $Y'_B = Y_B$.

Denoting $AB = b$ and writing the equilibrium equations (34) for the forces acting on the beam, we have:

$$\sum F_{kx} \equiv X_A - X'_B = 0,$$

$$\sum m_A(\boldsymbol{F}_k) \equiv -Y'_B b + N_C \cdot \frac{2}{3} b - Q\frac{b}{2} = 0,$$

$$\sum m_C(\boldsymbol{F}_k) \equiv -Y_A \cdot \frac{2}{3} b + Q\frac{b}{6} - Y'_B \frac{b}{3} = 0.$$

Assuming $X'_B = X_B$ and $Y'_B = Y_B$ and solving the equations, we obtain:

$$X_A = X_B = 15 \text{ kgf}, \quad Y_A = \frac{1}{4}Q - \frac{1}{2}Y_B = -7.5 \text{ kgf},$$

$$N_C = \frac{3}{4}Q + \frac{3}{2}Y_B = 52.5 \text{ kgf}.$$

We see that all the reactions except $\boldsymbol{Y}_A$ are directed as in Fig. 70; the true direction of $\boldsymbol{Y}_A$ is vertically downwards.

In solving problems by this method it should be remembered that *if the action of one body on another is denoted by a force* $\boldsymbol{R}$ *or its rectangular components* $\boldsymbol{X}$ *and* $\boldsymbol{Y}$, *then according to the 4th principle the action of the second body on the first must be denoted by a force* $\boldsymbol{R}'$ *equal to* $\boldsymbol{R}$ *in magnitude and opposite in sense or by its rectangular components* $\boldsymbol{X}'$ *and* $\boldsymbol{Y}'$ *respectively equal to* $\boldsymbol{X}$ *and* $\boldsymbol{Y}$ *in magnitude and opposite in sense.*

Beware of the following two mistakes which are often made in solving problems by this method: (1) the student fails to draw forces $\boldsymbol{X}'$ and $\boldsymbol{Y}'$ in the opposite direction of $\boldsymbol{X}$ and $\boldsymbol{Y}$, which makes for a wrong answer; (2) having drawn the forces $\boldsymbol{X}'$ and $\boldsymbol{Y}'$ correctly, the student assumes in solving the equations that $X' = -X$ and $Y' = -Y$, obtaining a wrong answer; actually, the same error is made as in the first case.

To preclude the possibility of such mistakes, it is suggested that, as a rule, the method of solution of Problem 26 be adopted. Furthermore, it usually gives simpler systems of equations, as the internal forces do not enter the equations for the structure as a whole (see the solution of Problem 26).

Problem 28. A horizontal force $\boldsymbol{F}$ acts on the three-pin arch in Fig. 71. Show that in determining the reactions of supports A and B force $\boldsymbol{F}$ cannot be transferred along its line of action to E.

Solution. Isolating the arch from its external supports A and B, we obtain a deformable structure which cannot be treated as rigid.

Consequently, the point where the force acts on the structure cannot be transferred along DE even to determine the conditions for the equilibrium of the structure (see § 3, corollary of the 1st and 2nd principles).

Let us demonstrate this by solving the problem (the weight of the arch is neglected). Consider first the right-hand member of the arch as a free body. Acting on it are only two forces, the reactions $\boldsymbol{R}_B$ and $\boldsymbol{R}_C$ of the pins B and C (force $\boldsymbol{R}_C$ is not shown in the diagram).

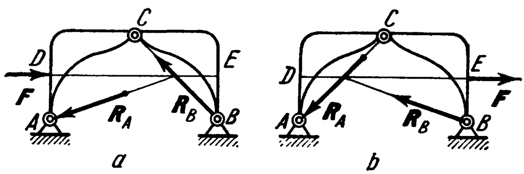


Fig. 71

To be in equilibrium, these two forces must be directed along the same line, i.e., along BC, and consequently the reaction $\boldsymbol{R}_B$ is directed along BC.

Investigating now the equilibrium of the arch as a whole, we find that acting on it are three forces, the given force $\boldsymbol{F}$ and the reactions of the supports $\boldsymbol{R}_B$ (whose direction we have established) and $\boldsymbol{R}_A$. From the theorem of three forces we know that if the system is in equilibrium the forces must be concurrent. Thus we obtain the direction of $\boldsymbol{R}_A$. The magnitudes of $\boldsymbol{R}_A$ and $\boldsymbol{R}_B$ can be found by the triangle rule.

If we apply force $\boldsymbol{F}$ at E and, reasoning in the same way, make the necessary constructions (Fig. 71*b*), we shall find that the reactions of the supports $\boldsymbol{R}_A$ and $\boldsymbol{R}_B$ are different both in magnitude and in direction.

§ 27*. Determination of Internal Forces (Stresses)

The forces with which the parts of a body or structure (beam, arch, etc.) act on each other are called the *internal forces*, or *stresses*. They can be determined by the method used in studying the equilibrium of systems of bodies. First the equilibrium of the body (structure) as a whole is considered and the reactions of the external constraints are determined. Then a section is taken in the body at the point where the internal forces have to be determined, dividing it into two parts, and the equilibrium of one of them is considered. If the system of external forces acting on the body is coplanar, the action of the discarded section can, in the most general case, be replaced by a coplanar system of forces distributed across the cut; as in the case of a rigid clamp or embedding (see Fig. 62), these forces can be represented by a single force applied at the centre of the section, with two unknown components X, Y, and a couple with an unknown moment M. An example of the calculation is presented in Problem 29.

Problem 29. Assuming the length of the beam AB in Problem 27 (Fig. 70) to be 1 m, determine the forces acting on a section through the beam at a distance $AE =$ $= 0.6$ m from point A.

Solution. The beam's external constraints are the support C and the pins at A and B. Their reactions were determined in Problem 27. Make a cut ab through the beam and consider the equilibrium of the part of the beam to the left (Fig. 72). As said before, the action of the discarded portion can be replaced by two forces X_E and Y_E applied at the centre E of the section, and by a couple with a moment M_E. Writing the equilibrium conditions (36) for the forces X_A, Y_A, Q, X_E, Y_E and the couple with a moment M_E, we obtain:

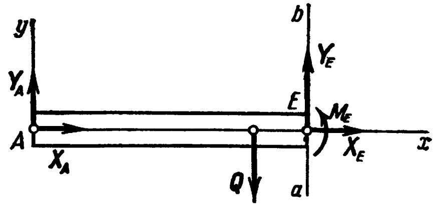


Fig. 72

$$\sum F_{kx} \equiv X_A + X_E = 0, \quad \sum F_{ky} \equiv Y_A + Y_E - Q = 0,$$

$$\sum m_A (F_k) \equiv M_E + 0.6 Y_E - 0.5 Q = 0.$$

From Problem 27, $Q = 20$ kgf, $X_A = 15$ kgf, $Y_A = -7.5$ kgf. Substituting these values, the equations yield:

$$X_E = -15 \text{ kgf}, \ Y_E = 27.5 \text{ kgf}, \ M_E = -6.5 \text{ kgf-m}.$$

Thus, acting on the left-hand part of the beam at section ab are: (1) a *longitudinal force* X_E, which in the problem exerts a compressive stress; (2) a *lateral force* Y_E, which strives to displace the portion of the beam adjoining the section along the line ab; (3) a couple with a moment M_E, called the *bending moment*, which in our case causes tensile stress in the upper part of the beam and compressive stress in the lower part.

§ 28*. Distributed Forces

In engineering problems we often have to deal with loads distributed over an area according to a known mathematical law. Let us examine some simple cases of distributed coplanar forces.

A plane system of distributed forces is characterised by the load per unit length of the line of application, which is called the *inten-*

sity q. The dimension of intensity is newtons per metre (N/m) or kilograms per metre (kgf/m).

(1) Forces Uniformly Distributed Along a Straight Line (Fig. 73*a*). The intensity q of such a system is a constant quantity. In solving problems of statics such a force system can be replaced by its resultant Q of magnitude

$$Q = aq \tag{39}$$

applied at the middle of AB.

(2) Forces Distributed Along a Straight Line According to a Linear Law (Fig. 73*b*). An example of such a load is the pressure of water against a dam, which drops from a maximum at the bottom

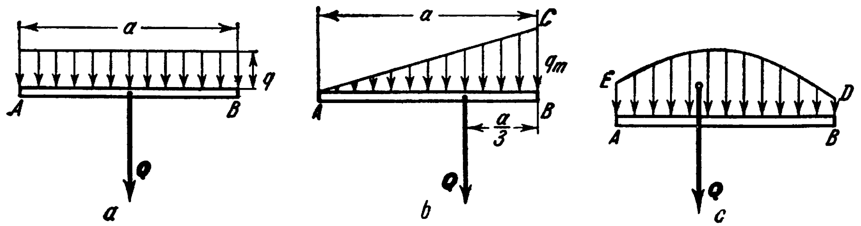


Fig. 73

to zero at the surface. For such forces the intensity q varies from zero to q_m. The resultant Q is determined in the same manner as the resultant of the gravity forces acting on a homogeneous triangular lamina ABC. As the weight of a homogeneous lamina is proportional to its area, the magnitude of Q is

$$Q = \frac{1}{2} aq_m \tag{40}$$

and is applied at a point at a distance of $\frac{a}{3}$ from side BC of triangle ABC [see § 57 item (2)].

(3) Forces Distributed Along a Straight Line According to an Arbitrary Law (Fig. 73*c*). The magnitude of the resultant Q of such forces is, by analogy with the force of gravity, equal to the area of the figure $ABDE$ drawn to scale, and Q passes through the centre of gravity of that area (determination of centre of gravity will be examined in § 55).

(4) Forces Unirformly Distributed Along the Arc of a Circle (Fig. 74). An example of such forces is the hydrostatic pressure on the sides of a cylindrical vessel. It is apparent from the laws of symmetry that the sum of the projections on the Oy-axis, which is perpendicular to the axis of symmetry Ox, is zero and, consequently, their resultant Q is directed along Ox. In magnitude $Q = Q_x =$

$= \Sigma (q\Delta l_k) \cos \varphi_k$, where $q\Delta l_k$ is the pressure on an arc element of length Δl_k and φ_k is the angle between the force and axis x. But from the diagram it is apparent that $\Delta l_k \cos \varphi_k = \Delta y_k$. Taking the common multiplier q outside of the summation sign, we obtain $Q = \Sigma q\Delta y_k = q\Sigma \Delta y_k = q \cdot AB$, whence

$$Q = qh, \tag{41}$$

where h is the length of the chord intersecting arc AB.

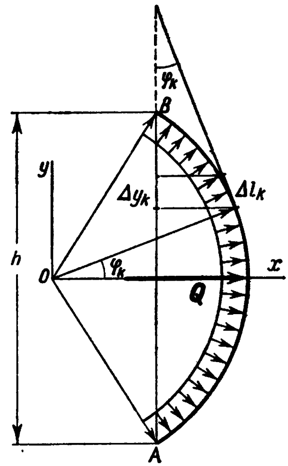


Fig. 74

Problem 30. An evenly distributed force of intensity q_0 kgf/m acts on a cantilever beam whose dimensions are shown in Fig. 75. Neglecting the weight of the beam and assuming the forces acting on the embedded portion to be distributed according to a linear law, determine the magnitude of the maximum intensities q_m and q_m' of the given forces if $b = 2a$ (compare with the diagram for Problem 14, § 17).

Solution. Replace the distributed forces by their resultants $\boldsymbol{Q}$, $\boldsymbol{R}$, and $\boldsymbol{R}'$. By Eqs. (39) and (40),

$$Q = q_0 b, \quad R = \frac{1}{2} q_m a, \quad R' = \frac{1}{2} q_m' a.$$

Now write the equilibrium conditions (37) for the parallel forces acting on the beam:

$$\sum F_{ky} \equiv Q + R - R' = 0,$$

$$\sum m_C (\boldsymbol{F}_k) \equiv R \frac{a}{3} - Q\left(\frac{b}{2} + \frac{a}{3}\right) = 0.$$

Substituting the values of Q, R, and R' and solving the equations, we obtain:

$$q_m = \left(3\frac{b^2}{a^2} + 2\frac{b}{a}\right) q_0; \quad q_m' = \left(3\frac{b^2}{a^2} + 4\frac{b}{a}\right) q_0.$$

At $b = 2a$ we have $q_m = 16q_0$ and $q_m' = 20q_0$.

Problem 31. A gas cylinder of height H with an internal diameter d contains gas compressed to p kgf/m². The walls of the cylinder are of thickness a. Determine: (1) the longitudinal and (2) the lateral stresses in the walls (stress is the ratio of the expanding force to the area of the cross section).

Solution. (1) Let us cut the cylinder with a plane perpendicular to its axis and investigate the equilibrium of one portion of the cylinder (Fig. 76*a*). Acting on it parallel to the cylinder axis are two forces: the pressure on the bottom $F = \frac{\pi d^2}{4} p$ and the resultant Q of the set of distributed forces acting on the cross section (the action of the removed portion). For equilibrium we have $Q = F = \frac{\pi d^2}{4} p$. Assuming the area of the cross section to be approximately πda, we obtain the following expansion stress σ_1:

$$\sigma_1 = \frac{Q}{\pi da} = \frac{1}{4} \frac{d}{a} p \text{ kgf/m}^2.$$

(2) Now cut the cylinder with a plane passing through the axis of the cylinder and examine the equilibrium of one half of it, assu-

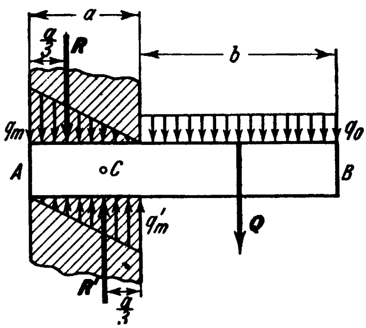


Fig. 75

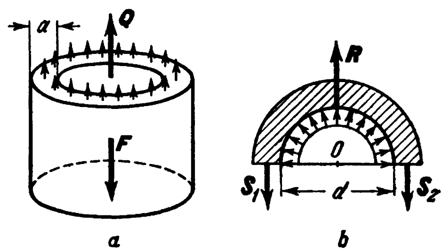


Fig. 76

ming all the forces to be acting on it in the plane of the median section (Fig. 76*b*). Acting on this half of the cylinder are: (a) the pressure of the gas of intensity $q = pH$ distributed uniformly along a semicircle [from Eq. (41) its resultant is $R = qd = pHd$] and (b) the forces distributed along sections A and B (the action of the removed half) whose resultants are S_1 and S_2; by virtue of symmetry, $S_1 = S_2 = S$. From the conditions of equilibrium we have $S_1 + S_2 = R$, whence $S = 1/2\ pdH$. As the area of the section, along which force S is distributed is aH (neglecting the area of the section of the cylinder bottom), we find the expansion stress

$$\sigma_2 = \frac{S}{aH} = \frac{1}{2} \frac{d}{a} p \text{ kgf/m}^2.$$

It will be noticed that the lateral stress is twice as large as the longitudinal stress.

Chapter 5
Elements of Graphical Statics

§ 29. Force and String Polygons. Reduction of a Coplanar Force System to Two Forces

In engineering problems graphical methods are often used which, though less accurate than analytical methods, produce faster and more easily visualised results.

The graphical method of solving problems of statics for coplanar force systems is based on the construction of force and string (or funicular) polygons.

Consider a system of three forces $\boldsymbol{F}_1$, $\boldsymbol{F}_2$, $\boldsymbol{F}_3$ acting on a rigid body (Fig. 77*a*). In Fig. 77*b* is constructed a *force polygon abcd* with the

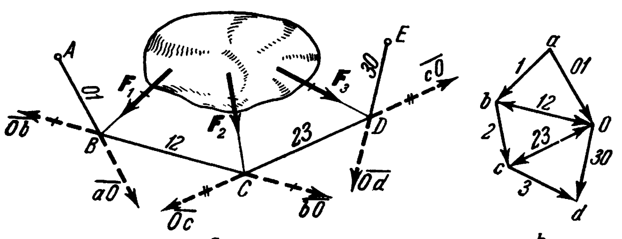


Fig. 77

given forces as its sides*). It will be recalled that when the end of the last force coincides with the beginning of the first force the polygon is said to be *closed*; otherwise it is said to be *open*.

From an arbitrary point *O* (the pole) in the plane of the force polygon and not collinear with the sides of the polygon draw to the vertices of the polygon rays *Oa*, *Ob*, *Oc*, *Od*, which we shall number *01*, *12*, *23*, and *30* (reading them "zero-one", "one-two", etc., as they

*) We shall denote the forces of a force polygon by numbers, e.g., *1* for $\boldsymbol{F}_1$, *2* for $\boldsymbol{F}_2$, etc.

denote the numbers of the forces coincident at the respective vertices).

Now take in Fig. 77*a* an arbitrary point A and draw through it a line parallel to ray *01* till its intersection with the action line of force F_1 at point B. From B draw a line parallel to ray *12* till its intersection with the action line of force F_2 at C, etc. The resulting figure $ABCDE$ is called a *string* (*funicular*) *polygon*. (If we take a string, pin it down at A and E and apply forces F_1, F_2, and F_3 to it at points B, C, and D, then, when equilibrium is maintained, it will take the shape of the broken line $ABCDE$. Hence the name of the polygon.)

A string polygon is *closed* if its first and last strings (AB and DE in our example) coincide, i.e., if they are collinear; otherwise the string polygon is said to be *open*.

By such a construction it is possible to replace any coplanar force system with two forces directed along the first and the last strings (AB and DE) of the polygon.

To prove this, consider first one of the forces acting on the body, e.g., force F_1 (Fig. 77*a*). By representing this force separately as vector $\overline{ab}$ (Fig. 77*b*) and then joining points a and b with an arbitrary point O, we thereby resolve force F_1 into two forces $\overline{aO}$ and $\overline{Ob}$, since, from the force triangle aOb, $F_1 = \overline{ab} = \overline{aO} + \overline{Ob}$ (Fig. 77*b*). But from the parallelogram rule, if $F_1 = \overline{aO} + \overline{Ob}$, force F_1 acting on the body can be replaced by forces equal to $\overline{aO}$ and $\overline{Ob}$ and applied at any point along the line of action of F_1, notably at point B in Fig. 77*a*. And, as side AB of the string polygon was drawn parallel to aO, and BC parallel to bO, force $\overline{aO}$ will be directed along string AB of the polygon, and force $\overline{Ob}$ along string BC. A similar result obtains for the other forces.

Thus, by drawing rays Oa, Ob, etc. in Fig. 77*b*, each of the forces F_1, F_2, F_3 is resolved into two forces, as

$$F_1 = \overline{ab} = \overline{aO} + \overline{Ob}, \quad F_2 = \overline{bc} = \overline{bO} + \overline{Oc}, \quad F_3 = \overline{cd} = \overline{cO} + \overline{Od}.$$

But the forces equal to $\overline{aO}$ and $\overline{Ob}$ replace force F_1 when applied at point B which was obtained by the construction of the string polygon (Fig. 77*a*). Similarly, let us replace forces F_2 and F_3 by forces $\overline{bO}$ and $\overline{Oc}$, and $\overline{cO}$ and $\overline{Od}$, applying them at points C and D respectively. Note that forces $\overline{Ob}$, $\overline{bO}$ and $\overline{Oc}$, $\overline{cO}$ directed along lines BC and CD cancel out, as (from Fig. 77*b*) $\overline{Ob} = -\overline{bO}$ and $\overline{Oc} = -\overline{cO}$. Thus the system of forces F_1, F_2, and F_3 is replaced by two forces $\overline{aO}$ and $\overline{Od}$ directed along the first and last strings AB and DE of the string polygon.

Analogous results are obtainable for any number of forces.

§ 30. Graphical Determination of a Resultant

If a force polygon constructed for a given coplanar system is not closed (the principal vector $\boldsymbol{R} \neq 0$) the system, by § 23, can be reduced to a single resultant.

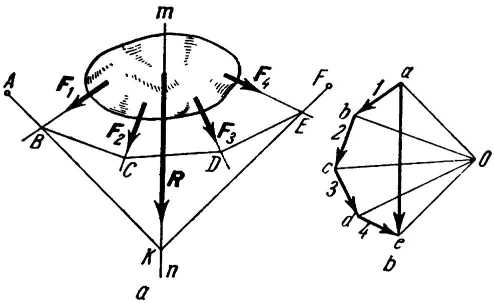


Fig. 78

Graphically the resultant can be determined by successively applying the parallelogram rule. With many forces, however, this becomes too cumbersome. Construction of force and string polygons simplifies the problem.

Let a set of forces $\boldsymbol{F}_1$, $\boldsymbol{F}_2$, $\boldsymbol{F}_3$, $\boldsymbol{F}_4$ be acting on a rigid body (Fig. 78*a*) and let *abcde* in Fig. 78*b* be a force polygon drawn to

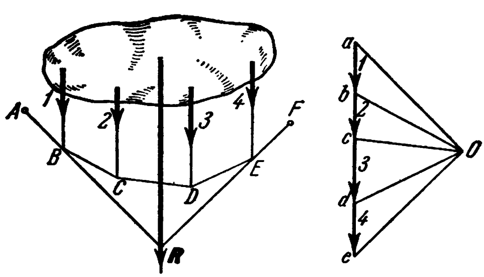


Fig. 79

scale. Its closing side *ae* represents the magnitude and direction of the resultant $\boldsymbol{R}$. To determine its point of application, connect the vertices *a*, *b*, *c*, *d*, *e* with an arbitrary pole *O* and construct a string polygon *ABCDEF*, where $AB \| aO$, $BC \| bO$, etc. (Fig. 78*a*). The given forces, we know, can be replaced by two forces directed along *AB* and *EF*. Hence, their resultant (and consequently the resultant of the forces $\boldsymbol{F}_1$, $\boldsymbol{F}_2$, $\boldsymbol{F}_3$, $\boldsymbol{F}_4$) passes through the point of intersection of *AB* and *EF*. Thus, by constructing a string polygon and continuing the first and the last strings till their intersection, we obtain a point K through which the resultant of the force system passes. By drawing through K a line parallel to ae and applying force $\boldsymbol{R}$ at any point on it we obtain the resultant.

A similar construction for parallel forces is shown in Fig. 79. The force polygon in this case is a line segment and the resultant $\boldsymbol{R} = \overline{ae}$.

§ 31. Graphical Determination of a Resultant Couple

If a force polygon constructed for a given coplanar force system closes while the string polygon remains open, the system can be reduced to a resultant couple.

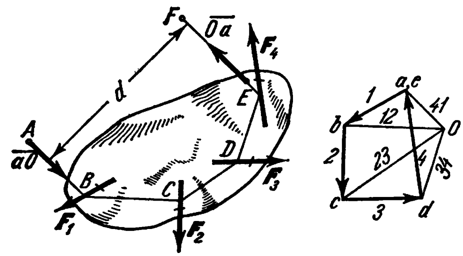


Fig. 80

For, if a force polygon $abcde$ constructed with the given forces $\boldsymbol{F}_1$, $\boldsymbol{F}_2$, $\boldsymbol{F}_3$, $\boldsymbol{F}_4$ is closed (Fig. 80), rays aO and Oe coincide*). Then the first and the last strings AB and EF of the string polygon, if it is not closed, are parallel.

It follows from § 29 that the given forces may be replaced by two forces equal, in the present case, to $\overline{aO}$ and $\overline{Oa}$ (as $\overline{Oe} = \overline{Oa}$) and directed along AB and EF. Thus, the force system $\boldsymbol{F}_1$, $\boldsymbol{F}_2$, $\boldsymbol{F}_3$, $\boldsymbol{F}_4$ is replaced by the couple $(\overline{aO}, \overline{Oa})$ with a moment arm d. The moment of this couple is $Oa \cdot d$, where Oa is measured to the scale of the forces in the force polygon and d is measured to the scale of the original diagram.

§ 32. Graphical Conditions of Equilibrium of a Coplanar Force System

The results obtained in the foregoing sections show that *for any coplanar system of forces acting on a rigid body to be in equilibrium it is necessary and sufficient for the force and string polygons constructed*

*) In this case we denote the rays Oa, Ob, etc., by numbers *12*, *23*, *34*, and *41*, as in a closed polygon we have two forces at every vertex.

with the given forces to be closed (the graphical conditions of equilibrium).

For, if any of the two polygons is not closed, a system can be reduced either to a resultant force or to a resultant couple, hence it will not be in equilibrium. If on the other hand both polygons are closed, the forces acting on the body may, apparently, be reduced to two equal collinear forces of opposite sense (Fig. 80, when $d = 0$), and the body is in equilibrium.

§ 33. Determination of the Reactions of Constraints

Let us determine graphically the reactions of the supports A and K of the truss in Fig. 81*a*. First draw the truss to a suitable scale (e.g., 0.4 m in 1 cm) and denote the given forces $\boldsymbol{F}_1$, $\boldsymbol{F}_2$, $\boldsymbol{F}_3$ acting on it and the reactions of the supports $\boldsymbol{R}_4$ and $\boldsymbol{R}_5$. The direction of

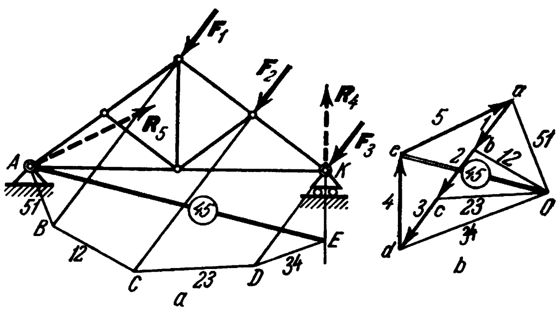


Fig. 81

$\boldsymbol{R}_4$ is known, and of $\boldsymbol{R}_5$ unknown. Now choose a scale for the forces (e.g., 0.5 tonf to 1 cm) and construct a force polygon (Fig. 81*b*), starting with forces *1*, *2*, and *3* ($\boldsymbol{F}_1 = \overline{ab}$, $\boldsymbol{F}_2 = \overline{bc}$, $\boldsymbol{F}_3 = \overline{cd}$). The construction ends with laying off the direction of force $\boldsymbol{R}_4$, as we do not know its magnitude (i.e., the location of point e). We do know however, that the end of force $\boldsymbol{R}_5$ must be at point a because, with the system in equilibrium, the force polygon must be closed.

To continue our solution, take a pole O and draw rays *12*, *23*, *34*, *51*. The direction of ray *45* is unknown because we do not know the location of vertex e of the force polygon. To find this ray, construct a string polygon next to the diagram in Fig. 81*a*, *starting with point A where the force* $\boldsymbol{R}_5$ *of unknown direction is applied* (otherwise you will be unable to close the polygon, as A is the only known point on the action line of $\boldsymbol{R}_5$). From point A, where the force $\boldsymbol{R}_5$ is applied, draw string *51* to its intersection with the action line of $\boldsymbol{F}_1$ at B, from there string *12* till its intersection with the action line of $\boldsymbol{F}_2$

at C, string *23* till its intersection with the action line of F_3 at D, and string *34* till its intersection with the action line of $\boldsymbol{R}_4$ at E. The closing line EA of the string polygon gives, by virtue of the conditions of equilibrium, the direction of string *45*.

In Fig. 81*b* we can now draw ray *45* from O parallel to EA. Its point of intersection with the direction of force *4* gives us the required vertex e of the force polygon. Vector $\overline{de}$ denotes the required force $\boldsymbol{R}_4$, and vector $\overline{ea}$ the required force $\boldsymbol{R}_5$ to scale, which solves the problem.

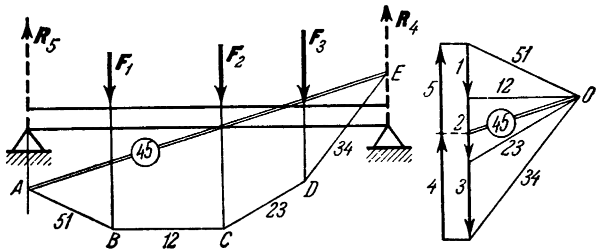


Fig. 82

An example of the graphical determination of the reactions of supports when the given forces are parallel is shown in Fig. 82. In this case the construction of the string polygon can start at any point, as the direction of both reactions is immediately known. The required ray *45* is shown in both diagrams by a double line.

Chapter 6
Solution of Trusses

§ 34. Trusses. Analytical Analysis of Plane Trusses

A truss is a rigid structure composed of straight members connected at their ends by pins. If the members of a truss are coplanar, it is called a *plane truss.* The points of intersection of the members are called *joints.* All external loads on the truss act only at the joints. In solving trusses, the friction at the joints and the weight of the members are assumed to be negligible in comparison with the external forces and are neglected (or the weight of the members is distributed to the joints). Hence, acting on every truss member are two forces applied at its ends; being in equilibrium, the forces are directed only along the members. It follows that *the members of a truss work only in tension or in compression.* We shall restrict ourselves to the consideration of rigid plane trusses composed of triangles. In such trusses the relation between the number of members k and the number of joints n is expressed by the equation:

$$k = 2n - 3. \tag{42}$$

In fact, any rigid triangle composed of three members has three joints (see, for example, the triangle ABD in Fig. 84 formed by members *1*, *2* , *3*). Each new joint requires only two new members (e.g, joint C in Fig. 84 is attached by members *4* and *5*, joint E, by members *6* and *7*, etc.). Hence, for the remaining $(n - 3)$ joints $2(n - 3)$ members are required. The total number of members of a truss is thus $k = 3 + 2(n - 3) = 2n - 3$. A truss with fewer members will not be rigid, one with more members will be statically indeterminate.

By the solution of a truss is meant the determination of the reactions of the constraints and the stresses in the members.

The reactions of the constraints can be determined by the conventional methods of statics (§ 25), treating the truss as a rigid body. Let us examine how the stresses in the members are determined.

Method of Isolation of Joints. This is convenient when the stresses in all members of the truss have to be determined. It involves consecutive examination of the equilibrium conditions of the forces intersecting at each joint. The calculations are best explained on a concrete example.

Consider the truss in Fig. 83*a*, which is made up of equal isosceles right triangles. The forces acting on it are parallel to the x-axis and equal: $F_1 = F_2 = F_3 = F = 2$ tonf.

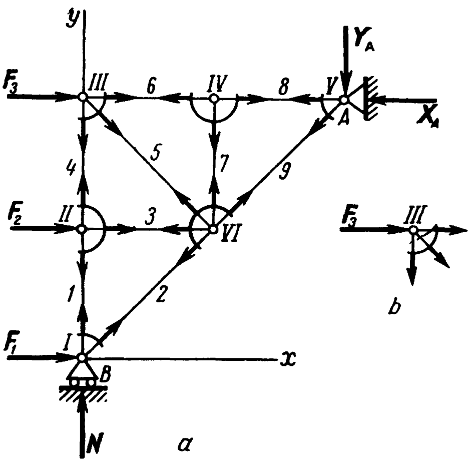


Fig. 83

The number of joints $n = 6$, the number of members $k = 9$. As this satisfies Eq. (42), the truss is rigid and there are no superfluous members.

Writing the equilibrium equation (33) for the truss as a whole, we find that the reactions of the supports are directed as shown in the figure and are equal to:

$$X_A = 3F = 6 \text{ tonf},$$

$$Y_A = N = \frac{3}{2} F = 3 \text{ tonf}.$$

Now determine the stresses in the members. Number the joints with Roman numerals and the members with Arabic. Denote the required stresses S_1 (in member *1*), S_2 (in member *2*), etc. Mentally isolate all the joints with the intersecting members from the rest of the truss. Replace the action of the discarded sections of the members with forces directed along the corresponding members and numerically equal to the required stresses S_1, S_2, ... Denote all these forces in the diagram, directing them *away from the joints*, i.e., regarding all the members as *being in tension* (Fig. 83*a*; each joint visualised separately, as joint *III* in Fig. 83*b*). If a calculation yields a negative stress, it means the member concerned is in compression.*) We do not introduce any special notation for the forces acting along the members in Fig. 83 as it is obvious that the forces acting along member *1* are numerically equal to S_1, those along member *2* are equal to S_2, etc.

*) Which is why, regardless of the methods employed, tensile stress is conventionally denoted by a "+" sign, and compressive stress by a "—" sign.

Now write the equilibrium equations (12) for the forces intersecting at each joint:

$$\sum F_{kx}=0, \quad \sum F_{ky}=0,$$

starting with joint *I*, where *tw omembers* meet, since two equations are sufficient to determine only two unknown stresses.

Writing the equilibrium equations for joint *I*, we obtain:

$$F_1 + S_2 \cos 45° = 0, \quad N + S_1 + S_2 \sin 45° = 0,$$

whence

$$S_2 = -F\sqrt{2} = -2.82 \text{ tonf}, \quad S_1 = -N - S_2\frac{\sqrt{2}}{2} = -\frac{F}{2} = -1 \text{ tonf}.$$

Now, knowing S_1, we can consider joint *II*, for which the equilibrium equations give:

$$S_3 + F_2 = 0, \quad S_4 - S_1 = 0,$$

whence

$$S_3 = -F = -2 \text{ tonf}, \quad S_4 = S_1 = -1 \text{ tonf}.$$

Having determined S_4, we can write the equilibrium equations, first for joint *III* and then for joint *IV*, which yield:

$$S_5 = -S_4\sqrt{2} = 1.41 \text{ tonf}, \quad S_6 = S_8 = -3 \text{ tonf}, \quad S_7 = 0.$$

Finally, to determine S_9 write the equilibrium equation for the forces intersecting at joint *V*, projecting them on axis *By*. This yields $Y_A + S_9 \cos 45° = 0$, whence $S_9 = -3\sqrt{2} = -4.23$ tonf.

For verification we can write the second equilibrium equation for joint *V* and two equations for joint *VI*. These equations were not needed to determine the stresses in the members, as the three equilibrium equations for the truss as a whole were used in determining N, X_A and Y_A (see § 26).

The final results of the computation can be tabulated:

Member No.	1	2	3	4	5	6	7	8	9
Stress, tonf	−1	−2.82	−2	−1	+1.41	−3	0	−3	−4.23

As the signs indicate, member *5* is in tension, the others, with the exception of No. 7, are in compression, and member *7* carries no load (a *zero member*).

The existence of zero members in a truss can be spotted immediately, since *if two members of a three-member joint are collinear, the stress in the third member is zero.* The result is obtained from the equilibrium equations for the projections on the axis perpendicular to the stated two members. For example, in the truss in Fig. 84, in the absence of force P_4, member *15* is a zero member and, hence, also member *13*. When force P_4 is acting, neither one is a zero member.

If in the calculations a joint occurs with more than two unknown quantities, the method of cuts can be employed.

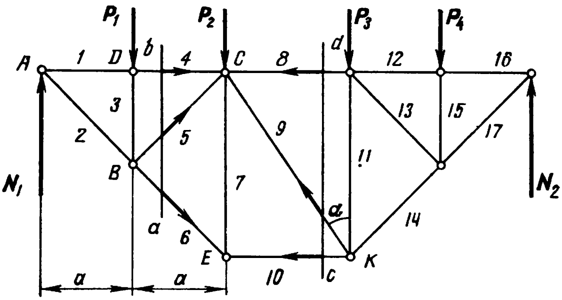


Fig. 84

Method of Cuts (Ritter's method). This method is convenient in determining the stresses in individual members, notably for verifying results. The truss is divided into two by a cut through *three* members for which (or for one of which) the stresses have to be determined, and the equilibrium of one of the parts is examined. The action of the discarded part is replaced by the corresponding forces directed away from the joints along the cut members, i.e., treating them as being in tension (as in the previous method). Then the equations (35) or (34) are written, taking the centres of the moments (or the projection axis) so that *there would be only one unknown stress in each equation.*

Example. Determine the stress in member *6* of the truss in Fig. 84. The acting vertical forces $P_1 = P_2 = P_3 = P_4 = 2$ tonf, the reactions of the supports $N_1 = N_2 = 4$ tonf. Draw a cut *ab* through members *4*, *5* and *6* and consider the equilibrium of the left-hand section of the truss, replacing the action of the right-hand section on it with forces directed along the cut members. To find S_6, write the moments equation about point C, where members *4* and *5* intersect. Assuming $AD = DC = a$ and $BC \perp BE$, we obtain,

$$-N_1 \cdot 2a + P_1 a + S_6 \cdot CB = 0,$$

whence we calculate S_6.

The arm CB can be calculated from the data defining the geometrical dimensions of the truss [if the method is employed for an approximate verification of a graphical analysis (§ 35) and the truss is drawn to scale, CB can be taken from the figure]. In our example $\angle ABC = 90°$ and $CB = a\sqrt{2}$. Hence, $S_6 = 3\sqrt{2} = 4.23$ tonf, and the member is in tension.

The stresses in members *4* and *5* can be determined by writing the moment equations about centres B (where members *5* and *6* intersect) and A (where members *4* and *6* intersect).

To determine the stress in member 9 of the truss, draw a cut dc through members *8*, *9* and *10* and, considering the equilibrium of the right-hand part, write the equations for the projections on an axis perpendicular to members *8* and *10*. We obtain:

$$S_9 \cos \alpha - P_3 - P_4 + N_2 = 0,$$

whence we determine S_9.

The stresses in members *8* and *10* can be determined by writing the moments equations about centres K and C.

§ 35*. Graphical Analysis of Plane Trusses

The method of isolation of joints can also be used for graphical analysis of trusses. For this the reactions of the constraints are determined as described in § 33. Then each joint is successively isolated and the stresses in the members attached to it are determined by constructing closed force polygons. The construction is carried out to scale, which must be chosen beforehand (§ 33), starting with a two-member joint (otherwise it would be impossible to determine the unknown stresses.)

As an example let us consider the truss in Fig. 85a. It has $n = 6$ joints and $k = 9$ members. It satisfies Eq. (42) and therefore is rigid and has no redundant members. The reactions of the supports $\boldsymbol{R}_4$ and $\boldsymbol{R}_5$ were determined in § 33, and forces $\boldsymbol{F}_1, \boldsymbol{F}_2, \boldsymbol{F}_3$ are known.

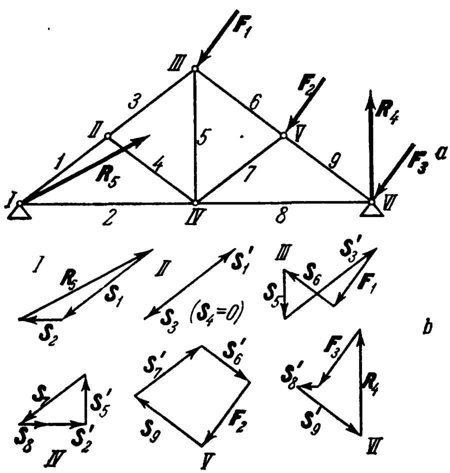


Fig. 85

Now consider the equilibrium of the members attached to joint I (we shall denote the joints by Roman and the members by Arabic numerals). Assume the members to be severed from the truss*). The action of the removed portion of the truss is shown by forces S_1 and S_2 directed along members *1* and *2*. We construct a closed triangle with forces R_5, S_1, and S_2, which intersect at joint *I* (Fig. 85*b*). For this, first lay off the known force R_5 to scale and then draw through its head and tail lines parallel to members *1* and *2*. This gives us forces S_1 and S_2 acting on members *1* and *2*. Now consider the equilibrium of the members attached to joint *II*. The action of the removed portion of the truss on them is denoted by forces S_1', S_2, and S_4 directed along the respective members. Force S_1' is known from the principle of action and reaction, for $S_1' = -S_1$. Constructing a closed triangle with the forces intersecting at joint *II* (starting with S_1'), we find the magnitudes of S_3 and S_4 (in this case $S_4 = 0$). Similarly we determine the stresses in all the other members. The force polygons for all the joints are shown in Fig. 85*b*. The last polygon (for joint *VI*) is constructed only to check the solution, as all the forces in it are already known.

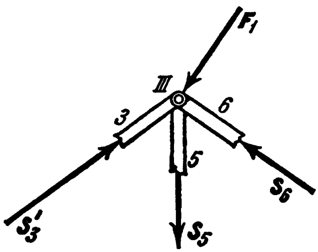


Fig. 86

The type of stress in each member may be determined as follows. "Cut out" a joint with portions of the members attached to it (for example, joint *III*) and apply the found forces to the sections of the members (Fig. 86). The force directed away from the joint (S_5 in Fig. 86) causes tension in the member; the force directed towards the joint (S_3' and S_6 in Fig. 86) causes compression. Conventionally tension is denoted by a "+" sign and compression by a "—" sign. Thus, in the present case $S_5 = +0.2$ tonf and $S_6 = -0.3$ tonf. In the example in Fig. 85 members *1*, *2*, *3*, *6*, *7*, and *9* are in compression and members *5* and *8* are in tension.

§ 36*. The Maxwell-Cremona Diagram

Trusses can be solved graphically much quicker and the results represented in more compact form if the force polygons for all the joints are constructed in a single diagram of forces called the *Maxwell-Cremona diagram*.

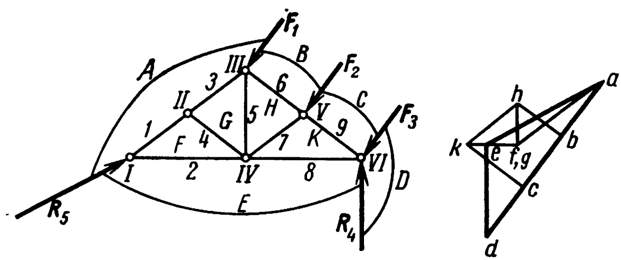
Very important in solving trusses by this method is a strict order of approach. The following steps are suggested:

*) Fig. 86 shows the members of joint *III* severed for investigation of its equilibrium. All the other joints should be considered as cut out in the same way.

(1) Determine the reactions of the constraints of the truss.

(2) Draw the given forces and the reactions of the constraints acting on the truss, placing all the vectors outside the periphery of the truss (Fig. 87*a*). Denote the regions on either side of the forces and between the members of the truss by $A, B, C, \ldots, K$.

(3) Construct a closed force polygon with the external forces (i.e., the loads and the reactions of the constraints) to scale, laying them off in the order in which they occur clockwise around the periphery of the truss (the heavy lines in Fig. 87*b*). It will be found convenient to denote the head and tail of each force vector by lower-case



a b

Fig. 87

letters corresponding to the labels of the regions on either side of it taken in clockwise direction (thus force $\boldsymbol{F}_1$ is denoted by $\overline{ab}$, force $\boldsymbol{F}_2$ by $\overline{bc}$, etc.). We thus obtain the polygon *abcdea* of the external forces acting on the truss (arrowheads are conventionally not drawn in the diagram).

(4) Now successively attach to this polygon the force polygons for all the joints of the truss, starting from a two-member joint. Construction of each polygon should start with a known force, and the forces should be laid off in the order in which they occur passing clockwise round the joint. (The stresses in the members are denoted like the external forces: the stress in member *1* by *af*, in member *2* by *fe*, etc.)

For joint *VI* in Fig. 87*a*, for instance, the given forces $\boldsymbol{F}_3$ and $\boldsymbol{R}_4$ are already denoted in the diagram as *cd* and *de* (Fig. 87*b*). From point *e* draw a line parallel to member *8*, and from point *c* a line parallel to member *9*. The intersection of the lines gives us the vertex *k* of the closed force polygon *cdekc* for joint *VI*. The force polygons for the other joints are constructed similarly.

The diagram of forces for the whole truss is shown in Fig. 87*b*. In order to determine the stress, say, in member *1*, we "cut out" joint *I* and, reading clockwise, find the name of the required force: *af*.

In the diagram we find the vector $\overline{af}$ and determine its magnitude. Applying now the vector to the section of member *1* (see Fig. 86), we find that the member is in compression. We get the same result if joint *II* is cut—only, reading clockwise, we find that the required vector is denoted by $\overline{fa}$. This notation of forces thus automatically takes account of the law of action and reaction.

Note the following special cases:

(1) If attached to a joint not subjected to external loads are three members, two of which are collinear (joint *II* in Fig. 87*a*), then,

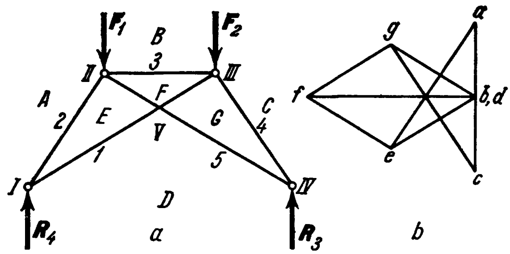


Fig. 88

as was shown in § 34, the stress in the third member (member *4* in Fig. 87*a*) is zero (*zero member*). That is why in the diagram in Fig. 87*b* points *g* and *f* coincide ($gf = 0$).

(2) If a truss has intersecting members (members *1* and *5* in Fig. 88*a*), the force diagram for them can be constructed in the ordinary way, their point of intersection being considered as a joint. The stresses in members *1* and *5* are equal in magnitude and sign and will appear twice in the diagram, as shown in Fig. 88*b*. Lines *de* and *fg* denote the stress in member *1*, and lines *ef* and *gd*, in member *5*.

(3) If in constructing a diagram you come upon a joint with more than two unknown quantities, try and construct the diagram simultaneously from two ends of the truss (if the truss is not symmetrical) or determine the stress in some members by the method of cuts. This method can be used to verify the correctness or accuracy of the graphical solution of a truss.

Chapter 7
Friction

§ 37. Laws of Static Friction

We know from experience that when two bodies tend to slide on each other, a resisting force appears at their surface of contact which opposes their relative motion. This force is called sliding friction.

Friction is due primarily to minute irregularities on the contacting surfaces, which resist their relative motion, and to forces of adhesion between contacting surfaces. A detailed examination of the nature of friction is a complex physico-mechanical problem lying beyond the scope of theoretical mechanics.

Engineering calculations are based on several general laws deduced from experimental evidence, which reflect the principal features of friction with an accuracy sufficient for practical purposes. These laws, the laws of sliding friction, can be formulated as follows:

(1) When two bodies tend to slide on each other, a frictional force is developed at the surface of contact, the magnitude of which can have any value from zero to a maximum value F_l which is called *limiting friction*, or *friction of impending motion*.

Frictional force is opposite in direction to the force which tends to move a body.

(2) Limiting friction is equal in magnitude to the product of the coefficient of static friction (or friction of rest) f_0 and the normal pressure or normal reaction N:

$$F_l = f_0 N, \tag{43}$$

The coefficient of static friction f_0 is a dimensionless quantity which is determined experimentally and depends on the material of the contacting bodies and the conditions of the surfaces (their finish, temperature, humidity, lubrication, etc.).

(3) Within fairly broad limits, the value of limiting friction does not depend on the area of the surface of contact.

Taken together, the first and second laws state that for conditions of equilibrium the static friction (adhesive force) $F \leqslant F_l$, or

$$F \leqslant f_0 N. \qquad (44)$$

The coefficient of friction can be determined experimentally by means of a simple device shown schematically in Fig. 89. The horizontal plate *AB* and rectangular block *D* are made of materials for which the coefficient of friction is to be determined. Acting on block *D* is the force of gravity **P**, which is balanced by the normal reaction of the plate **N**, and the applied force **Q**, which when the system is at rest, is balanced by the frictional force **F** (the magnitude of *Q* is equal to the weight of the pan *E* with the weights). By gradually loading the pan we determine the load Q^* at which the block starts moving. Obviously, the limiting friction $F_l = Q^*$. Hence, as in this case $N = P$, we find from Eq. (43):

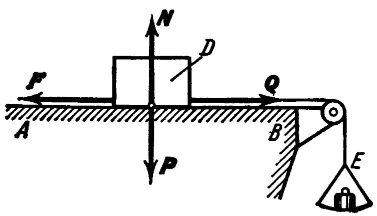


Fig. 89

$$f_0 = \frac{F_l}{N} = \frac{Q^*}{P}.$$

A series of such experiments demonstrates that, within certain limits of the weight *P* of the block, Q^* is proportional to *P* and f_0 is constant. The coefficient of static friction f_0 is also independent of the magnitude of the area of contact, within certain limits. This confirms the validity of the second and third laws of friction. The validity of the first law follows from the fact that at any value of *Q* less than Q^* the block remains at rest. Hence, the frictional force **F**, which balances force *Q*, can actually assume any value from zero (at $Q = 0$) to F_l (at $Q \geqslant Q^*$).

Attention should be called to the fact that, as long as the block remains at rest, the frictional force is equal to the applied force *Q*, and not to $F_l = f_0 N$. The force of friction becomes equal to $f_0 N$ only when slipping is impending.

The following table offers an idea of the values of the coefficient of static friction for various materials:

Wood on wood	0.4 to 0.7
Metal on metal	0.15 to 0.25
Steel on ice	0.027

For more detailed information the student is invited to consult engineering handbooks.

The foregoing refers to sliding friction of rest. When motion occurs, the frictional force is directed opposite to the motion and equals the product of the coefficient of kinetic, or sliding, friction and the normal pressure:

$$F = fN.$$

The coefficient of kinetic friction f is also a dimensionless quantity which is determined experimentally. The value of f depends not only on the material and conditions of the contacting surfaces but also, to some degree, on the relative velocity of the bodies. In most cases the value of f at first decreases with velocity and then attains a practically constant value.

§ 38. Reactions of Rough Constraints. Angle of Friction

Up till now, in solving problems of statics, we neglected friction and regarded the surfaces of constraints as smooth and their reactions as normal to the surface. The reactions of real (rough) constraints consist of two components: the normal reaction $\boldsymbol{N}$ and the frictional

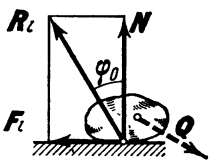


Fig. 90

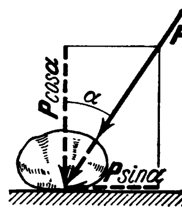


Fig. 91

force $\boldsymbol{F}$ perpendicular to it. Consequently, the total reaction $\boldsymbol{R}$ forms an angle with the normal to the surface. As the friction increases from zero to $\boldsymbol{F}_l$, force $\boldsymbol{R}$ changes from $\boldsymbol{N}$ to $\boldsymbol{R}_l$, its angle with the normal increasing from zero to a maximum value φ_0 (Fig. 90). The maximum angle φ_0 which the total reaction of a rough support makes with the normal to the surface is called the *angle of static friction*, or *angle of repose*.

From the diagram we have:

$$\tan\varphi_0 = \frac{F_l}{N}.$$

Since $F_l = f_0 N$, we have the following relation between the angle of friction and the coefficient of friction:

$$\tan\varphi_0 = f_0. \tag{45}$$

When a system is in equilibrium the total reaction $\boldsymbol{R}$ can pass anywhere within the angle of friction, depending on the applied forces. When motion impends, the angle between the reaction and the normal is φ_0.

If to a body lying on a rough surface is applied a force $\boldsymbol{P}$ making an angle α with the normal (Fig. 91), the body will move only if the shearing force $P \sin\alpha$ is greater than $F_l = f_0 P \cos\alpha$ (neglecting the weight of the body and considering $N = P \cos \alpha$). But the inequality $P \sin \alpha > f_0 P \cos \alpha$, where $f_0 = \tan \varphi_0$, is satisfied only if $\tan \alpha > \tan \varphi_0$, i.e., if $\alpha > \varphi_0$. Consequently, if angle α is less than φ_0 the body will remain at rest no matter how great the applied force. This explains the well-known phenomena of wedging and self-locking.

§ 39. Equilibrium With Friction

Examination of the conditions for the equilibrium of a body taking friction into account is usually limited to a consideration of the conditions when motion is impending and the frictional force acquires its maximum value F_l. For the analytical solution of problems the reaction of a rough constraint is denoted by its two components N and F_l, where $F_l = f_0 N$. The known equations of static equilibrium are then written, substituting $f_0 N$ for F_l, and solved for the required values.

If the problem requires that all possible positions of equilibrium be determined, it is sufficient to solve only for the position of impending motion. Other positions of equilibrium can then be found by reducing the coefficient of friction f_0 in the obtained solution to zero *).

It is important to note that in positions of equilibrium when motion does not impend the force of friction F is not equal to F_l, and its magnitude, if it is required, should be determined from the conditions of equilibrium as a new unknown quantity (see second part of Problem 32).

In graphical solutions it is more convenient to denote the reaction of a rough constraint by a single force R, which in the position of impending motion will be inclined at an angle φ_0 to the normal to the surface.

Problem 32. A load of weight $P = 10$ kgf rests on a horizontal surface (Fig. 92). Determine the force Q that should be applied at an angle $\alpha = 30°$ to the horizontal to move the load from its place, if the coefficient of static friction for the surfaces of contact is $f_0 = 0.6$.

Solution. According to the conditions of the problem we have to consider the position of impending motion of the load. In this position acting on it are forces P, Q, N, and F_l. Writing the equilibrium equations in terms of the projections on the coordinate axes, we obtain:

$$Q \cos \alpha - F_l = 0, \quad N + Q \sin \alpha - P = 0.$$

*) For, when motion is impending, the force of friction $F = F_l = f_0 N$. In other positions of equilibrium $F < f_0 N$; hence these positions can be found by reducing the value of f_0 in the equation $F = f_0 N$. At $f_0 = 0$ equilibrium is for the case of an absolutely smooth constraint.

From the second equation $N = P - Q \sin \alpha$, whence:

$$F_l = f_0 N = f_0 (P - Q \sin \alpha).$$

Substituting this value of F_l in the first equation, we finally obtain:

$$Q = \frac{f_0 P}{\cos \alpha + f_0 \sin \alpha} \approx 5.2 \text{ kgf}.$$

If a smaller force is applied to the load, say $Q' = 4$ kgf, the shearing force will equal $Q' \cos 30° = 2\sqrt{3} = 3.46$ kgf. The maximum force of friction which can develop in this case is $F_l = f_0 (P - Q' \sin 30°) = 4.8$ kgf, and the load will remain at rest;

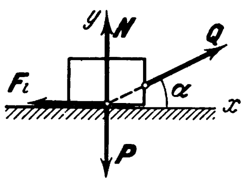


Fig. 92

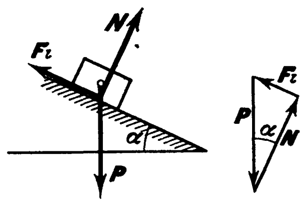


Fig. 93

the friction which keeps it in equilibrium can be found from the equilibrium equations in terms of the x projections, and will be equal to the shearing force ($F' = Q' \cos 30° = 3.46$ kgf) and not to F_l.

Attention is drawn to the fact that in all the computations F_l should be determined from the formula $F_l = f_0 N$, N being found from the conditions of equilibrium. A frequent mistake in solving problems of this type is made in assuming $F_l = f_0 P$, though actually the pressure on the surface is not equal to the weight of the load P.

Problem 33. Determine the angle α to the horizontal at which the load on the inclined plane in Fig. 93 remains in equilibrium if the coefficient of friction is f_0.

Solution. The problem requires that all possible positions for the equilibrium of the load be determined. For this, let us first establish the position of impending motion at which $\alpha = \alpha_l$. In that position acting on the load are its weight P, the normal reaction N and the limiting friction F_l. Constructing a closed triangle with these forces, we find that $F_l = N \tan \alpha_l$. But, on the other hand, $F_l = f_0 N$. Consequently,

$$\tan \alpha_l = f_0. \qquad \text{(a)}$$

In this equation α_l decreases as f_0 decreases. We conclude, therefore, that equilibrium is also possible at $\alpha < \alpha_l$. Finally, all the values of α at which the load remains in equilibrium are determined by the inequality

$$\tan \alpha \leqslant f_0. \quad \text{(b)}$$

If there is no friction ($f_0 = 0$), equilibrium is possible only at $\alpha = 0$. Consequently, with friction equilibrium is possible at all values of the angle of inclination of the plane between zero and α_l, whereas without friction equilibrium is possible at only one value of the angle α, which is zero. This is the difference between equilibrium with friction and the equilibrium of systems with absolutely smooth (frictionless) constraints.

The result expressed in equation (a) can be used for the determination of the coefficient of friction by experimentally finding angle α_l.

Note also that, from $f_0 = \tan \varphi_0$, where φ_0 is the angle of friction, it follows that $\alpha_l = \varphi_0$, i.e., the largest angle α at which a load resting on an inclined plane remains in equilibrium (the angle of repose) is equal to the angle of friction.

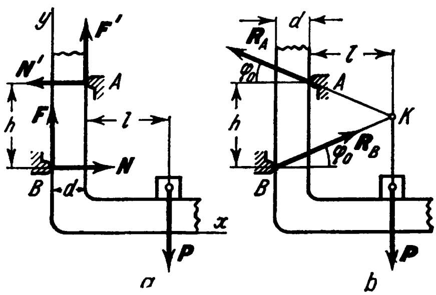


Fig. 94

Problem 34. A bent bar whose members are at right angles is constrained at A and B as shown in Fig. 94. The vertical distance between A and B is h. Neglecting the weight of the bar, determine the thickness d at which the bar with a load lying on its horizontal member will remain in equilibrium regardless of the location of the load. The coefficient of static friction of the bar on the constraints is f_0.

Solution. Let us denote the weight of the load by $\boldsymbol{P}$ and its distance from the vertical member of the bar by l. Now consider the position of impending slip of the bar, when $d = d_l$. In this position acting on it are the forces $\boldsymbol{P}$, $\boldsymbol{N}$, $\boldsymbol{F}$, $\boldsymbol{N}'$, and $\boldsymbol{F}'$, where $\boldsymbol{F}$ and $\boldsymbol{F}'$ are the forces of limiting friction. Writing the equilibrium equations (33) and taking the moments about A, we obtain:

$$N - N' = 0, \quad F + F' - P = 0, \quad Nh - Fd_l - Pl = 0,$$

where $F = f_0 N$ and $F' = f_0 N'$. From the first two equations we find:

$$N = N', \quad P = 2f_0 N.$$

Substituting these values in the third equation and eliminating N, we have:

$$h - f_0 d_l - 2f_0 l = 0,$$

whence

$$d_l = \frac{h}{f_0} - 2l.$$

If in this equation we reduce f_0 to zero, the right-hand side will tend to infinity. Hence, equilibrium is possible at any value of $d > d_l$. The maximum value of d_l is at $l = 0$. Thus, the bar will remain in equilibrium wherever the load is placed (at $l \geqslant 0$) if the inequality

$$d \geqslant \frac{h}{f_0}$$

is satisfied. The less the friction, the greater must d be. If there is no friction ($f_0 = 0$), equilibrium is obviously impossible, as $d = \infty$.

Here now is a graphical solution of the problem. Instead of the normal reactions and the frictional forces we denote at points A and B the total reactions $\boldsymbol{R}_A$ and $\boldsymbol{R}_B$ whose angles with the normals in the condition of impending slip will be equal to the angle of friction φ_0 (Fig. 94*b*). Thus, acting on the bar are three forces $\boldsymbol{R}_A$, $\boldsymbol{R}_B$, and $\boldsymbol{P}$. For equilibrium they must all intersect at K, where forces $\boldsymbol{R}_A$ and $\boldsymbol{R}_B$ intersect. We obtain the equation $h = (l + d_l) \tan \varphi_0 + l \tan \varphi_0$, or $h = (2l + d_l) f_0$, as $\tan \varphi_0 = f_0$. Thus we obtain the same result for d_l as in the analytical solution.

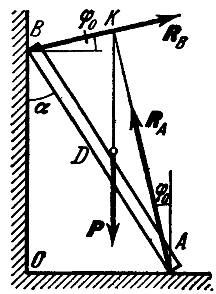


Fig. 95

The problem offers an example of a self-locking device which is often used in practice.

Problem 35. Neglecting the weight of the ladder AB in Fig. 95, determine the values of angle α at which a man can climb to the top of the ladder at B if the angle of friction for the contacts at the floor and the wall is φ_0.

Solution. Let us examine the position of impending slip of the ladder by the graphical method. For impending motion the forces acting on the ladder are the reactions of the floor and wall $\boldsymbol{R}_A$ and $\boldsymbol{R}_B$ which are inclined at the angle of friction φ_0 to the normals to the surfaces. The action lines of the reactions intersect at K. Thus, for the system to be in equilibrium the third force $\boldsymbol{P}$ (the weight of the man) acting on the ladder must also pass through K. Hence, in the position shown in the diagram the man cannot climb higher than D. For him to reach B the action lines of $\boldsymbol{R}_A$ and $\boldsymbol{R}_B$ must intersect somewhere along BO, which is possible only if force $\boldsymbol{R}_A$ is directed along AB, i.e., when $\alpha \leqslant \varphi_0$.

Thus a man can climb to the top of a ladder only if its angle with the wall does not exceed the angle of friction with the floor. The friction on the wall is irrelevant, i.e., the wall may be smooth.

§ 40*. Belt Friction

A force $\boldsymbol{P}$ is applied at the end of a string passing over a cylindrical shaft (Fig. 96). Let us determine the least force $\boldsymbol{Q}$ that must be applied at the other end of the string to maintain equilibrium at a given angle $AOB = \alpha$.

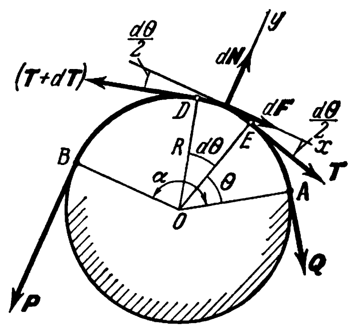


Fig. 96

Consider the equilibrium of an element DE of the string of length $dl = R\,d\theta$, where R is the radius of the shaft. The difference dT between the tensions in the string at D and E is balanced by the frictional force $dF = f_0\,dN$ ($d\boldsymbol{N}$ is the normal reaction), since at the lowest value of Q motion is impending. Consequently,

$$dT = f_0\,dN.$$

The value of dN is determined from the equilibrium equation derived for the force components parallel to axis Oy. The sine of a very small angle approximately equals the angle itself, and, neglecting small quantities of higher order, we have:

$$dN = T\sin\frac{d\theta}{2} + (T + dT)\sin\frac{d\theta}{2} = 2T\frac{d\theta}{2} = T\,d\theta.$$

Substituting this value of dN in the preceding equation we obtain:

$$dT = f_0 T\,d\theta.$$

Dividing both members of the equation by T and integrating the right-hand member in the interval from 0 to α and the left-hand member from Q to P (as the tension in the string is Q at the point where $\theta = 0$ and P at the point where $\theta = \alpha$), we obtain:

$$\int_Q^P \frac{dT}{T} = f_0 \int_0^\alpha d\theta \quad \text{or} \quad \ln\frac{P}{Q} = f_0\alpha,$$

from which it follows that $P/Q = e^{f_0\alpha}$, or

$$Q = Pe^{-f_0\alpha}. \qquad (46)$$

We see that the required force Q depends only on the coefficient of friction f_0 and the angle α; it does not depend on the radius of the shaft. If there is no friction ($f_0 = 0$), we have, as could be expected, $Q = P$. Of great practical importance is the fact that by increasing angle α (wrapping the string around the shaft) it is possible substantially to reduce the force Q required to balance force P. Table 1 shows, for example, that a tension of one ton can be supported by only 2 kgf by wrapping a hemp rope twice about a wooden post.

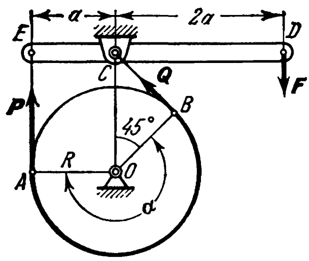


Fig. 97

Eq. (46) also gives the relation between the tensions in the driving part (P) and the driven part (Q) of a belt uniformly rotating a pulley without slippage. Assuming, for instance, $\alpha = \pi$ and, for a leather belt on a cast-iron pulley, $f_0 = 0.3$ the ratio of the tensions $Q/P = e^{-0.3\pi} \approx 0.4$.

Table 1

Values for Q/P at $f_0 = 0.5$
(Hemp Rope on Wood)

Number of turns	α	$Q/P = e^{-f_0\alpha}$
$\frac{1}{2}$	π	0.208
1	2π	0.043
$1\frac{1}{2}$	3π	0.009
2	4π	0.002

Problem 36. A force F is applied to the lever DE of the band-brake in Fig. 97. Determine the braking torque M_T exerted on the drum of radius R, if $CD = 2CE$ and the coefficient of friction of the band on the drum is $f_0 = 0.5$.

Solution. Acting on the drum and band AB wrapped around it is a force P (evidently $P = 2F$) applied at A and a force Q applied at B which is determined by Eq. (46). We also have $f_0 = 0.5$ and $\alpha = \frac{5}{4}\pi = 3.93$ radians. Hence,

$$Q = 2Fe^{-\frac{5}{8}\pi} \approx 0.28F.$$

The required torque is

$$M_T = (P - Q)\,R = 1.72FR \text{ kgf-m.}$$

The less the value of Q, i.e., the greater the coefficient of friction f_0 and the angle α, the greater the torque.

§ 41*. Rolling Friction and Pivot Friction

Rolling friction is defined as the resistance offered by a surface to a body rolling on it.

Consider a roller of radius R and weight P resting on a rough horizontal surface (Fig. 98*a*). If we apply to the axle of the roller a force

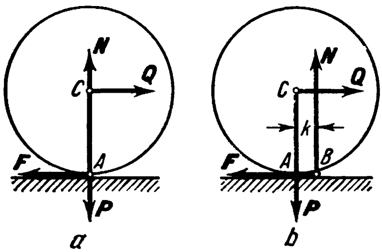


Fig. 98

$\boldsymbol{Q} < \boldsymbol{F}_l$, there will be developed at A a frictional force $\boldsymbol{F}$, equal in magnitude to Q, which prevents the roller from slipping on the surface. If the normal reaction $\boldsymbol{N}$ is also assumed to be applied at A, it will balance force $\boldsymbol{P}$, with forces $\boldsymbol{Q}$ and $\boldsymbol{F}$ making a couple which turns the roller. If these assumptions were correct, we could expect the roller to move, howsoever small the force $\boldsymbol{Q}$.

Experience tells us however, that this is not the case; for, due to deformation, the bodies contact over a certain surface AB (Fig. 98*b*). When force $\boldsymbol{Q}$ acts, the pressure at A decreases and at B increases. As a result, the reaction $\boldsymbol{N}$ is shifted in the direction of the action of force $\boldsymbol{Q}$. As $\boldsymbol{Q}$ increases, this displacement grows till it reaches a certain limit k. Thus, in the position of impending motion, acting on the roller will be a couple $(\boldsymbol{Q}_l, \boldsymbol{F})$ with a moment $Q_l R$ balanced by a couple $(\boldsymbol{N}, \boldsymbol{P})$ of moment Nk. As the moments are equal, we have $Q_l R = Nk$, or

$$Q_l = \frac{k}{R} N. \tag{47}$$

As long as $Q < Q_l$ the roller remains at rest; when $Q > Q_l$ it starts to roll.

The linear quantity k in Eq. (47) is called the *coefficient of rolling friction*, or *resistance*, and is generally measured in centimetres. The value of k depends on the material of the bodies and is determined experimentally. The following list offers an idea of some typical values of k:

Wood on wood	0.05 to 0.08 cm
Mild steel on steel (wheel on rail)	0.005 cm
Hardened steel on steel (ball bearing) . . .	0.001 cm

The ratio k/R for most materials is much less than the coefficient of static friction f_0. That is why in mechanisms rolling parts (wheels, rollers, ball bearings, etc.) are preferred to sliding parts.

Problem 37. Determine the values of the angle α at which a cylinder of radius R will remain at rest on an inclined plane if the coefficient of rolling friction is k (Fig. 99).

Solution. Consider the position of impending motion, when $\alpha = \alpha_1$. Resolving force $\boldsymbol{P}$ into components $\boldsymbol{P}_1$ and $\boldsymbol{P}_2$ (see Fig. 99), we find that the moving force $Q_l = P_1 = P \sin \alpha_1$ and the normal reaction $N = P_2 = P \cos \alpha_1$. From Eq. (47) we have:

$$P \sin \alpha_1 = \frac{k}{R} P \cos \alpha_1,$$

or

$$\tan \alpha_1 = \frac{k}{R}.$$

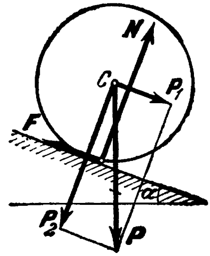


Fig. 99

If k tends to zero the value of α_1 also tends to zero. We conclude from this that equilibrium is maintained at any angle $\alpha < \alpha_1$. This result may be used for determining coefficient k by experimentally finding angle α_1.

Concept of Pivot Friction. Consider a sphere at rest on a horizontal plane. If a horizontal couple with a moment M is applied to the sphere it will tend to rotate the sphere about its vertical axis. We know from experience that the sphere will start turning only when M exceeds some specific value M_l which is determined by the equation

$$M_l = \lambda N, \tag{48}$$

where N is the normal pressure of the sphere on the surface—in this case equal to the sphere's weight. This result is explained by the development of so-called pivot friction, i.e., resistance to rotation due to the friction of the sphere on the surface. This is the type of friction developed in step bearings (pivots). The factor λ in Eq. (48) is a linear quantity called the *coefficient of pivot friction*. The value of λ is very small (one-fifth to one-tenth of the coefficient of rolling friction k).

Chapter 8
Couples and Forces in Space*)

§ 42. Moment of a Force About a Point as a Vector

Before proceeding with the solution of problems of statics for force systems in space, we should elaborate some of the concepts introduced in the preceding chapters. Let us begin with the concept of moment of a force.

1. Vector Expression of a Moment. The moment of a force $\boldsymbol{F}$ about a centre O (see Fig. 100), as a measure of the tendency of that force to turn a body, is characterised by the following three elements: (1) magnitude of the moment, which is equal to the product of the force and the moment arm, i.e., Fh; (2) the plane of rotation OAB through the line of action of the force $\boldsymbol{F}$ and the centre O; and (3) the sense of the rotation in that plane. When all the given forces and the centre O are coplanar there is no need to specify the plane of rotation OAB, and the moment can be defined as a scalar algebraic quantity equal to $\pm Fh$, where the sign indicates the sense of rotation (see § 14).

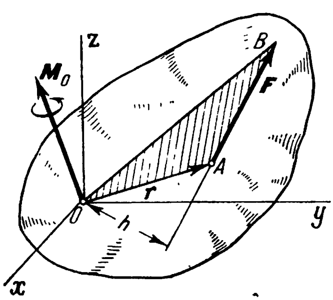


Fig. 100

If, however, the given forces are not coplanar, the planes of rotation have different aspects for different forces and have to be specified additionally. The position of a plane in space (its aspect) can be specified by specifying a line (vector) normal to it. If, furthermore, the modulus of this vector is taken as representing the magnitude of the force moment, and the direction of the vector is made to denote the

*) The student who is interested only in the methods of solving problems involving the equilibrium of bodies subjected to non-coplanar force systems may leave out §§ 45-48, which deal with the reduction of couples and forces in space.

sense of rotation, such a vector completely specifies the three elements which characterise the moment of a force with respect to a given centre O.

Thus, in the general case *we shall denote the moment* $\boldsymbol{M}_O(\boldsymbol{F})$ *of a force* $\boldsymbol{F}$ *about a point* O (Fig. 100) *by a vector* $\boldsymbol{M}_O$ *applied at* O, *equal in magnitude* (to some scale) *to the product of the force* $\boldsymbol{F}$ *and the moment arm* h, *and normal to the plane* OAB *through* O *and* $\boldsymbol{F}$. *We shall direct vector* $\boldsymbol{M}_O$ *so that the rotation viewed from the arrowhead is observed as counterclockwise.* Vector $\boldsymbol{M}_O$ will thus specify the magnitude of the moment, the plane of rotation OAB, which may be different for different forces, and the sense of rotation in that plane. The point at which vector $\boldsymbol{M}_O$ is applied defines the position of the moment centre.

2*. Expression of Moment of a Force in Terms of a Vector (Cross) Product. Consider the cross product of vectors $\overline{OA}$ and $\boldsymbol{F}$ (Fig. 100). From the definition*),

$$|\overline{OA} \times \boldsymbol{F}| = 2 \text{ areas of } \triangle OAB = M_O,$$

as vector $\boldsymbol{M}_O$ is equal in magnitude to twice the area of triangle OAB. Vector $(\overline{OA} \times \boldsymbol{F})$ is perpendicular to plane OAB in the direction from which a counterclockwise rotation would be seen to carry $\overline{OA}$ into $\boldsymbol{F}$ through the smaller angle between them (when the two forces are laid off from the same point), i.e., it is in the same direction as vector $\boldsymbol{M}_O$. Hence, vectors $(\overline{OA} \times \boldsymbol{F})$ and $\boldsymbol{M}_O$ are equal in magnitude and direction and, as can be readily verified, in dimension also, i.e., they both denote the same quantity. Therefore,

$$\boldsymbol{M}_O = \overline{OA} \times \boldsymbol{F} \text{ or } \boldsymbol{M}_O = \boldsymbol{r} \times \boldsymbol{F}, \tag{49}$$

where vector $\boldsymbol{r} = \overline{OA}$ is the radius vector from O to A.

Thus, *the moment of a force* $\boldsymbol{F}$ *about a centre* O *is equal to the cross product of the radius vector* $\boldsymbol{r} = \overline{OA}$, *from* O *to the point of application* A *of the force, and the force itself.* This expression of moment of a force will be found convenient in proving some theorems.

Equation (49) can also be used to compute the moment $\boldsymbol{M}_O$ analytically. Suppose coordinate axes $Oxyz$ are drawn through origin O (see Fig. 100) and the projections F_x, F_y, F_z of force $\boldsymbol{F}$ on the axes and the x, y, z coordinates of its point of application A are known. Then, as $r_x = x$, $r_y = y$, and $r_z = z$, from the known formula in

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*) The cross product $\boldsymbol{a} \times \boldsymbol{b}$ of vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is itself a vector $\boldsymbol{c}$ equal in magnitude to the area of a parallelogram constructed with vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ as its sides. Vector $\boldsymbol{c}$ is perpendicular to the plane through the two vectors in the direction from which a counterclockwise rotation would be seen to carry $\boldsymbol{a}$ into $\boldsymbol{b}$ through the smaller of the angles between them.

vector algebra,

$$M_O = r \times F = \begin{vmatrix} i & j & k \\ x & y & z \\ F_x & F_y & F_z \end{vmatrix}, \qquad (49')$$

where i, j and k are the unit vectors on the coordinate axes. If the determinant in the right-hand part of the equation is expanded according to the first row, the factors of i, j and k will be equal to the projections M_x, M_y, M_z of vector M_O on the coordinate axes (as $M_O = M_x i + M_y j + M_z k$). Consequently,

$$M_x = yF_z - zF_y, \quad M_y = zF_x - xF_z, \quad M_z = xF_y - yF_x. \qquad (50)$$

Eqs. (50) make it possible, after analytically calculating the projections M_x, M_y, M_z, to determine the vector M_O from them and, in particular, its magnitude.

Example. Let F be a force applied at a point with coordinates $x = 0.3$ m, $y = -0.6$ m, $z = 0.2$ m, and let the projections of F on the coordinate axes be $F_x = -20$ kgf, $F_y = 10$ kgf, $F_z = 0$. Determine the moment of the force about the origin O. From Eqs. (50) we have:

$$M_x = -0.2 \cdot 10 = -2, \quad M_y = -0.2 \cdot 20 = -4,$$
$$M_z = 0.3 \cdot 10 - 0.6 \cdot 20 = -9.$$

Hence,

$$M_O = \sqrt{M_x^2 + M_y^2 + M_z^2} = \sqrt{101} \approx 10 \text{ kgf-m}.$$

§ 43. Moment of a Force With Respect to an Axis

Before proceeding with the solution of problems of statics for any force system in space we must introduce the concept of moment of a force about an axis.

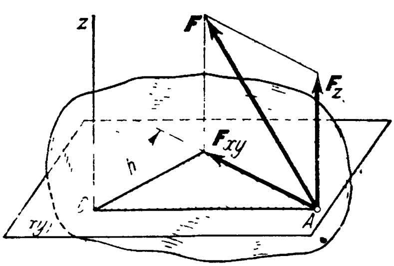

Fig. 101

The moment of a force about an axis is the measure of the tendency of the force to produce rotation about that axis. Consider a rigid body free to rotate about an axis z (Fig. 101). Let a force F applied at A be acting on the body. Let us now pass a plane xy through point A normal to the axis z and let us resolve the force F into rectangular components F_z parallel to the z-axis and F_{xy} in the plane xy (F_{xy} is in fact the projection of force F on the plane xy). Obviously, force F_z, being parallel to axis z, cannot turn the body about that axis (it only tends to translate the body *along* it). Thus we find that the total tendency of force $\boldsymbol{F}$ to rotate the body is the same as that of its component $\boldsymbol{F}_{xy}$. We conclude, then, that

$$m_z(\boldsymbol{F}) = m_z(\boldsymbol{F}_{xy}), \tag{50}$$

where $m_z(\boldsymbol{F})$ denotes the moment of force $\boldsymbol{F}$ with respect to axis z. But the rotational effect of force $\boldsymbol{F}_{xy}$, which lies in a plane perpendicular to axis z, is the product of the magnitude of this force and its distance h from the axis. The moment of force $\boldsymbol{F}_{xy}$ with respect to point O, where the axis pierces the plane xy, is the same. Hence, $m_z(\boldsymbol{F}_{xy}) = m_O(\boldsymbol{F}_{xy})$ or, by Eq. (50)*),

$$m_z(\boldsymbol{F}) = m_O(\boldsymbol{F}_{xy}) = \pm F_{xy}h. \tag{51}$$

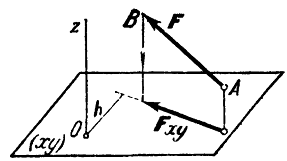

Fig. 102

From this we deduce the following definition: *The moment of a force about an axis is an algebraic quantity equal to the moment of the projection of that force on a plane normal to the axis with respect to the point of intersection of the axis and the plane.*

We shall call a moment positive if the rotation induced by a force $\boldsymbol{F}_{xy}$ is seen as counterclockwise when viewed from the positive end of the axis, and negative if it is seen as clockwise.

It is evident from Fig. 101 that in computing a moment according to Eq. (51) plane xy may pass through any point on axis z. Thus, in order to determine the moment of a force about axis z in Fig. 102 we have to: (1) pass an arbitrary plane xy normal to the axis; (2) project force $\boldsymbol{F}$ on the plane and compute the magnitude of $\boldsymbol{F}_{xy}$; (3) erect a perpendicular from point O, where the plane and axis intersect, to the action line of $\boldsymbol{F}_{xy}$ and determine its length h; (4) compute the product $F_{xy}h$; (5) determine the sense of the moment.

In determining moments the following special cases should be borne in mind:

(1) If a force is parallel to an axis, its moment about that axis is zero (since $F_{xy} = 0$).

(2) If the line of action of a force intersects with the axis, its moment with respect to that axis is zero (since $h = 0$).

Combining the two cases, we conclude that *the moment of a force with respect to an axis is zero if the force and the axis are coplanar.*

(3) If a force is perpendicular to an axis, its moment about that axis is equal to the product of the force magnitude and the perpendicular distance from the force to the axis.

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*) The symbol $m_O(\boldsymbol{F})$, which we used for coplanar force systems and which we shall continue to use, denotes the (algebraic) magnitude of the moment.

Problem 38. Determine the moments with respect to the coordinate axes of forces $\boldsymbol{P}$ and $\boldsymbol{Q}$ acting on a horizontal plate as shown in Fig. 103.

Solution. (1) Force $\boldsymbol{P}$ is parallel to axis z perpendicular to the x and y axes, and passes at a distance of $b/2$ and $a/2$ from them respectively. Hence, taking into account the signs,

$$m_x(\boldsymbol{P}) = -P\frac{b}{2}, \quad m_y(\boldsymbol{P}) = P\frac{a}{2}, \quad m_z(\boldsymbol{P}) = 0.$$

(2) To determine $m_x(\boldsymbol{Q})$, project force $\boldsymbol{Q}$ on the plane yz. We obtain:

$$Q_{yz} = Q\sin\alpha.$$

The moment arm of force $\boldsymbol{Q}_{yz}$ with respect to O is b and the rotation induced by it, when observed from the end of axis x, is counterclockwise. Consequently,

$$m_x(\boldsymbol{Q}) = bQ\sin\alpha.$$

Now let us determine $m_y(\boldsymbol{Q})$. Force $\boldsymbol{Q}$ lies in plane ABD, which is normal to axis y and intersects it at B. Consequently, $\boldsymbol{Q}_{xz} = \boldsymbol{Q}$.

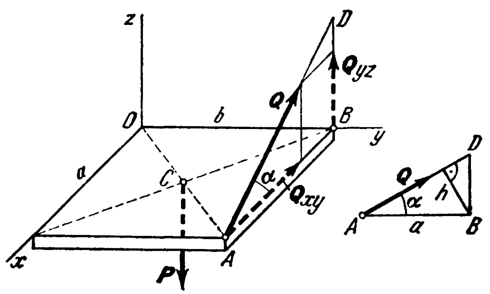

Fig. 103

Erecting a perpendicular from B to the line of action of $\boldsymbol{Q}$ (see supplementary diagram in Fig. 103), we find that it is of length $h = a\sin\alpha$. Finally, taking into account the sense of rotation, we obtain

$$m_y(\boldsymbol{Q}) = -Qa\sin\alpha.$$

To determine $m_z(\boldsymbol{Q})$ we project force $\boldsymbol{Q}$ on the plane xy and find that $Q_{xy} = Q\cos\alpha$ and that the moment arm of $\boldsymbol{Q}_{xy}$ with respect to O is b. Hence, taking into account the sign,

$$m_z(\boldsymbol{Q}) = bQ\cos\alpha.$$

Analytical Expression of the Moment of a Force About the Coordinate Axes. Consider a rectangular coordinate system with an arbitrary origin O (Fig. 104) and a force $\boldsymbol{F}$ applied at a point A whose coordinates are x, y, z. Let us compute analytically the moment of force $\boldsymbol{F}$ with respect to axis z. For this we project force $\boldsymbol{F}$ on the plane xy and resolve the projection $\boldsymbol{F}_{xy}$ into rectangular components $\boldsymbol{F}_x$ and $\boldsymbol{F}_y$. These components are, obviously, equal in magnitude to the projections of the force $\boldsymbol{F}$ on the x and y axes. But, from the definition,

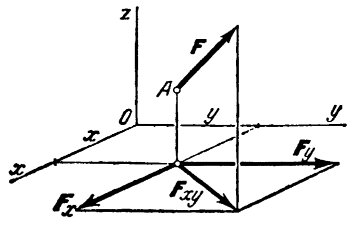

Fig. 104

$$m_z(\boldsymbol{F}) = m_O(\boldsymbol{F}_{xy}) = m_O(\boldsymbol{F}_x) + m_O(\boldsymbol{F}_y),$$

which also follows from Varignon's theorem. Also, from the diagram, $m_O(\boldsymbol{F}_x) = -yF_x$ and $m_O(\boldsymbol{F}_y) = xF_y$. Hence,

$$m_z(\boldsymbol{F}) = xF_y - yF_x.$$

We obtain the moments about the other two axes in the same way, and finally,

$$\left.\begin{aligned} m_x(\boldsymbol{F}) &= yF_z - zF_y, \\ m_y(\boldsymbol{F}) &= zF_x - xF_z, \\ m_z(\boldsymbol{F}) &= xF_y - yF_x. \end{aligned}\right\} \qquad (52)$$

Eqs. (52) give the *analytical expression* of the moments of a force about the axes of a Cartesian coordinate system. Using these equations, we can determine the moments if we know the projections of a force on the coordinate axes and the coordinates of its point of application.

Problem 39. Compute analytically the moments of force $\boldsymbol{Q}$ in Fig. 103 with respect to the coordinate axes.

Solution. Force $\boldsymbol{Q}$ is applied at a point A whose coordinates are $x = a$, $y = b$, $z = 0$. Its projections on the coordinate axes are

$$Q_x = -Q\cos\alpha,\ Q_y = 0,\ Q_z = Q\sin\alpha.$$

Substituting these expressions in Eqs. (52), we have:

$$m_x(\boldsymbol{Q}) = bQ\sin\alpha,\ m_y(\boldsymbol{Q}) = -aQ\sin\alpha,\ m_z(\boldsymbol{Q}) = bQ\cos\alpha.$$

§ 44. Relation Between the Moments of a Force About a Point and an Axis

Consider a force $\boldsymbol{F}$ acting on a body at a point A (Fig. 105). Let us draw an axis z and take an arbitrary point O on it. The moment of $\boldsymbol{F}$ about O is denoted by a vector $\boldsymbol{M}_O$, normal to plane OAB, of magnitude

$$M_O = Fh = 2 \text{ areas } \triangle\ OAB.$$

Drawing now a plane xy through a point O_1 normal to axis z; projecting force F on it, we find from Eq. (51) that

$$m_z(F) = m_{O_1}(F_{xy}) = 2 \text{ areas } \triangle\ O_1A_1B_1.$$

But triangle $O_1A_1B_1$ is the projection of triangle OAB on the plane xy. The angle between the planes of the two triangles is equal to the angle between the normals to the two planes, i.e., γ. Then, from the well-known geometrical formula, area $\triangle\ O_1A_1B_1$ = area $\triangle\ OAB \cos\gamma$.

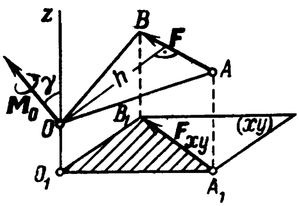

Fig. 105

Doubling both sides of the equation and observing that twice the areas of triangles $O_1A_1B_1$ and OAB are respectively equal to $m_z(F)$ and M_O, we obtain finally:

$$m_z(F) = M_O \cos\gamma. \tag{53}$$

As the product $M_O \cos\gamma$ gives the projection of vector $\boldsymbol{M}_O = \boldsymbol{m}_O(F)$ on axis z, Eq. (53) may also be written in the form:

$$m_z(F) = M_z \text{ or } m_z(F) = [\boldsymbol{m}_O(F)]_z. \tag{54}$$

Changing the position of O on the z-axis will, as is apparent from Fig. 105, affect the magnitude and direction of vector $\boldsymbol{M}_O$ (because triangle OAB changes); however, the projection of $\boldsymbol{M}_O$ on the z-axis, measured by the area of triangle $O_1A_1B_1$, remains the same.

We have thus proved the following relation between the moment of a force about an axis and its moment about a point on that axis: *The moment of a force F with respect to an axis is equal to the projection on that axis of the vector denoting the moment of that force with respect to any point on the given axis.*

It follows that the equations (50) in § 42 at the same time yield the analytical expressions of the force moments about the coordinate axes, i.e., Eqs. (52), as by this theorem $m_x(F) = M_x$, etc.

§ 45. Vector Expression of the Moment of a Couple

The action of a couple on a body is characterised by: (1) the magnitude of the moment of the couple, (2) the aspect of the plane of action, and (3) the sense of rotation in that plane. In considering couples in space all three characteristics must be specified in order to define any couple. This can be done if, by analogy with the moment of a force, *the moment of a couple is denoted by a vector* $\boldsymbol{m}$ *or*

$\boldsymbol{M}$ *whose modulus* (to some scale) *is equal to the magnitude of the moment of that couple, i.e., the product of one of its forces and the moment arm. The vector is normal to the plane of action of the couple in the direction from which the rotation induced by the couple would be observed as counterclockwise* (Fig. 106).

Since a couple may be located anywhere in its plane of action or in a parallel plane (see § 19), it follows that vector $\boldsymbol{m}$ can be attached to any point of the body (such a vector is known as a *free vector*).

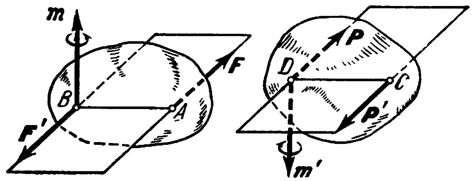

Fig. 106

It is evident that vector $\boldsymbol{m}$ does, in fact, define the given couple as, if we know $\boldsymbol{m}$, by passing an arbitrary plane normal to $\boldsymbol{m}$, we obtain the plane of action of the couple; by measuring the length of $\boldsymbol{m}$ we obtain the magnitude of the couple moment; and the direction of $\boldsymbol{m}$ shows the sense of rotation of the couple.

In magnitude the moment of a couple is equal to the moment of one of its forces with respect to the point of application of the other force (see § 18), i.e., $m = m_B(F)$; the two vectors have the same direction (compare Figs. 106 and 100). Consequently,

$$\boldsymbol{m} = \boldsymbol{m}_B(\boldsymbol{F}) = \boldsymbol{m}_A(\boldsymbol{F}').$$

§ 46*. Composition of Couples in Space. Conditions of Equilibrium of Couples

Couples in space are compounded according to the following theorem: *Any system of couples acting on a rigid body is equivalent to a single couple with a moment equal to the geometrical sum of the moments of the component couples.*

Let us first prove the theorem for two couples with moments $\boldsymbol{m}_1$ and $\boldsymbol{m}_2$ acting on a body in planes (*I*) and (*II*) (Fig. 107). Take on the line of intersection of the two planes a segment $AB = d$. By virtue of the properties of couples proved in § 19, we can represent the couple with a moment $\boldsymbol{m}_1$ in terms of forces $\boldsymbol{F}_1$ and $\boldsymbol{F}_1'$, and the couple with a moment $\boldsymbol{m}_2$ in terms of forces $\boldsymbol{F}_2$ and $\boldsymbol{F}_2'$, applied at points A and B respectively. Evidently, $F_1 d = m_1$ and $F_2 d = m_2$.

By compounding the forces applied at points A and B we ascertain that couples $(\boldsymbol{F}_1, \boldsymbol{F}_1')$ and $(\boldsymbol{F}_2, \boldsymbol{F}_2')$ are really replaced by $(\boldsymbol{R}, \boldsymbol{R}_1)$. Let us determine the moment $\boldsymbol{M}$ of this couple. As $\boldsymbol{R} = \boldsymbol{F}_1 + \boldsymbol{F}_2$, and the moment of a couple is equal to the moment of one of its forces with respect to the point of application of the other force, we obtain from Eq. (49):

$$\boldsymbol{M} = \overline{AB} \times \boldsymbol{R} = \overline{AB} \times (\boldsymbol{F}_1 + \boldsymbol{F}_2) = (\overline{AB} \times \boldsymbol{F}_1) + (\overline{AB} \times \boldsymbol{F}_2).$$

But $\overline{AB} \times \boldsymbol{F}_1 = \boldsymbol{m}_1$ and $\overline{AB} \times \boldsymbol{F}_2 = \boldsymbol{m}_2$. Hence,

$$\boldsymbol{M} = \boldsymbol{m}_1 + \boldsymbol{m}_2, \tag{55}$$

i.e., vector $\boldsymbol{M}$ is represented by the diagonal of a parallelogram with vectors $\boldsymbol{m}_1$ and $\boldsymbol{m}_2$ as its sides, which proves the theorem for two couples.

If there are n couples with moments $\boldsymbol{m}_1, \boldsymbol{m}_2, \ldots, \boldsymbol{m}_n$ acting on a body, we can apply Eq. (55) successively with the result that we

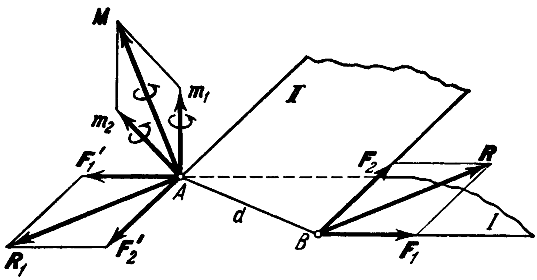

Fig. 107

shall have replaced a system of couples by a single couple with a moment

$$\boldsymbol{M} = \boldsymbol{m}_1 + \boldsymbol{m}_2 + \ldots + \boldsymbol{m}_n = \sum \boldsymbol{m}_k. \tag{56}$$

Vector $\boldsymbol{M}$ can be determined as the closing side of a polygon constructed with the component vectors as its sides.

If the component vectors are non-coplanar, the problem is best solved by the analytical method. For this draw a coordinate system. From the theorem of the projection of a vector sum on an axis, and from Eq. (56), we have:

$$M_x = \sum m_{kx}, \quad M_y = \sum m_{ky}, \quad M_z = \sum m_{kz}. \tag{57}$$

With these projections we can construct vector $\boldsymbol{M}$. Its magnitude is given by the expression

$$M = \sqrt{M_x^2 + M_y^2 + M_z^2}.$$

The above results readily give *the conditions for the equilibrium of a system of couples acting on a rigid body*. Any system of couples can be reduced to a single couple with a moment determined by Eq. (56), but for equilibrium we must have $\boldsymbol{M} = 0$, or

$$\sum \boldsymbol{m}_k = 0,$$

i.e., the polygon constructed with the moment vectors of the component couples acting on a body must be closed.

The analytical conditions of equilibrium can be found if we take into account that $M = 0$ only if $M_x = 0$, $M_y = 0$, and $M_z = 0$. This, by virtue of Eqs. (57), is possible if

$$\sum m_{kx} = 0, \quad \sum m_{ky} = 0, \quad \sum m_{kz} = 0. \tag{58}$$

In conclusion it should be noted that if all the couples lie in the same or in parallel planes, the vectors of their moments will be col-

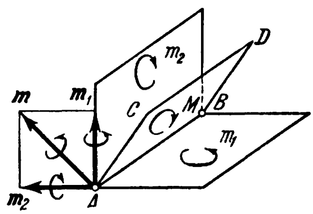

Fig. 108

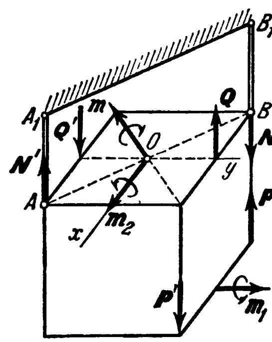

Fig. 109

linear and their composition is reduced to an algebraic operation. This result was obtained in § 20.

Problem 40. Acting on a rigid body are two couples in mutually perpendicular planes (Fig. 108). The moment of each is 3 kgf-m. Determine the resultant couple.

Solution. Denote the moments of the two couples by vectors $\boldsymbol{m}_1$ and $\boldsymbol{m}_2$ applied at an arbitrary point A; the moment of the resultant couple is denoted by the vector $\boldsymbol{m}$. The resultant couple is located in plane $ABCD$ normal to $\boldsymbol{m}$ and the magnitude of the resultant moment is $3\sqrt{2}$ kgf-m.

If the sense of rotation of one of the given couples is reversed, the resultant couple will occupy a plane normal to $ABCD$.

Problem 41. The cube in Fig. 109 hangs from two vertical rods AA_1 and BB_1 so that its diagonal AB is horizontal. Applied to the cube are couples $(\boldsymbol{P}, \boldsymbol{P}')$ and $(\boldsymbol{Q}, \boldsymbol{Q}')$. Neglecting the weight of the cube, determine the relation between forces P and Q at which it will be in equilibrium and the reactions of the rods.

Solution. The system of couples $(\boldsymbol{P}, \boldsymbol{P}')$ and $(\boldsymbol{Q}, \boldsymbol{Q}')$ is equivalent to a couple and can be balanced only by a couple. Hence, the required reactions $\boldsymbol{N}$ and $\boldsymbol{N}'$ must form a couple. Let us denote its moment $\boldsymbol{m}$ normal to diagonal AB as shown in the diagram. In magnitude $m = Na\sqrt{2}$, where a is the length of the edge of the cube. Denote the moments of the given couples by the symbols $\boldsymbol{m}_1$ and $\boldsymbol{m}_2$; their magnitudes are $m_1 = Pa$ and $m_2 = Qa$ and their directions are as shown.

Now draw a coordinate system and write the equilibrium equations (58):

$$\sum m_{kx} \equiv m_2 - m\cos 45^\circ = 0, \quad \sum m_{ky} \equiv m_1 - m\cos 45^\circ = 0.$$

The third condition is satisfied similarly.

It follows from the obtained equations that we must have $m_1 = m_2$, i.e., $Q = P$. We find, further, that

$$m = \frac{m_1}{\cos 45^\circ} = m_1\sqrt{2} = Pa\sqrt{2}.$$

But $m = Na\sqrt{2}$, hence $N = P$.

Thus, equilibrium is possible when $Q = P$. The reactions of the rods are equal to P in magnitude and are directed as shown.

§ 47. Reduction of a Force System in Space to a Given Centre

The results obtained above make it possible to solve the problem of reducing an arbitrary force system to a given centre. This problem, which is analogous to the one examined in § 22, is solved by applying the theorem of the translation of a force to a parallel position.

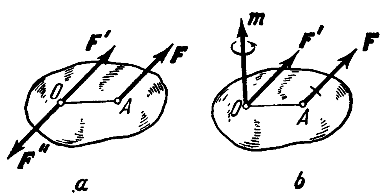

Fig. 110

In order to transfer a force $\boldsymbol{F}$ acting on a rigid body from a point A (Fig. 110a) to a point O, we apply at O forces $\boldsymbol{F}' = \boldsymbol{F}$ and $\boldsymbol{F}'' = -\boldsymbol{F}$. Force $\boldsymbol{F}' = \boldsymbol{F}$ will be applied at O together with the couple $(\boldsymbol{F}, \boldsymbol{F}'')$ with a moment $\boldsymbol{m}$, which can also be shown as in Fig. 110b. We have:

$$\boldsymbol{m} = \boldsymbol{m}_O(\boldsymbol{F}). \tag{59}$$

Consider now a rigid body on which an arbitrary system of forces $\boldsymbol{F}_1, \boldsymbol{F}_2, \ldots, \boldsymbol{F}_n$ is acting (Fig. 111a). Take any point O as the centre of reduction and transfer all the forces of the system to it, adding the corresponding couples. We have then acting on the body a system of forces

$$F'_1 = F_1,\ F'_2 = F_2,\ \ldots,\ F'_n = F_n \qquad (60)$$

applied at O and a system of couples whose moments, by Eq. (59), are:

$$m_1 = m_O(F_1),\ m_2 = m_O(F_2),\ \ldots,\ m_n = m_O(F_n). \qquad (61)$$

The forces applied at O can be replaced by a single force R applied at the same point. Its magnitude is $R = \sum F'_k$, or, by Eqs. (60),

$$R = \sum F_k. \qquad (62)$$

We can compound all the obtained couples by geometrically adding the vectors of their moments. The system of couples will be replaced by a couple of moment $M_O = \Sigma m_k$, or, by Eqs. (61),

$$M_O = \sum m_O(F_k). \qquad (63)$$

As in the case of a coplanar system, the quantity R (the geometrical sum of all the forces) is called the *principal vector of the system*;

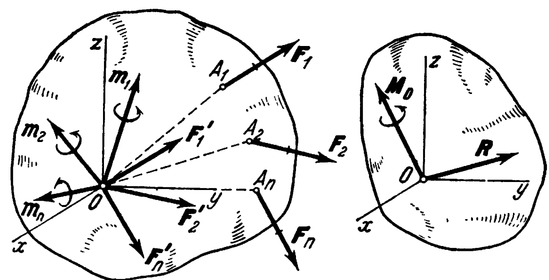

Fig. 111

vector M_O (the geometrical sum of the moments of all the forces with respect to O) is called the *principal moment of the system* with respect to O.

We have thus proved the following theorem: *Any system of forces acting on a rigid body can be reduced to an arbitrary centre O and replaced by a single force* R, *equal to the principal vector of the system applied at the centre of reduction, and a couple with a moment* M_O, *equal to the principal moment of the system with respect to O* (Fig. 111*b*).

Vectors R and M_O are usually determined analytically, i.e., according to their projections on the coordinate axes.

We know the expressions of R_x, R_y and R_z from § 10. We shall denote the projections of M_O on the coordinate axes by the symbols

M_x, M_y, M_z. From the theorem of the projection of a vector sum on an axis we have $M_x = \sum[m_O(F_k)]_x$, or, from Eq. (54), $M_x = \sum m_x(F_k)$. Similarly we obtain M_y and M_z.

Thus the formulas for determining the projections of the principal vector $\boldsymbol{R}$ and the principal moment $\boldsymbol{M}_O$ are:

$$R_x = \sum F_{kx}, \qquad R_y = \sum F_{ky}, \qquad R_z = \sum F_{kz}; \tag{64}$$

$$M_x = \sum m_x(\boldsymbol{F}_k), \quad M_y = \sum m_y(\boldsymbol{F}_k), \quad M_z = \sum m_z(\boldsymbol{F}_k). \tag{65}$$

It follows from this theorem that two systems of forces, for which $\boldsymbol{R}$ and $\boldsymbol{M}_O$ are the same, are statically equivalent. Hence, to define a force system acting on a rigid body it is sufficient to define its principal vector and its principal moment with respect to a given centre, i.e., to specify the six quantities in Eqs. (64) and (65).

§ 48*. Reduction of a Force System in Space to the Simplest Possible Form

The theorem proved in § 47 makes it possible to establish the simplest form to which a given force system in space can be reduced. For this it is necessary to determine the principal vector and the principal moment of the system with respect to an arbitrary point and investigate the result.

The following cases are possible:

(1) If $\boldsymbol{R} = 0$ and $\boldsymbol{M}_O = 0$, the system is in equilibrium. This case will be examined in § 49.

(2) If $\boldsymbol{R} = 0$ and $\boldsymbol{M}_O \neq 0$, the system can be reduced to a couple of moment computed according to Eqs. (65). In this case, as in the case of a coplanar force system (§ 23), M_O does not depend upon the choice of point O. A free body subjected to the action of such a force system can be (though not always) in pure rotational motion.

(3) If $\boldsymbol{R} \neq 0$ and $\boldsymbol{M}_O = 0$, the system can be reduced to a resultant $\boldsymbol{R}$ passing through point O. The magnitude of $\boldsymbol{R}$ is computed according to Eqs. (64).

A free body subjected to the action of such a force system can be in pure translatory motion (if the resultant $\boldsymbol{R}$ passes through the centre of gravity of the body).

(4) If $\boldsymbol{R} \neq 0$ and $\boldsymbol{M}_O \neq 0$ and $\boldsymbol{M}_O \perp \boldsymbol{R}$, the system can be reduced to a single resultant $\boldsymbol{R}$ not passing through point O.

For, if $\boldsymbol{M}_O \perp \boldsymbol{R}$, the couple denoted by vector $\boldsymbol{M}_O$ and the force $\boldsymbol{R}$ are coplanar (Fig. 112). If we take the couple forces $\boldsymbol{R}'$ and $\boldsymbol{R}''$ equal in magnitude to R and place them as shown in Fig. 112, we obtain forces $\boldsymbol{R}$ and $\boldsymbol{R}''$ which cancel each other, and the system is replaced by a resultant $\boldsymbol{R}' = \boldsymbol{R}$ through point O' [see § 23, case

(3b)]. The distance OO' ($\overline{OO'} \perp \boldsymbol{R}$) is determined from Eq. (31), where $d = OO'$.

It will be readily noticed that this case applies, in particular, to any system of parallel or coplanar forces whose principal vector $\boldsymbol{R} \neq 0$.

(5) If $\boldsymbol{R} \neq 0$ and $\boldsymbol{M}_O \neq 0$ and vector $\boldsymbol{M}_O$ is parallel to $\boldsymbol{R}$ (Fig. 113*a*), the system can be reduced to a resultant force $\boldsymbol{R}$ and a couple ($\boldsymbol{P}$, $\boldsymbol{P}'$) in a plane normal to the resultant force (Fig. 113*b*). Such a force-and-couple system is called a *wrench*; the line of action of force $\boldsymbol{R}$ is the *axis of the wrench*. No further reduction of the system is possible. For, though the system will not change if the couple is transferred, if force $\boldsymbol{R}$ is transferred from centre O to any point C (Fig. 113*a*), there will be added to the moment $\boldsymbol{M}_O$ a moment $\boldsymbol{M}'_C = \boldsymbol{m}_C(\boldsymbol{R})$, perpendicular to $\boldsymbol{R}$ and, hence, to $\boldsymbol{M}_O$. The moment $\boldsymbol{M}_C = \boldsymbol{M}_O + \boldsymbol{M}'_C$ of the resulting couple will increase. Thus, this type of force system cannot be reduced to a single resultant or to a single couple.

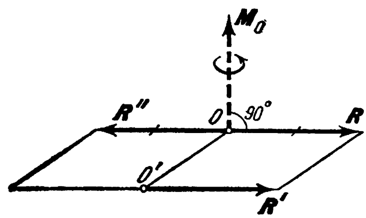

Fig. 112

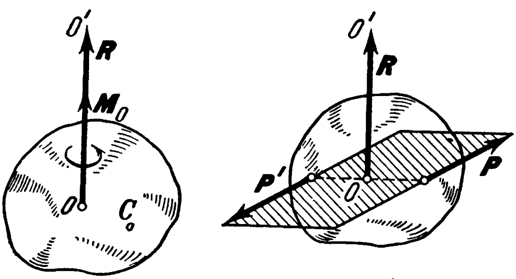

Fig. 113

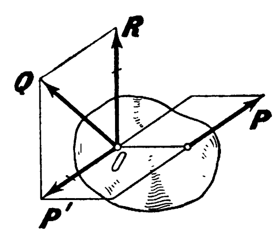

Fig. 114

A free rigid body subjected to the action of such a force system can perform only a compound (screw) motion.

If one of the forces of the couple, say $\boldsymbol{P}'$, is added to force $\boldsymbol{R}$, the given system can be replaced by two non-coplanar forces $\boldsymbol{Q}$ and $\boldsymbol{P}$ (Fig. 114). As the new force system is equivalent to a wrench, it also has no resultant.

(6) If $\boldsymbol{R} \neq 0$ and $\boldsymbol{M}_O \neq 0$ and vectors $\boldsymbol{M}_O$ and $\boldsymbol{R}$ are neither perpendicular nor parallel, the system can be reduced to a wrench whose axis does not pass through point O.

To prove this, let us resolve vector $\boldsymbol{M}_O$ into two components, $\boldsymbol{M}_1$ along $\boldsymbol{R}$ and $\boldsymbol{M}_2$ perpendicular to $\boldsymbol{R}$ (Fig. 115). We have $M_1 =$

$= M_O \cos\alpha$ and $M_2 = M_O \sin\alpha$, where α is the angle between vectors $\boldsymbol{M}_O$ and $\boldsymbol{R}$. The couple denoted by the vector $\boldsymbol{M}_2$ ($\boldsymbol{M}_2 \perp \boldsymbol{R}$) and force $\boldsymbol{R}$ can be replaced, as in the case shown in Fig. 112, with

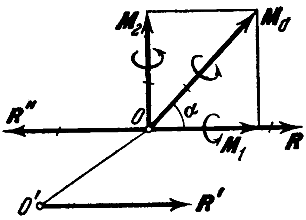

Fig. 115

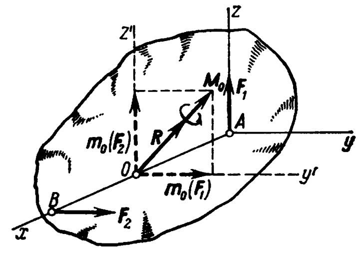

Fig. 116

a single resultant $\boldsymbol{R}'$ applied at point O'. As a result, the given system is replaced by a force $\boldsymbol{R}' = \boldsymbol{R}$ and a couple of moment $\boldsymbol{M}_1$ parallel to $\boldsymbol{R}'$, i.e., a wrench whose axis passes through point O'.

Problem 42. Determine the form to which the force system $\boldsymbol{F}_1$, $\boldsymbol{F}_2$ in Fig. 6 can be reduced, if $F_1 = F_2 = F$ and $AB = 2a$.

Solution. Reduce the forces $\boldsymbol{F}_1$, $\boldsymbol{F}_2$ with respect to point O in the middle of segment AB (Fig. 116). The principal vector of the system $\boldsymbol{R} = \boldsymbol{F}_1 + \boldsymbol{F}_2$ is directed along the bisector of angle $y'Oz'$, and its magnitude $R = F\sqrt{2}$. The principal moment of the system $\boldsymbol{M}_O = \boldsymbol{m}_O(\boldsymbol{F}_1) + \boldsymbol{m}_O(\boldsymbol{F}_2)$. Vector $\boldsymbol{m}_O(\boldsymbol{F}_1)$ is directed along axis y', and vector $\boldsymbol{m}_O(\boldsymbol{F}_2)$ along axis z'; in magnitude both are equal to Fa. Consequently, in magnitude $M_O = Fa\sqrt{2}$, and vector $\boldsymbol{M}_O$ is also directed along the bisector of angle $y'Oz'$. Thus, the system of forces $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ is reduced to a wrench and, as mentioned in § 3, has no resultant.

§ 49. Conditions of Equilibrium of an Arbitrary Force System in Space. The Case of Parallel Forces

Like a coplanar force system, any force system in space can be reduced to a point O and replaced by a resultant force $\boldsymbol{R}$ and couple of moment $\boldsymbol{M}_O$ [the values of $\boldsymbol{R}$ and $\boldsymbol{M}_O$ are determined from Eqs. (62) and (63)]. Reasoning as in the beginning of § 24, we come to the conclusion that the necessary and sufficient conditions for the given system of forces to be in equilibrium are that $\boldsymbol{R} = 0$ and $\boldsymbol{M}_O = 0$. But vectors $\boldsymbol{R}$ and $\boldsymbol{M}_O$ can be zero only if all their projections on the coordinate axes are zero, i.e., when $R_x = R_y = R_z = 0$ and $M_x = M_y = M_z = 0$, or, by Eqs. (64) and (65), when the acting forces satisfy the conditions

$$\left.\begin{array}{lll}\sum F_{kx}=0, & \sum F_{ky}=0, & \sum F_{kz}=0; \\ \sum m_x(\boldsymbol{F}_k)=0, & \sum m_y(\boldsymbol{F}_k)=0, & \sum m_z(\boldsymbol{F}_k)=0.\end{array}\right\} \quad (66)$$

Thus, *the necessary and sufficient conditions for the equilibrium of any force system in space are that the sums of the projections of all the forces on each of the three coordinate axes and the sums of the moments of all the forces about those axes must separately vanish**).

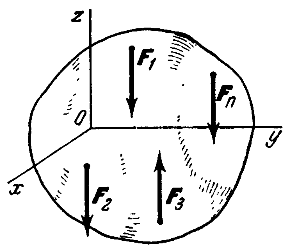

Fig. 117

Eqs. (66) express the simultaneously necessary conditions for the equilibrium of any rigid body subjected to the action of any force system in space. The first three of the equations express the conditions necessary for the body to have no translatory motion parallel to the coordinate axes; the latter three equations express the conditions of no rotation about the axes.

If, besides forces, there is also a couple defined by its moment $\boldsymbol{M}$ acting on a body, the form of the first three of Eqs. (66) will remain the same (the sum of the projections of the forces of a couple on any axis is zero), but the last three equations will take the following form:

$$\sum m_x(\boldsymbol{F}_k)+M_x=0, \quad \sum m_y(\boldsymbol{F}_k)+M_y=0, \quad \sum m_z(\boldsymbol{F}_k)+M_z=0. \quad (67)$$

The Case of Parallel Forces. If all the forces acting on a body are parallel, the coordinate axes can be chosen so that the axis z is parallel to the forces (Fig. 117). Then the x and y projections of all the forces will be zero, their moments about axis z will be zero, and the Eqs. (66) will be reduced to three conditions of equilibrium:

$$\sum F_{kz}=0, \quad \sum m_x(\boldsymbol{F}_k)=0, \quad \sum m_y(\boldsymbol{F}_k)=0. \quad (68)$$

The other equations will turn into identities $0 = 0$.

Thus, *the necessary and sufficient conditions for the equilibrium of a system of parallel forces in space are that the sum of the projections of all the forces on the coordinate axis parallel to the forces and the sums of the moments of all the forces about the other two coordinate axes must separately vanish.*

---

*) In writing conditions (66) you may take, if you find it expedient, one coordinate system to compute the force projections and another to compute the moments.

§ 50. Varignon's Theorem of the Moment of a Resultant With Respect to an Axis

Let there be acting on a rigid body a force system $\boldsymbol{F}_1, \boldsymbol{F}_2, \ldots, \boldsymbol{F}_n$ which can be reduced to a resultant $\boldsymbol{R}$ whose action line passes through any point C (Fig. 118). Let us apply at the same point a force $\boldsymbol{R}' = -\boldsymbol{R}$. The system $\boldsymbol{F}_1, \boldsymbol{F}_2, \ldots, \boldsymbol{F}_n, \boldsymbol{R}'$ will now be in equilibrium and will satisfy all the conditions (66). In particular, for any coordinate axis Ox we shall have:

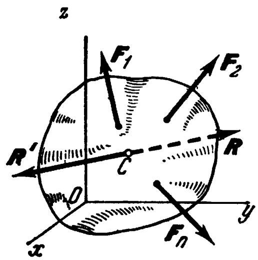

Fig. 118

$$\sum m_x(\boldsymbol{F}_k) + m_x(\boldsymbol{R}') = 0.$$

But as $\boldsymbol{R}' = -\boldsymbol{R}$ and both forces are collinear, it follows that $m_x(\boldsymbol{R}') = -m_x(\boldsymbol{R})$. Substituting this expression of $m_x(\boldsymbol{R}')$ in the previous equation, we find that

$$m_x(\boldsymbol{R}) = \sum m_x(\boldsymbol{F}_k). \qquad (69)$$

Thus, *if a given force system has a resultant, the moment of that resultant with respect to any axis is equal to the algebraic sum of the moments of the component forces with respect to the same axis* (Varignon's Theorem).

§ 51. Problems on Equilibrium of Bodies Subjected to Action of Force Systems in Space

The principle of solving the problems of this section is the same as for coplanar force systems. After isolating the body whose equilibrium has to be considered, the constraints attached to it are replaced by their reactions, the equilibrium equations are written as for a free body, and the required quantities are obtained.

In order to obtain simpler sets of equations, the student is advised to draw the coordinate axes so that they would intersect with, or be perpendicular, to, as many unknown forces as possible (if this does not complicate the computation of the projections and moments of other forces).

A new feature in writing equations is the computation of the moments of the forces with respect to the coordinate axes.

If, while examining the general diagram, the student finds difficulty in determining the moment of some force with respect to an axis, he is advised to draw an auxiliary diagram showing the projections of the given body and the required force on a plane normal to the axis under consideration.

If, in computing a moment, there is difficulty in determining the projection of a force on any plane, or the moment arm of the projection, resolve the force into two rectangular components (one of them parallel to a coordinate axis) and then apply Varignon's theorem. The moments can also be computed analytically from Eqs. (52).

Problem 43. Three workers lift a homogeneous rectangular plate whose dimensions are a by b (Fig. 119). If one worker is at A, determine the points B and D where the other two workers should stand so that they would all exert the same force.

Solution. The plate is a free body acted upon by four parallel forces $\boldsymbol{Q}_1$, $\boldsymbol{Q}_2$, $\boldsymbol{Q}_3$, and $\boldsymbol{P}$, where $\boldsymbol{P}$ is the weight of the plate. Assuming that the plate is horizontal and drawing the coordinate axes

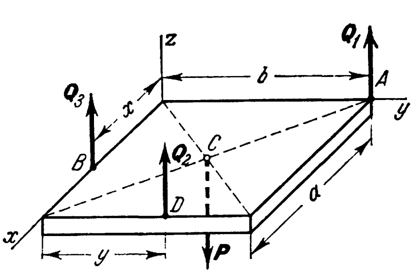

Fig. 119

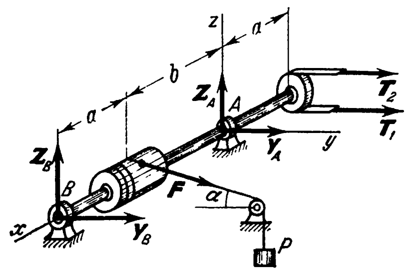

Fig. 120

as shown in the figure, we obtain from the equilibrium conditions (68):

$$Q_1 b + Q_2 y - P\frac{b}{2} = 0,$$

$$-Q_2 a - Q_3 x + P\frac{a}{2} = 0,$$

$$Q_1 + Q_2 + Q_3 = P.$$

According to the conditions of the problem, $Q_1 = Q_2 = Q_3 = Q$, hence, from the last equation, $P = 3Q$. Substituting this expression in the first two equations and eliminating Q, we have:

$$b + y = \frac{3}{2}b, \quad a + x = \frac{3}{2}a,$$

whence

$$x = \frac{a}{2}, \quad y = \frac{b}{2}.$$

Problem 44. A horizontal shaft supported in bearings A and B (Fig. 120) has attached at right angles to it a pulley of radius $r_1 =$

$= 20$ cm and a drum of radius $r_2 = 15$ cm. The shaft is driven by a belt passing over the pulley; attached to a cable wound on the drum is a load of weight $P = 180$ kgf, which is lifted with uniform motion when the shaft turns. Neglecting the weight of the construction, determine the reactions of the bearings and the tension T_1 in the driving portion of the belt, if it is known that it is double the tension T_2 in the driven portion and if $a = 40$ cm, $b = 60$ cm, and $\alpha = 30°$.

Solution. As the shaft rotates uniformly, the forces acting on it are in equilibrium and the equations of equilibrium can be applied. Let us write the equilibrium equations for these forces. Drawing the coordinate axes as shown and regarding the shaft as a free body, denote the forces acting on it: the tension $\boldsymbol{F}$ of the cable, which is equal to P in magnitude, the tensions $\boldsymbol{T}_1$ and $\boldsymbol{T}_2$ in the belt, and the reactions $\boldsymbol{Y}_A$, $\boldsymbol{Z}_A$, $\boldsymbol{Y}_B$, and $\boldsymbol{Z}_B$ of the bearings (each of the reactions $\boldsymbol{R}_A$ and $\boldsymbol{R}_B$ can have any direction in planes normal to axis x and they are therefore denoted by their rectangular components).

To write the equilibrium equations (66), calculate the projections of all the forces on, and their moments about, the coordinate axes (see table); as the x projections of all the forces are zero, they have been omitted*).

$\boldsymbol{F}_k$	$\boldsymbol{F}$	$\boldsymbol{T}_1$	$\boldsymbol{T}_2$	$\boldsymbol{R}_A$	$\boldsymbol{R}_B$
F_{ky}	$F\cos\alpha$	T_1	T_2	Y_A	Y_B
F_{kz}	$-F\sin\alpha$	0	0	Z_A	Z_B
$m_x(\boldsymbol{F}_k)$	$-Fr_2$	T_1r_1	$-T_2r_1$	0	0
$m_y(\boldsymbol{F}_k)$	$F\sin\alpha\cdot b$	0	0	0	$-Z_B(a+b)$
$m_z(\boldsymbol{F}_k)$	$F\cos\alpha\cdot b$ 7	$-T_1a$	$-T_2a$	0	$Y_B(a+b)$

*) Tables will be found especially convenient in solving the problems of this chapter. The table is filled column by column, i.e., first the projections and moments of force $\boldsymbol{F}$ are computed, then of force $\boldsymbol{T}_1$, etc. In this way attention is focused on every force in succession. If we were to write the equilibrium equations (66) at once, we should have to return to each force six times with a correspondingly greater chance of making a mistake—especially of omitting some force in this or that equation.

From the equilibrium equations, and noting that $F = P$, we obtain:

$$P\cos\alpha + T_1 + T_2 + Y_A + Y_B = 0, \quad \text{(I)}$$

$$-P\sin\alpha + Z_A + Z_B = 0, \quad \text{(II)}$$

$$-r_2P + r_1T_1 - r_1T_2 = 0, \quad \text{(III)}$$

$$bP\sin\alpha - (a+b)Z_B = 0, \quad \text{(IV)}$$

$$bP\cos\alpha - aT_1 - aT_2 + (a+b)Y_B = 0. \quad \text{(V)}$$

Remembering that $T_1 = 2T_2$, we find immediately from equations (III) and (IV) that

$$T_2 = \frac{r_2P}{r_1} = 135 \text{ kgf},$$

$$Z_B = \frac{bP}{a+b}\sin\alpha = 54 \text{ kgf}.$$

From equation (V) we obtain

$$Y_B = \frac{3aT_2 - bP\cos\alpha}{a+b} \approx 69 \text{ kgf}.$$

Substituting these values in the other equations we find:

$$Y_A = -P\cos\alpha - 3T_2 - Y_B \approx -630 \text{ kgf},$$

$$Z_A = P\sin\alpha - Z_B = 36 \text{ kgf},$$

and finally,

$$T_1 = 270 \text{ kgf},$$

$$Y_A \approx -630 \text{ kgf},$$

$$Z_A = 36 \text{ kgf}, \quad Y_B \approx 69 \text{ kgf},$$

$$Z_B = 54 \text{ kgf}.$$

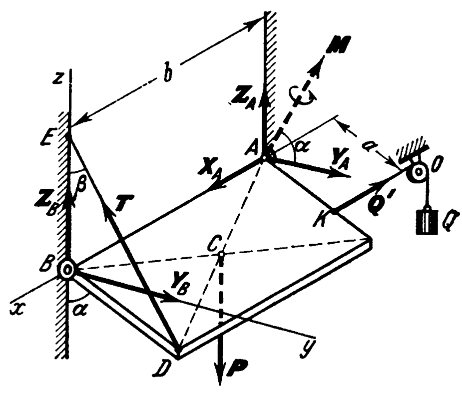

Fig. 121

Problem 45. A rectangular plate of weight $P = 12$ kgf making an angle $\alpha = 60°$ with the vertical is supported by a journal bearing at B and a step bearing at A (Fig. 121). The plate is kept in equilibrium by the action of a string DE; acting on the plate is a load $Q = 20$ kgf suspended by a string passing over pulley O and attached at K so that KO is parallel to AB. Determine the tension in string DE and the reactions of the bearings A and B, if $BD = BE$, $AK = a = 0.4$ m, and $AB = b = 1$ m.

Solution. Consider the equilibrium of the plate as a free body. Draw the coordinate axes with the origin at B (in which case force $\boldsymbol{T}$ intersects with the y and z axes, which simplifies the moment equations) and the acting forces and the reactions of the constraints as shown (the dashed vector $\boldsymbol{M}$ belongs to a different problem). For the equilibrium equations, calculate the projections and moments

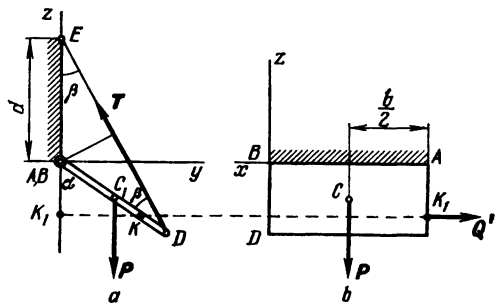

Fig. 122

of all the forces; for this we introduce angle β and denote $d = BD = BE$ (see table). Computation of some of the moments is explained in the auxiliary diagrams (Fig. 122*a* and *b*).

$\boldsymbol{F}_k$	$\boldsymbol{P}$	$\boldsymbol{Q}'$	$\boldsymbol{T}$	$\boldsymbol{R}_A$	$\boldsymbol{R}_B$
F_{kx}	0	$-Q'$	0	X_A	0
F_{ky}	0	0	$-T\sin\beta$	Y_A	Y_B
F_{kz}	$-P$	0	$T\cos\beta$	Z_A	Z_B
$m_x(\boldsymbol{F}_k)$	$-P\frac{d}{2}\sin\alpha$	0	$Td\sin\beta$	0	0
$m_y(\boldsymbol{F}_k)$	$-P\frac{b}{2}$	$Q'a\cos\alpha$	0	Z_Ab	0
$m_z(\boldsymbol{F}_k)$	0	$Q'a\sin\alpha$	0	$-Y_Ab$	0

Fig. 122*a* shows the projection on plane *Byz* from the positive end of axis *x*. This diagram is useful in computing the moments of forces $\boldsymbol{P}$ and $\boldsymbol{T}$ about axis *x*. It can be seen from the diagram that the projections of these forces on the *yz* plane are equal to the forces, and that the moment arm of force $\boldsymbol{P}$ with respect to point *B* is $BC_1 \sin\alpha = \frac{d}{2}\sin\alpha$; the moment arm of force $\boldsymbol{T}$ with respect to point *B* is $BE\sin\beta = d\sin\beta$.

Fig. 122*b* shows the projection on plane *Bxz* from the positive end of axis *y*. This diagram, together with Fig. 122*a*, helps to compute the moments of forces $\boldsymbol{P}$ and $\boldsymbol{Q}'$ about axis *y*. It can be seen that the projections of these forces on the *xz* plane are equal to the forces themselves and that the moment arm of force $\boldsymbol{P}$ with respect to point

B is $1/2\, AB = \frac{b}{2}$; the arm of force Q' with respect to B is AK_1, i.e., $AK \cos\alpha$, or $a \cos\alpha$, as is evident from Fig. 122*a*.

Writing the equilibrium equations and assuming $Q' = Q$, we obtain:

$$-Q + X_A = 0, \quad \text{(I)}$$

$$-T \sin\beta + Y_A + Y_B = 0, \quad \text{(II)}$$

$$-P + T \cos\beta + Z_A + Z_B = 0, \quad \text{(III)}$$

$$-P \frac{d}{2} \sin\alpha + Td \sin\beta = 0, \quad \text{(IV)}$$

$$-P \frac{b}{2} + Qa \cos\alpha + Z_A b = 0, \quad \text{(V)}$$

$$Qa \sin\alpha - Y_A b = 0. \quad \text{(VI)}$$

Taking into account that $\beta = \frac{\alpha}{2} = 30°$, we find from equations (I), (IV), (V), and (VI) that

$$X_A = Q = 20 \text{ kgf}, \qquad T = P \frac{\sin\alpha}{2\sin\beta} \approx 10.4 \text{ kgf},$$

$$Z_A = \frac{P}{2} - \frac{Qa}{b} \cos\alpha = 2 \text{ kgf}, \quad Y_A = \frac{Qa}{b} \sin\alpha \approx 6.9 \text{ kgf}.$$

Substituting these values into equations (II) and (III), we obtain: $Y_B = T \sin\beta - Y_A = -1.7$ kgf, $Z_B = P - T\cos\beta - Z_A = 1$ kgf, and finally,

$$T \approx 10.4 \text{ kgf}, \ X_A = 20 \text{ kgf}, \ Y_A \approx 6.9 \text{ kgf},$$

$$Z_A = 2 \text{ kgf}, \ Y_B = -1.7 \text{ kgf}, \ Z_B = 1 \text{ kgf}.$$

Problem 46. Solve Problem 45 for the case when the plate is additionally subjected to a couple of moment $M = 12$ kgf-m acting in the plane of the plate; the sense of rotation (viewed from the top of the plate) is counterclockwise.

Solution. Add to the forces in Fig. 121 the moment vector $\boldsymbol{M}$ of the couple applied at any point perpendicular to the plate, e.g., point A. Its projections are: $M_x = 0$, $M_y = M \cos\alpha$, and $M_z = M \sin\alpha$. Applying the equilibrium equations (67), we find that equations (I) to (IV) remain the same as for Problem 45 while the last two equations will be:

$$-P \frac{b}{2} + Z_A b + Qa \cos\alpha + M \cos\alpha = 0, \quad \text{(V')}$$

$$-Y_A b + Qa \sin\alpha + M \sin\alpha = 0. \quad \text{(VI')}$$

Note that the same result can be obtained without writing Eqs. (67), by denoting the couple as two forces directed, for example, along AB and KO (their magnitudes will, apparently, be M/a) and applying the usual equilibrium equations.

Solving equations (I)-(IV), (V′), and (VI′), we obtain results similar to those in Problem 45, the only difference being that in all equations Qa will be replaced by $Qa + M$. The answer is:

$$T \approx 10.4 \text{ kgf}, \ X_A = 20 \text{ kgf}, \ Y_A \approx 17.3 \text{ kgf},$$

$$Z_A = -4 \text{ kgf}, \ Y_B = -12.1 \text{ kgf}, \ Z_B = 7 \text{ kgf}.$$

Problem 47. A horizontal rod AB is attached to a wall by a ball-and-socket joint and is kept perpendicular to the wall by wires KE and CD as shown in Fig. 123a. Hanging from end B is a load of

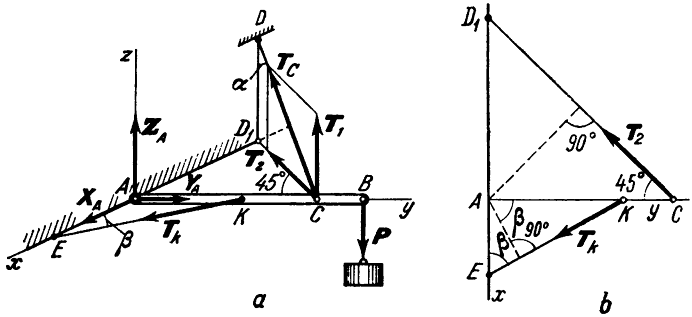

Fig. 123

weight $P = 36$ kgf. Determine the reaction of the ball-and-socket joint A and the tensions in the wires if $AB = a = 0.8$ m, $AC = AD_1 = b = 0.6$ m, $AK = \frac{a}{2}$, $\alpha = 30°$, and $\beta = 60°$. Neglect the weight of the rod.

Solution. Consider the equilibrium of the rod as a free body. Acting on it are force $\boldsymbol{P}$ and reactions $\boldsymbol{T}_K$, $\boldsymbol{T}_C$, $\boldsymbol{X}_A$, $\boldsymbol{Y}_A$, and $\boldsymbol{Z}_A$. Draw the coordinate axes and calculate the projections and moments of all the forces (see table).

As all the forces pass through axis y, their moments with respect to it are zero. To compute the moments of force $\boldsymbol{T}_C$ with respect to the coordinate axes, resolve it into components $\boldsymbol{T}_1$ and $\boldsymbol{T}_2$ ($T_1 = T_C \cos\alpha$, $T_2 = T_C \sin\alpha$) and apply Varignon's theorem*). We have $m_x(\boldsymbol{T}_C) = m_x(\boldsymbol{T}_1)$, as $m_x(\boldsymbol{T}_2) = 0$, and $m_z(\boldsymbol{T}_C) = m_z(\boldsymbol{T}_2)$, as $m_z(\boldsymbol{T}_1) = 0$. The computation of the moments of the forces with respect to axis z is explained in the auxiliary diagram (Fig. 123b) showing the projection on plane Axy.

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*) Attention is drawn to the fact that the angle between force T_C and plane Ayz is not 45°, as is sometimes erroneously assumed in such cases. Therefore, for example, in computing $m_x(T_C)$ in the usual way, it is first necessary to determine the angle, which complicates the computations. With the aid of Varignon's theorem, however, we find immediately that $m_x(T_C) = m_x(T_1) = T_1 \cdot AC$.

F_k	P	T_K	T_C	R_A
F_{kx}	0	$T_K \cos\beta$	$-T_2 \sin 45°$	X_A
F_{ky}	0	$-T_K \sin\beta$	$-T_2 \cos 45°$	Y_A
F_{kz}	$-P$	0	T_1	Z_A
$m_x(F_k)$	$-Pa$	0	$T_1 b$	0
$m_z(F_k)$	0	$-T_K \frac{a}{2} \cos\beta$	$T_2 b \sin 45°$	0

Substituting the values of T_1 and T_2 we obtain the following equations:

$$T_K \cos\beta - T_C \sin\alpha \sin 45° + X_A = 0, \quad \text{(I)}$$

$$-T_K \sin\beta - T_C \sin\alpha \cos 45° + Y_A = 0, \quad \text{(II)}$$

$$-P + T_C \cos\alpha + Z_A = 0, \quad \text{(III)}$$

$$-Pa + T_C b \cos\alpha = 0, \quad \text{(IV)}$$

$$-T_K \frac{a}{2} \cos\beta + T_C b \sin\alpha \sin 45° = 0, \quad \text{(V)}$$

solving which we find that $T_C \approx 55.4$ kgf, $T_K \approx 58.8$ kgf, $X_A \approx -9.8$ kgf, $Y_A \approx 70.5$ kgf, and $Z_A = -12$ kgf. Components X_A and Z_A thus actually act in the opposite direction to than shown in the diagram.

Problem 48. An equilateral triangular plate with sides of length a is supported in a horizontal plane by six bars as shown in Fig. 124. Each inclined bar makes an angle of $\alpha = 30°$ with the horizontal. Acting on the plate is a couple with a moment $\boldsymbol{M}$. Neglecting the weight of the plate, determine the stresses produced in the bars.

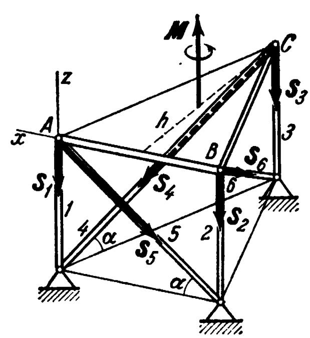

Fig. 124

Solution. Regarding the plate as a free body, draw, as shown in the figure, the vector of moment $\boldsymbol{M}$ of the couple and the reactions of the bars $S_1, S_2, \ldots, S_6$. Direct the reactions as if all the bars were in tension (i.e., we assume that the plate is being separated from the bars). As the body is in equilibrium, the sums of the moments of all the forces and couples acting on it with respect to any axis must be zero [see Eqs. (67)].

Drawing axis z along bar *1* and writing the equations of the moment with respect to that axis, we obtain, as $M_z = M$,

$$(S_6 \cos \alpha)\, h + M = 0,$$

where $h = \frac{a\sqrt{3}}{2}$ is the altitude of the triangle. From this we find:

$$S_6 = -\frac{2\sqrt{3}}{3} \frac{M}{a \cos \alpha}.$$

Writing the equations of the moments with respect to the axes along bars *2* and *3*, we obtain similar results for forces S_4 and S_5.

Now write the equations of the moments about axis x, which is directed along side AB of the triangle. Taking into account that $M_x = 0$, we obtain

$$S_3 h + (S_4 \sin \alpha)\, h = 0,$$

whence, as $S_4 = S_6$, we find

$$S_3 = -S_4 \sin \alpha = \frac{2\sqrt{3}}{3} \frac{M}{a} \tan \alpha.$$

Writing the moment equations with respect to axes AC and BC, we obtain similar results for S_1 and S_2.

Finally, for $\alpha = 30°$, we have:

$$S_1 = S_2 = S_3 = \frac{2}{3} \frac{M}{a}; \qquad S_4 = S_5 = S_6 = -\frac{4}{3} \frac{M}{a}.$$

The answer shows that the given couple creates tensions in the vertical bars and compressions in the inclined ones.

This solution suggests that it is not always necessary to apply equilibrium equations (66). There are several forms of equilibrium equations for non-coplanar force systems, just as for coplanar systems, of which Eqs. (66) are the principal form.

In particular, it can be proved that the necessary and sufficient conditions for the equilibrium of a force system in space are that the sums of the moments of all the forces with respect to each of six axes directed along the edges of any triangular pyramid or along the side and base edges of a triangular prism are each zero.

The latter conditions were applied in solving Problem 48.

Problem 49. Determine the stresses in section AA_1 of the multiple-loaded beam in Fig. 125*a*, where force $\boldsymbol{Q}$ is through the centre of the right-hand portion of the beam, force $\boldsymbol{F}$ is in plane Oxz, and force $\boldsymbol{P}$

i- parallel to axis Oy. The length of the right-hand portion of the beam is b, and the height h.

Solution. The required stresses are determined as in Problem 29 (§ 27). Draw a section AA_1 and consider the equilibrium of the right-hand portion of the beam (Fig. 125*b*); since there are no external constraints acting on it there is no need to consider the equilibrium of the beam as a whole to determine their reactions, as was the case in Problem 29. Draw the coordinate axes x, y, z through the

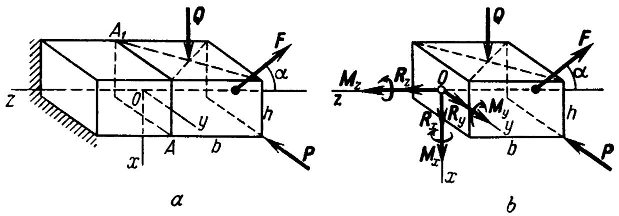

Fig. 125

centre O of the section. The action of the discarded portion of the beam can be replaced by an unknown spatial system of distributed forces in turn equivalent, as follows from § 47, to a force $\boldsymbol{R}$ applied at O, with unknown projections R_x, R_y, R_z, and a moment $\boldsymbol{M}_O$ whose projections M_x, M_y, M_z are also unknown. Drawing these forces and moments (Fig. 125*b*) and writing the first three equations (66) and Eqs. (67) for all the forces and couples acting on the right-hand portion of the beam, we obtain:

$$R_x - F\sin\alpha + Q = 0, \quad R_y - P = 0, \quad R_z - F\cos\alpha = 0;$$

$$M_x - bP = 0, \quad M_y + bF\sin\alpha - \frac{b}{2}Q = 0, \quad M_z - \frac{h}{2}P = 0.$$

Solving these equations yields:

$$R_x = F\sin\alpha - Q, \qquad R_y = P, \qquad R_z = F\cos\alpha;$$

$$M_x = bP, \qquad M_y = \frac{b}{2}Q - bF\sin\alpha, \qquad M_z = \frac{h}{2}P.$$

Thus, acting on cut AA_1 are: two lateral forces of magnitude R_x and R_y, a longitudinal tensile force of magnitude R_z, and three couples with moments M_x, M_y, and M_z, the first two bending the beam about the x and y axes, respectively, and the third creating a torque about axis z.

§ 52*. Conditions of Equilibrium of a Constrained Rigid Body. Concept of Stability of Equilibrium

In §§ 11, 24, 49 and others we obtained equations specifying the necessary conditions for the equilibrium of *free rigid bodies.* With regard to constrained bodies, these equations are applied on the basis of the axiom of constraints. The resulting equations define the reactions of the constraints.

The question of the conditions for the equilibrium of a constrained rigid body arises when its supports do not constrain it rigidly (see Problems 6 and 7 in § 13 and others). In this case only some of the

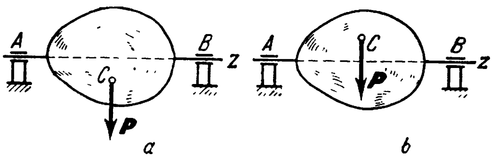

Fig. 126

equations based on the axiom of constraints contain the reactions of the constraints and can define those reactions. The other equations show the relations between the given forces (Problem 6) or the specific position (Problem 7) in which equilibrium is possible, i.e., they specify the conditions of equilibrium for the given body. Thus, *for the case of a constrained solid body, the conditions of equilibrium are specified only by those equations based on the axiom of constraints which do not include the reactions of the constraints.*

For instance, applying the axiom of constraints to a body with a fixed axis of rotation (Fig. 126) and writing Eqs. (66), we find that the reactions of bearings A and B are present in all the equations except the last one (see Problem 44). The reactions are absent from the equation $\Sigma m_z(\boldsymbol{F}_k) = 0$ as they intersect with axis z.

Hence, *the condition for the equilibrium of a body with a fixed axis of rotation is that the sum of the moments of all the acting forces with respect to that axis is zero*:

$$\Sigma m_z(\boldsymbol{F}_k) = 0.$$

For a body whose supports do not constrain it rigidly, the important question of *stability of equilibrium* arises. If the acting forces tend to return a body to its configuration of equilibrium (when it has been displaced from that configuration), the equilibrium is said to be *stable*; otherwise it is *unstable.* In real conditions a body can be in equilibrium only if it is in stable equilibrium.

Consider, for example, a body mounted on a horizontal axis. It will be in equilibrium under the action of the force of gravity $\boldsymbol{P}$ when $m_z(\boldsymbol{P}) = 0$, i.e., when the centre of gravity C of the body is in its lowest (Fig. 126*a*) or highest (Fig. 126*b*) configuration. In the first case, if the body is displaced, the moment of force $\boldsymbol{P}$ tends to return it to its position of equilibrium. In the second case, the slightest displacement of C will cause the moment of $\boldsymbol{P}$ to increase that displacement. Consequently, *equilibrium is stable if the centre of gravity of a body occupies the lowest possible configuration, and unstable when it occupies the highest configuration.* This applies to all cases of equilibrium of bodies in gravity conditions. If the centre of gravity lies on the axis of rotation, the equilibrium is said to be *indifferent* or *neutral.*

Let us consider also the equilibrium of the rod examined in Problem 7 (or in Problem 12). The equilibrium equation for the rod is $\Sigma m_A(\boldsymbol{F}_k) = 0$. In this case (see Fig. 38):

$$m_A(\boldsymbol{T}) = Qa\cos\frac{\alpha}{2}, \quad |m_A(\boldsymbol{P})| = Pa\sin\alpha.$$

If angle α is increased, $|m_A(\boldsymbol{P})|$ increases and $m_A(\boldsymbol{T})$ decreases, and under the action of force $\boldsymbol{P}$ angle α will continue to increase. If angle α is decreased, $m_A(\boldsymbol{T})$ increases and $m_A(\boldsymbol{P})$ decreases, and under the action of force $T = Q$ angle α will continue to decrease. Hence, the equilibrium of the rod as defined by the equation $\sin\frac{\alpha}{2} = \frac{Q}{2P}$ is *unstable.* At $\alpha = 180°$ the equilibrium is stable if $Q < 2P$, and unstable if $Q > 2P$, which can be verified by introducing $\beta = 180° - \alpha$.

This method of analysis is applicable only in the simplest cases. More complex cases are analysed by methods of dynamics.

Chapter 9
Centre of Gravity

§ 53. Centre of Parallel Forces

The concept of the centre of parallel forces comes in handy when we have to deal with solving certain problems of mechanics, in particular, when the centre of gravity of bodies has to be determined.

Consider a system of parallel forces F_1, F_2, . . ., F_n of same sense applied to a rigid body at points A_1, A_2, . . ., A_n (Fig. 127). The resultant $\boldsymbol{R}$ of this system is, evidently, parallel to the component forces in the same direction and its magnitude is

$$R = \Sigma F_k. \qquad (70)$$

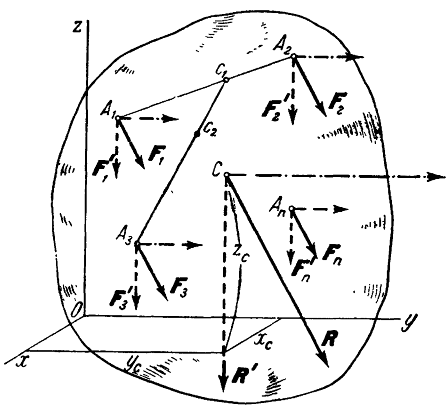

Fig. 127

If, now, the given forces are rotated in the same direction through the same angle about their points of application, we obtain new systems of parallel forces of same sense, whose magnitudes and points of application are the same as for the original system, but which act in a different direction (e.g., as shown by the dashed lines in Fig. 127). The resultant of each of these parallel-force systems will, evidently, be of same magnitude R, but differently directed. To determine this direction, we must find for each case a point through which the resultant passes. Let us show that *the action line of the resultant of parallel forces always passes through a certain point C, regardless of the direction of the forces.* Thus, resultant $\boldsymbol{R}_1$ (not shown in the diagram) of forces $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ will pass through a point c_1 on line A_1A_2 such that $F_1 \cdot A_1c_1 = F_2 \cdot A_2c_1$, regardless of the direction of the forces, since the latter does not affect the location of A_1A_2 or the form of the equation. The resultant of forces $\boldsymbol{R}_1$ and $\boldsymbol{F}_3$ (which is the resultant of forces $\boldsymbol{F}_1$, $\boldsymbol{F}_2$, and $\boldsymbol{F}_3$) will always pass through a point c_2 on line c_1A_3 determined in the same way as point c_1. By successively compounding all the forces, we find that their resultant $\boldsymbol{R}$ passes through a point C, which is fixed relative to points $A_1, A_2, \ldots, A_n$, and consequently, relative to the body as a whole.

The point C, through which the action line of the resultant of a system of parallel forces passes, no matter how those forces are rotated about their points of application through the same angle in the same direction, is called the centre of parallel forces.

Let us find the *coordinates of the centre of parallel forces*. The position of point C with regard to the body is constant and does not depend on the coordinate system. Let us, therefore, take an arbitrary coordinate system $Oxyz$ and denote the coordinates of the points in this system: $A_1(x_1, y_1, z_1)$, $A_2(x_2, y_2, z_2)$, $\ldots$, $C(x_C, y_C, z_C)$. As the position of point C does not depend on the direction of the forces, let us first rotate the forces about their points of application to a position parallel to axis z, and let us apply to the turned forces $\boldsymbol{F}'_1, \boldsymbol{F}'_2, \ldots, \boldsymbol{F}'_n$ Varignon's theorem (§ 50). As the resultant of these forces is $\boldsymbol{R}'$, then, taking the moments of the forces with respect to axis y, we obtain from Eq. (69):

$$m_y(\boldsymbol{R}') = \sum m_y(\boldsymbol{F}'_k). \tag{71}$$

But from the diagram [or from Eqs. (52)] we see that $m_y(\boldsymbol{R}') = Rx_C$, as $R' = R$; $m_y(\boldsymbol{F}'_1) = F_1x_1$, as $F'_1 = F_1$, etc. Consequently, $Rx_C = F_1x_1 + F_2x_2 + \ldots + F_nx_n$, whence

$$x_C = \frac{F_1x_1 + F_2x_2 + \ldots + F_nx_n}{R} = \frac{\sum F_kx_k}{R}.$$

We obtain a similar expression for coordinate y_C by taking the moments with respect to axis x. In order to determine z_C, we again rotate the forces to a position parallel to axis y (as indicated by dash-and-dot lines) and apply Varignon's theorem, taking the moments with respect to axis x. This gives us:

$$-Rz_C = -F_1z_1 + (-F_2z_2) + \ldots + (-F_nz_n),$$

whence we find z_C.

We thus obtain the following equations which specify the coordinates of the centre of parallel forces:

$$x_C = \frac{\sum F_kx_k}{R}, \quad y_C = \frac{\sum F_ky_k}{R}, \quad z_C = \frac{\sum F_kz_k}{R}, \tag{72}$$

where R is determined from Eq. (70).

Note that Eqs. (70) and (72) are valid for parallel forces acting in opposite directions, if F_k is taken as an algebraic quantity (i.e., plus for one sense and minus for the opposite sense) and if $R \neq 0$.

§ 54. Centre of Gravity of a Rigid Body

Every particle of a body near the surface of the earth is subjected to the action of a vertical downward force known as the force of gravity (this will be discussed in detail in § 121).

For a body that is very small as compared with the earth's radius, the gravity forces acting on its particles may be regarded as parallel to each other and constant in value however the body is turned. A field in which these conditions are satisfied is called a *uniform gravitational field*.

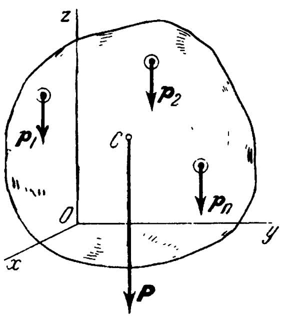

Fig. 128

We shall denote the resultant of gravity forces $\boldsymbol{p}_1, \boldsymbol{p}_2, \ldots, \boldsymbol{p}_n$ acting on the particles of a given body by the symbol $\boldsymbol{P}$ (Fig. 128). This force is equal to the weight of the body in magnitude and is specified by the equation*):

$$P = \Sigma p_k. \tag{73}$$

The forces $\boldsymbol{p}_k$ continue to act parallel to each other and are applied at the same points of the body regardless of how it is rotated, but their direction with respect to the body changes. It follows, then, as proved in § 53, that the resultant $\boldsymbol{P}$ of the forces $\boldsymbol{p}_k$ passes through one and only one point C, which is constant for the given body. This point is the centre of parallel gravity forces $\boldsymbol{p}_k$ and is called the *centre of gravity* of the body. Thus, *the centre of gravity of a body is a point, constant for every body, through which the resultant of the gravity forces acting on the particles of that body passes, regardless of how the body is orientated in space*. That such a point must always exist follows from the discussion in § 53.

The coordinates of the centre of gravity, as the centre of parallel forces, are specified by Eqs. (72) and are

$$x_C = \frac{\sum p_k x_k}{P}, \quad y_C = \frac{\sum p_k y_k}{P}, \quad z_C = \frac{\sum p_k z_k}{P}, \tag{74}$$

where x_k, y_k, z_k are the coordinates of the points of application of the gravity forces acting on the particles of the body.

*) The weight of a body is defined as the force with which a body at rest acts under the force of gravity on a constraint preventing its vertical fall (e.g., the tray of a balance).

It should be noted in conclusion that the centre of gravity, by definition, is a geometric point which is not necessarily a part of the body (e.g., the centre of gravity of a ring).

§ 55. Coordinates of Centres of Gravity of Homogeneous Bodies

The weight p_k of any part of a homogeneous body is proportional to the volume v_k of that part, i.e. $p_k = \gamma v_k$; the weight P of the whole body is proportional to its volume V, i.e., $P = \gamma V$, where γ is the weight of a unit volume.

Substituting the values of P and p_k into Eqs. (74), we find that the factor γ appears in both the numerator and the denominator and therefore cancels out. Thus we obtain:

$$x_C = \frac{\sum v_k x_k}{V}, \quad y_C = \frac{\sum v_k y_k}{V}, \quad z_C = \frac{\sum v_k z_k}{V}. \tag{75}$$

We see that the centre of gravity of a homogeneous body depends only on its geometric shape and does not depend on the value of γ. The point C, whose coordinates are specified by Eqs. (75), is called the *centre of gravity of the volume V*.

Reasoning in the same way, we may obtain for a homogeneous lamina the equations

$$x_C = \frac{\sum s_k x_k}{S}, \quad y_C = \frac{\sum s_k y_k}{S}, \tag{76}$$

where S is the area of the lamina and s_k the areas of its divisions.

The point whose coordinates are specified by Eqs. (76) is called the *centre of gravity of the area S*.

The equations specifying the coordinates of the *centre of gravity of a line* are deduced similarly:

$$x_C = \frac{\sum l_k x_k}{L}, \quad y_C = \frac{\sum l_k y_k}{L}, \quad z_C = \frac{\sum l_k z_k}{L}, \tag{77}$$

where L is the length of the line and l_k the lengths of its divisions.

Eqs. (77) are convenient for determining the centre of gravity of articles made of thin wire of uniform cross section.

Thus, the centre of gravity of a homogeneous body is determined as the centre of gravity of the respective volume, area or line.

§ 56. Methods of Determining the Coordinates of the Centre of Gravity of Bodies

Proceeding from the general formulas evolved above, the following methods are suggested for determining the coordinates of the centre of gravity of various bodies:

(1) Method of Symmetry. Let us prove that if a body has a plane axis, or centre of symmetry, its centre of gravity is coincident with the plane, axis or centre, respectively.

Consider a homogeneous body which has a plane of symmetry. This plane, then, divides the body into two parts of equal weight p_1 and p_2, the centres of gravity of which lie at equal distances from the plane of symmetry. Consequently, the centre of gravity of the body, as a point through which the resultant of the two equal and parallel forces p_1 and p_2 passes, must lie in the plane of symmetry. The proof is similar for the case of bodies with an axis or centre of symmetry.

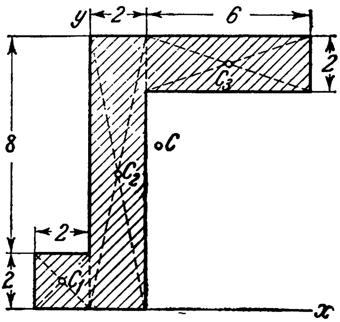

Fig. 129

It follows, by virtue of the properties of symmetry, that the centre of gravity of a homogeneous ring, disc, or rectangular lamina, rectangular parallelepiped, sphere, and other homogeneous bodies having a centre of symmetry, is coincident with their geometrical centre (centre of symmetry).

(2) Method of Division. If a body can be divided into a finite number of elements, for each of which the centre of gravity is known, the coordinates of the centre of gravity of the body as a whole can be directly calculated from Eqs. (74)-(77). The number of components in the numerators will be equal to the number of elements into which the body is divided.

Problem 50. Locate the centre of gravity of the homogeneous thin plate shown in Fig. 129 (the dimensions are given in centimetres).

Solution. After choosing the coordinate axes and dividing the plate into three rectangles as shown, we compute the coordinates of the centre of gravity and the area of each rectangle:

Nos	1	2	3
x_k	—1	1	5
y_k	1	5	9
s_k	4	20	12

The total area of the plate is

$$S = s_1 + s_2 + s_3 = 36 \text{ cm}^2.$$

Substituting the computed values in Eqs. (76), we have:

$$x_C = \frac{x_1 s_1 + x_2 s_2 + x_3 s_3}{S} = \frac{-4+20+60}{36} = 2\frac{1}{9} \text{ cm},$$

$$y_C = \frac{y_1 s_1 + y_2 s_2 + y_3 s_3}{S} = \frac{4+100+108}{36} = 5\frac{8}{9} \text{ cm}.$$

The location of the centre of gravity C is shown in the diagram, and in this case it is outside the plate. This example shows again that the centre of gravity of a body is a geometric point which is not necessarily a part of the body.

(3) **Method of Supplementation.** This method is a special case of the method of division and is used to locate the centres of gravity of bodies with holes or cavities, if the centres of the body as a whole without the cavity, and of the cavity itself, are known.

Problem 51. Determine the centre of gravity of a thin disc of radius R with a circle of radius r cut out of it (Fig. 130). The distance $C_1C_2 = a$.

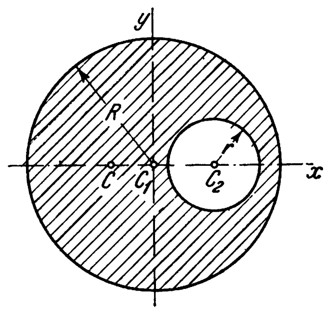

Fig. 130

Solution. The centre of gravity of the disc lies on the line C_1C_2, as it is the axis of symmetry. Draw the coordinate axes as shown. To find the coordinate x_C, first supplement the missing part of the plate to make a uniform disc and reduce the whole by the area of the cut-out circle, taking the latter with a negative sign. We have $s_1 = \pi R^2$, $x_1 = 0$, $s_2 = -\pi r^2$, $x_2 = a$, $S = s_1 + s_2 = \pi(R^2 - r^2)$.

Substituting these values into Eqs. (76), we obtain:

$$x_C = \frac{x_1 s_1 + x_2 s_2}{S} = -\frac{ar^2}{R^2 - r^2}; \quad y_C = 0.$$

The centre of gravity, we find, lies to the left of C_1.

(4) **Method of Integration.** If a body cannot be divided into a finite number of elements with known centres of gravity, it is first divided into infinitesimal elements of volume Δv_k, for which Eqs. (75) acquire the form

$$x_C = \frac{\sum x_k \Delta v_k}{V}, \text{ etc.,} \tag{78}$$

where x_k, y_k, z_k are the coordinates of a point inside the volume Δv_k. Passing to the limit in Eqs. (78), with $\Delta v_k \to 0$, i.e., reducing the elementary volumes to points, we replace the summations in the numerators by integrals over the volume of the body. In the limit, Eqs. (78) give

$$x_C = \frac{1}{V}\int\limits_{(V)} x\,dv; \quad y_C = \frac{1}{V}\int\limits_{(V)} y\,dv, \quad z_C = \frac{1}{V}\int\limits_{(V)} z\,dv. \tag{79}$$

Similarly, for the coordinates of the centre of gravity of an area or curve we obtain in the limit, from Eqs. (76) and (77),

$$x_C = \frac{1}{S}\int\limits_{(S)} x\,ds, \quad y_C = \frac{1}{S}\int\limits_{(S)} y\,ds \tag{80}$$

and

$$x_C = \frac{1}{L}\int\limits_{(L)} x\,dl, \quad y_C = \frac{1}{L}\int\limits_{(L)} y\,dl, \quad z_C = \frac{1}{L}\int\limits_{(L)} z\,dl. \tag{81}$$

An example of the use of the method of integration is given in the following section.

(5) **Experimental Method.** The centre of gravity of a composite non-homogeneous body, say an aircraft or locomotive, can be determined experimentally. In one method (the method of suspension), the body is suspended by strings or ropes attached to different points. The direction of the string supporting the body will each time give the direction of the force of gravity. The point of intersection of these directions locates the centre of gravity of the body.

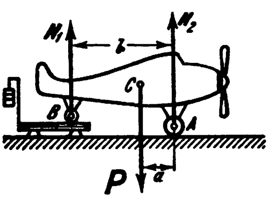

Fig. 131

Another method of experimentally locating the centre of gravity is the method of weighing, explained by the following example.

Example. Determine the centre of gravity of an aircraft (the distance a in Fig. 131) if distance $AB = l$ is known.

By placing wheel B on the platform of a balance, we determine the pressure exerted by the wheel, and hence the reaction N_1. Similarly, we determine the reaction N_2. Equating to zero the sums of the moments of all the forces with respect to the centre of gravity C, we obtain $N_2a - N_1(l - a) = 0$, whence

$$a = \frac{N_1 l}{N_1 + N_2}$$

Obviously $N_1 + N_2 = P$, where P is the weight of the aircraft. If P is immediately known, a single weighing is sufficient to find a.

§ 57. Centres of Gravity of Some Homogeneous Bodies

(1) **Centre of Gravity of a Circular Arc.** Consider an arc AB of radius R subtending a central angle $AOB = 2\alpha$. By virtue of its symmetry, the centre of gravity of the arc lies on axis x (Fig. 132). Let us determine the coordinate x_C by formulas (81). For this take an element MM' of length $dl = Rd\varphi$ on the arc AB. Its location is defined by angle φ. The x coordinate of the element MM' is $x = R\cos\varphi$. Substituting these values of x and dl in the first of Eqs. (81), and remembering that the integral must be over the whole length of the arc, we obtain:

$$x_C = \frac{1}{L}\int_A^B x\,dl = \frac{R^2}{L}\int_{-\alpha}^{\alpha}\cos\varphi\,d\varphi = 2\frac{R^2}{L}\sin\alpha,$$

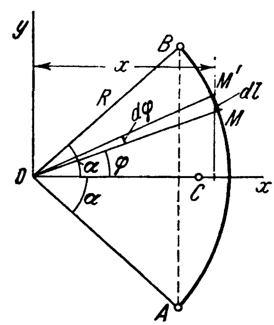

Fig. 132

where L is the length of the arc AB, which is equal to $R\cdot 2\alpha$. Hence, *the centre of gravity of a circular arc lies on its axis of symmetry at a distance from the centre*

$$x_C = R\frac{\sin\alpha}{\alpha}, \tag{82}$$

where angle α is measured in radians.

The same result can be obtained without direct integration. If we denote the length of an arc element by the symbol Δl_k, Eq. (77) gives

$$x_C = \frac{1}{L}\sum x_k\,\Delta l_k,$$

where x_k is the coordinate of element Δl_k, and with an accuracy to infinitesimals of higher order we have $x_k = R\cos\varphi_k$ (substituting φ_k for φ). Then (see Fig. 74, § 28), $x_k\Delta l_k = R\Delta l_k\cos\varphi_k = R\Delta y_k$, whence $\Sigma x_k\Delta l_k = R\Sigma\Delta y_k = R\cdot AB$. And finally, noting that $AB = 2R\sin\alpha$ and $L = R\cdot 2\alpha$, we arrive at Eq. (82).

(2) **Centre of Gravity of a Triangular Area.** Let us divide triangle ABD (Fig. 133) into n narrow strips parallel to side AD. The centres of gravity of these strips will, evidently, lie on median BE of the triangle. We conclude, then, that the centre of gravity of the triangle lies on this median. A similar result is obtained for the other two medians. We conclude therefore that *the centre of gravity of a triangular area lies at the intersection of its medians.* And, as is

known,

$$CE = \frac{1}{3} BE.$$

(3) **Centre of Gravity of a Circular Sector.** Consider a circular sector OAB of radius R whose central angle is 2α (Fig. 134). Let us divide the area of the sector OAB with radii drawn from O into n sectors. In the limit, when we increase n indefinitely, we can regard the sectors as triangular areas the centres of gravity of which lie on arc DE of radius $\frac{2}{3} R$. Consequently, the centre of gravity of the sector OAB is coincident with the centre of gravity of arc DE, the location of which is determined by Eq. (82).

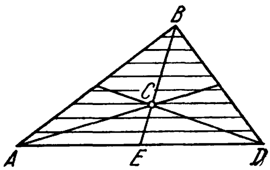

Fig. 133

Thus, *the centre of gravity of a circular sector lies on its axis of symmetry at a distance from its centre*

$$x_C = \frac{2}{3} R \frac{\sin \alpha}{\alpha}. \tag{83}$$

(4) **Centre of Gravity of a Pyramid.** Consider the triangular pyramid (tetrahedron) $ABDE$ in Fig. 135. To locate its centre of gravity, let us divide it with planes parallel to base ABD into n infinitesimal truncated pyramids, which can be regarded in the limit,

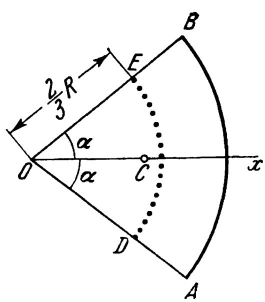

Fig. 134

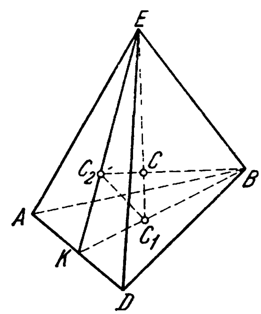

Fig. 135

when n is increased indefinitely, as plane triangles. The centres of gravity of these triangles lie on line EC_1 joining the vertex E of the pyramid and the centre of gravity C_1 of its base. Consequently, the centre of gravity of the pyramid lies on line EC_1.

Reasoning in the same way, we find that the centre of gravity also lies on line BC_2 joining vertex B with the centre of gravity of face ADE. Hence, the required centre of gravity lies at point C where EC_1 and BC_2 intersect.

Let us now locate point C. As C_1C_2 and BE divide the sides of angle BKE into proportional segments, they are parallel, and triangle C_1CC_2 is similar to triangle ECB. Furthermore, $C_1C_2 = \frac{1}{3} BE$, as $KC_1 = \frac{1}{3} KB$. We thus find:

$$\frac{CC_1}{CE} = \frac{C_1C_2}{BE} = \frac{1}{3},$$

whence

$$CC_1 = \frac{1}{3} CE = \frac{1}{4} C_1E. \qquad (84)$$

This formula is valid for any polyhedral pyramid and, in the limit, for a cone.

Thus, *the centre of gravity of a pyramid (or cone) lies on the line joining the vertex of the pyramid (cone) and the centre of gravity of the base, at a distance from the base equal to one quarter of the length of the whole line.*

Formulas for the coordinates of the centres of gravity of other homogeneous bodies can be found in various technical and mathematical handbooks.

Part 2

KINEMATICS OF A PARTICLE AND A RIGID BODY

Chapter 10

Kinematics of a Particle

§ 58. Introduction to Kinematics

Kinematics is the section of mechanics which treats of the geometry of the motion of bodies, without taking into account their inertia (mass) or the forces acting on them.

On the one hand, kinematics is an introduction to dynamics, insofar as the fundamental concepts and relationships of kinematics have to be understood before studying the motion of bodies taking into account the action of forces. On the other hand, the methods of kinematics are in themselves of practical importance, for example in studying the transmission of motion in mechanisms. That is why the demands of the developing machine-building industry led to the emergence of kinematics as a separate division of mechanics (in the first half of the 19th century).

By motion in mechanics is meant the relative displacement with time of a body in space with respect to other bodies.

In order to locate a moving body (or particle) we assume a coordinate system, which we call the *frame of reference* or reference system, to be fixed relative to the body with respect to which the motion is being considered. If the coordinates of all the points of a body remain constant within a given frame of reference, the body is said to be at *rest* relative to that reference system. If, on the other hand, the coordinates of any points of the body change with time, the body is said to be in motion relative to the given frame of reference (and consequently, relative to a body which is fixed with respect to the frame of reference). When we speak of the motion of a body with respect to a given frame of reference, we shall mean its motion relative to a body fixed with respect to that frame of reference.

Any motion in space takes place with time. In mechanics we deal with three-dimensional Euclidean space in which all dimensions are measured by the methods of Euclidean geometry. The unit of length, by which distance is measured, is the metre. Time in mechanics is considered as universal, i.e., as passing simultaneously in all our frames of reference. The unit of time is one second.*)

Euclidean space and universal time reflect only approximately the actual properties of space and time. Our daily experience shows, however, that for the motions considered in mechanics (at velocities much below the velocity of light) the approximation is sufficiently accurate for all practical purposes.

Time is a continuously varying quantity. In problems of kinematics, time t is taken as an independent variable (the argument). All other variables (distance, speed, etc.) are regarded as changing with time, i.e., as functions of time t. Time is measured from some *initial instant* ($t = 0$) which is specified for every problem. Any given *instant of time* t is specified by the number of seconds that has passed between the initial and the given time. The difference between successive instants of time is called the *time interval.*

The principles of kinematics, evolved from and confirmed by practical experience, are based on the axioms of geometry. No other laws or axioms are necessary for the kinematic study of motion.

For the solution of problems of kinematics, the specific motion under consideration has to be described.

To describe the motion, or the law of motion, of a given body (particle) kinematically means to specify the position of that body (particle) relative to a given frame of reference for any moment of time. One of the main problems of kinematics is that of describing the motion of particles or bodies in terms of mathematical expressions. Hence, we shall commence the investigation of the motion of any object with determining the ways of describing that motion.

The *principal problem of kinematics* is that of determining all the kinematic characteristics of the motion of a body as a whole or of any of its particles (path, velocity, acceleration, etc.) when the law of motion for the given body is known. For the solution of this problem we must know either the equations of motion for the given body or for another body kinematically associated with it.

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*) The International System of Units (see § 101) defines the *metre* as the length equal to 1 650 763,73 wave lengths in vacuo of the radiation corresponding to the transition between the levels of $2p_{10}$ and $5d_5$ of the krypton-86 atom; the *second* is defined as 1/31 556 925.9747 of the tropical year for 1900 January 0 at 12 hous ephemeris time.

We shall start the study of kinematics with an investigation of the motion of the simplest body—a particle (kinematics of a particle), proceeding later to the examination of the kinematics of rigid bodies.

§ 59. Methods of Describing Motion of a Particle. Path

We shall begin the study of kinematics with examining the methods of describing motion. To describe the motion of a particle, it is necessary to specify its position in a chosen frame of reference at any given time. There are three methods of describing motion: (1) the natural method, (2) the coordinate method, (3) the vector method.

(1) **Natural Method of Describing Motion.** The continuous curve described by a particle moving with respect to a given frame of reference is called the *path* of that particle. If the path is a straight line, the motion is said to be *rectilinear*, if the path is a curve, the motion is *curvilinear*.

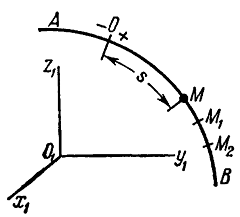

Fig. 136

The natural method of describing motion is convenient when the particle's path is known at once. Let the curve AB in Fig. 136 be the path of a particle M moving with respect to a frame of reference $O_1x_1y_1z_1$. Take any fixed point O on the path as the origin of another frame of reference; now, taking the path as an arc-coordinate axis, assume the positive and negative directions, as is done with rectangular axes. The position of the particle M on the path is now specified by a single coordinate s, equal to the distance from O to M measured along the arc of the path and taken with the appropriate sign. The displacement of particle M carries it through positions M_1, M_2, . . ., i.e., the distance s changes with time. In order to know the position of M on the path at any instant, we must know the relation

$$s = f(t). \tag{1}$$

Eq. (1) expresses the *law of motion of particle M* along its path.

Thus, in order to describe the motion of a particle by the natural method, a problem must state: (1) *the path of the particle*; (2) *the origin on the path*, showing the positive and negative *directions*; (3) *the equation of the particle's motion along the path in the form* $s = f(t)$.

For example, if a particle is moving from an origin O along a curve so that its distance from O increases in proportion to the square of the time, the equation of motion will be

$$s = at^2,$$

where a is the displacement of the particle in the first second. At time $t_2 = 2$ s the particle will be at distance $4a$ from O, etc. Consequently, knowing Eq. (1) we can determine the position of a moving particle at any instant.

Note that s in Eq. (1) denotes the moving particle's *position*, not the distance travelled by it. For example, if the particle in

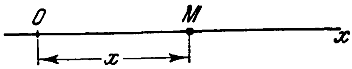

Fig. 137

Fig. 136 travels from O to M_1 and then reverses its motion to point M, its coordinate at that moment is $s = \overset{\frown}{OM}$, but the distance it travelled is $\overset{\frown}{OM_1} + \overset{\frown}{M_1M}$, i.e., not s.

In the case of *rectilinear motion*, if axis Ox is directed along the particle's path (Fig. 137), we have $s = x$, yielding the *law of rectilinear motion* of a particle as

$$x = f(t). \tag{2}$$

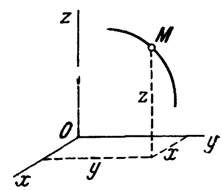

Fig. 138

(2) Coordinate Method of Describing Motion. The natural method of describing motion offers a very clear picture, but a particle's path may not be known, which is why the coordinate method is employed more frequently.

The position of a particle with respect to a given frame of reference $Oxyz$ can be specified by its Cartesian coordinates x, y, z (Fig. 138). When motion takes place, the three coordinates will change with time. If we want to know the equation of motion of a particle, i.e., its location in space at any instant, we must know its coordinates for any moment of time, i.e., the relations

$$x = f_1(t), \quad y = f_2(t), \quad z = f_3(t) \tag{3}$$

should be known.

Eqs. (3) are the *equations of motion of a particle in terms of the Cartesian rectangular coordinates.* They describe the curvilinear motion of a particle by the coordinate method *).

If a particle moves in one plane, then, taking the plane for the xy plane, we obtain two equations of motion:

$$x = f_1(t), \quad y = f_2(t) \tag{4}$$

*) The motion of a particle can be described in other coordinate systems: polar (see § 71), spherical, etc.

Finally, in the case of rectilinear motion, if axis x is directed along the path, the motion is described by the single equation (2) obtained before (in this case the coordinate and natural methods of describing motion coincide).

Eqs. (3) or (4) are, at the same time, the equations of the particle's *path* in parametric form, where the time t is the parameter. By eliminating time t from the equations of motion we can obtain the equation of the path in the usual form, i.e., in the form of a relation between the particle's coordinates.

Examples. (1) Let a particle's motion in a plane Oxy be given by the equations

$$x = 2t, \;\; y = 12t^2. \tag{a}$$

From these equations at time $t = 0$ the particle is at M_0 (0, 0), i.e., at the origin of the coordinate system; at time $t_1 = 1$ s it is at M_1 (2, 12), etc. Thus, equations (a) actually define the particle's position at any instant. By assigning t different values and drawing a graph of the particle's displacement we can construct its path.

Another way of determining the path is by eliminating t from the equations (a). From the first equation $t = \frac{x}{2}$; substituting it for t in the second equation, we obtain $y = 3x^2$. Hence the path is a parabola with the apex at the origin of the coordinate axes and the axis parallel to axis Oy.

(2) Consider the case when a particle's motion is described by the equations:

$$x = a \sin \pi t, \quad y = a \cos \pi t, \quad z = a \cos \pi t. \tag{b}$$

Squaring the first two equations and adding them, we obtain: $x^2 + y^2 = a^2$. Also, from the second and third equations $y = z$. Thus, the path lies on the line of intersection of a circular cylinder of radius a, whose axis coincides with axis Oz, with a plane $y = z$ bisecting the spatial angle between planes Oxy and Oxz, i.e., an ellipse with semiaxes a and $a\sqrt{2}$ lying in the plane $y = z$.

For other examples of determining path see Problems 53, 54, 56 (§ 65).

(3) **Vector Method of Describing Motion.** Let a particle M be moving relative to any frame of reference $Oxyz$. The position of the particle at any instant can be specified by a vector $\boldsymbol{r}$ drawn from the origin O to the particle M (Fig. 139). Vector $\boldsymbol{r}$ is called the *radius vector* of the particle M.

When the particle moves, the vector $\boldsymbol{r}$ changes with time both in magnitude and direction. Thus, $\boldsymbol{r}$ is a variable vector (a vector function) depending on the argument t:

$$\boldsymbol{r} = \boldsymbol{r}\,(t). \tag{5}$$

Eq. (5) describes the curvilinear motion of a particle in vector form and can be used to construct a vector $\boldsymbol{r}$ for any particular moment of time and to determine the position of the moving particle at that moment.

The locus of the tips of vector $\boldsymbol{r}$ defines the path of the moving particle.

The vector method is convenient for establishing general dependencies, as it describes a particle's motion in terms of one vector equation (5) instead of the three scalar equations (3).

The relationship between the coordinate and vector methods of describing motion can easily be established by introducing unit vectors $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$ directed along the x, y, z axes, respectively (see Fig. 139). As the projections of vector $\boldsymbol{r}$ on the coordinate axes are equal to the coordinates of the particle M, i.e. $r_x = x$, $r_y = y$, $r_z = z$, we obtain

$$\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}. \qquad (6)$$

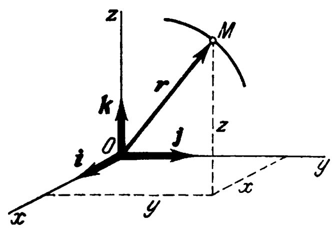

Fig. 139

Hence if, for example, the motion of a particle in plane Oxy is given in coordinate form by the equations $x = 2t$, $y = 12t^2$, the vector equation (5) of that motion will be

$$\boldsymbol{r} = 2t\boldsymbol{i} + 12t^2\boldsymbol{j}.$$

Using this equation we can construct vector $\boldsymbol{r}$ and determine the particle's position at any instant t. For example, at $t_1 = 1$ s, $\boldsymbol{r}_1 = 2\boldsymbol{i} + 12\boldsymbol{j}$ and can be constructed as the diagonal of the corresponding parallelogram, etc.

Conversely, if a particle's motion is described in vector form by, for example, the equation $\boldsymbol{r} = (1-t)\,\boldsymbol{i} + 2t^2\boldsymbol{j} - 3t\boldsymbol{k}$, the equation of motion in coordinate form will be $x = (1-t)$, $y = 2t^2$, $z = -3t$.

§ 60*. Conversion From Coordinate to Natural Method of Describing Motion

It has been shown that if motion is described by the equations (3) or (4), the path of the particle can be determined. It is, furthermore, known that $ds^2 = dx^2 + dy^2 + dz^2$, or $ds = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}\, dt$, where $\dot{x} = \frac{dx}{dt}$, etc. Hence, assuming that at $t = 0$ the displacement $s = 0$, we obtain*)

$$s=\int_0^t \sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2}\,dt. \tag{7}$$

After integrating, Eq. (7) yields the equation of motion along the path in the form (1). If the motion is described by Eqs. (4), the equation (7) will lack the member with the derivative of z.

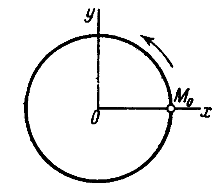

Fig. 140

Problem 52. The motion of a point in plane Oxy is described by the equations

$$x = a\cos\omega t,\quad y = a\sin\omega t, \tag{a}$$

where a and ω are constants. Determine the path and the equation of motion along it.

Solution. Squaring the equations (a) and adding them, we obtain: $x^2 + y^2 = a^2$.

Hence, the path is a circle of radius a with the centre at the origin of the coordinate system (Fig. 140). Computing the derivatives of x and y with respect to t, we obtain

$$\dot{x} = -a\omega\sin\omega t,\quad \dot{y} = a\omega\cos\omega t.$$

Substituting into Eq. (7), we have

$$s=\int_0^t a\omega\,dt \quad \text{or} \quad s=a\omega t. \tag{b}$$

Equation (b) describes the particle's motion along the path in the form (1). From equations (a), when $t = 0$, we have $x = a$, and $y = 0$, i.e., the particle is at M_0; as t increases x decreases and y increases, assuming positive values. Consequently, the counting off of s starts at point M_0 and the displacement along the circle is in the direction indicated by the arrow in Fig. 140. It will be observed from equation (b) that as the particle moves the displacement s increases in proportion to the time, the increment being $a\omega$ each second, such motion is called uniform.

In this case the conversion from the natural method made for a more graphic visualisation of the motion than could be presented by equations (a) directly.

*) By taking the square root with the plus sign we thereby determine the positive direction of the displacement s (in the direction the point starts moving at time $t = 0$).

§ 61. Velocity Vector of a Particle

One of the basic kinematic characteristics of motion of a particle is a vector quantity called velocity. First let us introduce the concept of average velocity of a particle in a given time interval. Let a moving particle occupy at time t a position M defined by the radius vector $\boldsymbol{r}$, and at time t_1 a position M_1 defined by the radius vector $\boldsymbol{r}_1$ (Fig. 141). The displacement during the time interval $\Delta t = t_1 - t$ is defined by a vector $\overline{MM_1}$ which we shall call the *displacement vector of the particle*. The vector is directed along a chord if the particle is

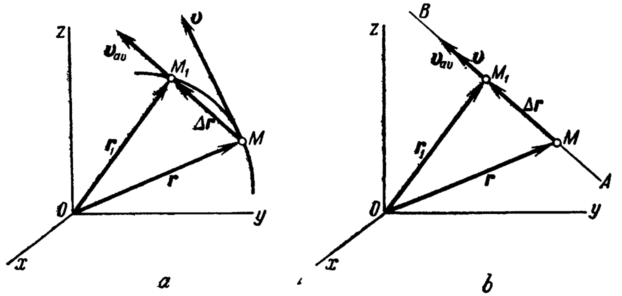

Fig. 141

in curvilinear motion (Fig. 141*a*), and along the path AB in rectilinear motion (Fig. 141*b*).

From triangle OMM_1 we have $\boldsymbol{r} + \overline{MM_1} = \boldsymbol{r}_1$, whence

$$\overline{MM_1} = \boldsymbol{r}_1 - \boldsymbol{r} = \Delta \boldsymbol{r}.$$

The ratio of the displacement vector of a particle to the corresponding time interval defines a vector quantity called the *average* (both magnitude and direction) *velocity* of the particle during the given time interval Δt:

$$\boldsymbol{v}_{av} = \frac{\overline{MM_1}}{\Delta t} = \frac{\Delta \boldsymbol{r}}{\Delta t}. \qquad (8)$$

The magnitude of the average velocity given by Eq. (8) is

$$v_{av} = \frac{MM_1}{\Delta t}. \qquad (8')$$

Vector $\boldsymbol{v}_{av}$ has the same direction as vector $\overline{MM_1}$, i.e., along the chord MM_1 in the direction of the motion of the particle in the case of curvilinear motion, and along the path itself in the case of rectilinear motion (the direction of the vector is not altered by being divided by Δt).

Obviously, the smaller the time interval $\Delta t = t_1 - t$ for which the average velocity has been calculated, the more precisely will $\boldsymbol{v}_{av}$ characterise the particle's motion. To obtain a characteristic of motion independent of the choice of the time interval Δt, the concept of *instantaneous velocity of a particle* is introduced.

The instantaneous velocity of a particle at any time t is defined as the vector quantity $\boldsymbol{v}$ towards which the average velocity $\boldsymbol{v}_{av}$ tends when the time interval Δt tends to zero:

$$\boldsymbol{v} = \lim_{\Delta t \to 0} (\boldsymbol{v}_{av}) = \lim_{\Delta t \to 0} \frac{\Delta \boldsymbol{r}}{\Delta t}.$$

The limit of the ratio $\Delta \boldsymbol{r}/\Delta t$ as $\Delta t \to 0$ is the first derivative of the vector $\boldsymbol{r}$ with respect to t and is denoted, like the derivative of a scalar function, by the symbol $d\boldsymbol{r}/dt$*).

Finally we obtain:

$$\boldsymbol{v} = \frac{d\boldsymbol{r}}{dt}. \tag{9}$$

Thus, *the vector of instantaneous velocity of a particle is equal to the first derivative of the radius vector of the particle with respect to time.*

As the limiting direction of the secant MM_1 is a tangent, the vector of instantaneous velocity is tangent to the path of the particle in the direction of motion.

Eq. (9) also shows that the velocity vector $\boldsymbol{v}$ is equal to the ratio of the infinitesimal displacement $d\boldsymbol{r}$ of the particle tangent to its path to the corresponding time interval dt.

In rectilinear motion the velocity vector $\boldsymbol{v}$ is always directed along the straight line in which the particle is moving and can change only in magnitude; in curvilinear motion the direction of the velocity vector changes continuously. The dimension of velocity is displacement/time, and the customary units are m/s or km/h.

§ 62. Acceleration Vector of a Particle

Acceleration characterises the time rate of change of velocity in magnitude and direction.

Let a moving particle occupy a position M and have a velocity $\boldsymbol{v}$ at a given time t, and let it at time t_1 occupy a position M_1 and have a velocity $\boldsymbol{v}_1$ (Fig. 142). The increase in velocity in the time interval $\Delta t = t_1 - t$ is $\Delta \boldsymbol{v} = \boldsymbol{v}_1 - \boldsymbol{v}$. To construct vector $\Delta \boldsymbol{v}$, lay off vector

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*) In general, for any variable vector $\boldsymbol{u}$ depending on an argument t

$$\lim_{\Delta t \to 0} \frac{\Delta \boldsymbol{u}}{\Delta t} = \frac{d\boldsymbol{u}}{dt}.$$

$\boldsymbol{v}_1$ from point M and construct a parallelogram with $\boldsymbol{v}_1$ as its diagonal and $\boldsymbol{v}$ as one of its sides. It is evident that the other side will represent vector $\Delta\boldsymbol{v}$. Note that vector $\Delta\boldsymbol{v}$ is always directed towards the inside of the path. The ratio of the velocity increment vector $\Delta\boldsymbol{v}$ to the corresponding time interval Δt defines the *vector of average acceleration of the particle* in the given time interval:

$$\boldsymbol{w}_{\text{av}} = \frac{\Delta \boldsymbol{v}}{\Delta t}. \tag{10}$$

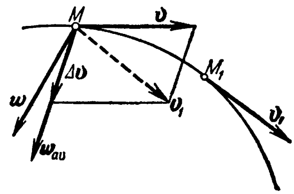

Fig. 142

Obviously, the vector of average acceleration has the same direction as vector $\Delta\boldsymbol{v}$, i.e., towards the inside of the path.

The instantaneous acceleration of a particle at a given time t is defined as the vector quantity $\boldsymbol{w}$ towards which the average acceleration $\boldsymbol{w}_{\text{av}}$ tends when the time interval Δt tends to zero:

$$\boldsymbol{w} = \lim_{\Delta t \to 0} \frac{\Delta \boldsymbol{v}}{\Delta t} = \frac{d\boldsymbol{v}}{dt},$$

or, taking into account Eq. (9),

$$\boldsymbol{w} = \frac{d\boldsymbol{v}}{dt} = \frac{d^2\boldsymbol{r}}{dt^2}. \tag{11}$$

Thus, *the vector of instantaneous acceleration of a particle is equal to the first derivative of the velocity vector or the second derivative of the radius vector of the particle with respect to time.*

The dimension of acceleration is displacement/(time)^2, and the commonly used unit is m/s^2.

It follows from Eq. (11) that the acceleration vector $\boldsymbol{w}$ is equal to the ratio of the increment of the velocity vector $d\boldsymbol{v}$ to the corresponding time interval dt.

Let us see how vector $\boldsymbol{w}$ is directed with respect to the path of the particle. In rectilinear motion vector $\boldsymbol{w}$ is, obviously, directed along the straight line in which the particle is moving. If the path is a plane curve, the acceleration vector $\boldsymbol{w}$, like the vector $\boldsymbol{w}_{\text{av}}$, lies in the plane of the curve and is directed towards the inside of the curve. If the path is a curve in space, the vector $\boldsymbol{w}_{\text{av}}$ is directed towards its inside, in a plane through the tangent to the path at point M and a line parallel to the tangent through the neighbouring point M_1 (see Fig. 142). In the limit, when point M_1 tends to M, this plane coincides with the so-called *osculating plane**). Hence, in the general

*) The osculating plane through a point M on a curve may also be defined as the limiting position of a plane through points M, M_1 and M_2 of the given curve when points M_1 and M_2 tend to M. Of all the planes passing through point M, the osculating plane has the highest order of osculation (has the greatest contact with the curve). Every point of a three-dimensional curve (e.g., a helix) has its own osculating plane. The osculating plane of a plane curve is coincident with the plane of the curve and is common for all its points.

case, *the acceleration vector* $\boldsymbol{w}$ *lies in the osculating plane and is directed towards the inside of the curve.*

Eqs. (9) and (11) obtained in §§ 61-62 give the general expressions of the basic kinematic characteristics of motion in vector form and are essential for developing all other formulas and relationships in particle kinematics.

§ 63. Theorem of the Projection of the Derivative of a Vector

The following theorem will be found useful in solving vector equations containing derivatives, when it is necessary to go over from relations between vectors to relations between their projections: *The projection of the derivative of a vector on a fixed axis is equal to the derivative of the projection of the differentiated vector on the same axis*).*

Let there be a variable vector $\boldsymbol{p}$ depending on an argument t. The derivative of $\boldsymbol{p}$ with respect to t is also a vector $\boldsymbol{q}$. By analogy with Eq. (6), vector $\boldsymbol{p}$ can be represented in terms of its projections in the form $\boldsymbol{p} = p_x\boldsymbol{i} + p_y\boldsymbol{j} + p_z\boldsymbol{k}$. As vectors $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ are constant in magnitude ($|\boldsymbol{i}| = |\boldsymbol{j}| = |\boldsymbol{k}| = 1$) and direction (the axes x, y, z are stationary),

$$\boldsymbol{q} = \frac{d\boldsymbol{p}}{dt} = \frac{dp_x}{dt}\boldsymbol{i} + \frac{dp_y}{dt}\boldsymbol{j} + \frac{dp_z}{dt}\boldsymbol{k}. \tag{12}$$

On the other hand, vector $\boldsymbol{q}$ can also be represented in the form

$$q = q_x\boldsymbol{i} + q_y\boldsymbol{j} + q_z\boldsymbol{k}. \tag{13}$$

As the left-hand sides of equations (12) and (13) are equal, so are their right-hand sides, whence,

$$\text{if } \boldsymbol{q} = \frac{d\boldsymbol{p}}{dt}, \text{ then } q_x = \frac{dp_x}{dt}, \quad q_y = \frac{dp_y}{dt}, \quad q_z = \frac{dp_z}{dt}, \tag{14}$$

and the theorem is proved.

It should be noted that the relation of the form (14) is not valid for the magnitudes of vectors, i.e., if $\boldsymbol{q} = \frac{d\boldsymbol{p}}{dt}$ then generally $|\boldsymbol{q}| \neq \frac{d|\boldsymbol{p}|}{dt}$. In particular, we have proved that $\boldsymbol{v} = \frac{d\boldsymbol{r}}{dt}$ and $\boldsymbol{w} = \frac{d\boldsymbol{v}}{dt}$; here, too, in the general case $|\boldsymbol{v}| \neq \frac{d|\boldsymbol{r}|}{dt}$ and $|\boldsymbol{w}| \neq \frac{d|\boldsymbol{v}|}{dt}$.

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*) If this purely mathematical theorem expressed by Eqs. (14) is known, the proof can be omitted.

For, say, if a particle moves in a circle whose centre is at the origin of a coordinate system, its velocity is $\boldsymbol{v}=\frac{d\boldsymbol{r}}{dt}$ (the direction of $\boldsymbol{r}$ changes with time), but in magnitude $|\boldsymbol{r}|$ = const, and consequenly, $\frac{d|\boldsymbol{r}|}{dt}=0$. It is evident, then, that $|\boldsymbol{v}|\neq\frac{d|\boldsymbol{r}|}{dt}$. That $|\boldsymbol{w}|\neq\frac{d|\boldsymbol{v}|}{dt}$ will be shown in § 67.

The methods of determining the magnitudes of the velocity and acceleration of a particle are discussed further on.

§ 64. Determination of the Velocity and Acceleration of a Particle When its Motion is Described by the Coordinate Method

Let us see how a particle's velocity and acceleration can be calculated when its motion is described by Eqs. (3) or (4). The determination of path was discussed in § 59.

(1) Determination of the Velocity of a Particle. The velocity vector of a particle is $\boldsymbol{v}=\frac{d\boldsymbol{r}}{dt}$. Hence, using relation (14), and taking into account that $\boldsymbol{r}_x=x$, $\boldsymbol{r}_y=y$, $\boldsymbol{r}_z=z$ (see Fig. 139), we have:

$$v_x=\frac{dx}{dt}, \quad v_y=\frac{dy}{dt}, \quad v_z=\frac{dz}{dt}, \tag{15}$$

or

$$v_x=\dot{x}, \; v_y=\dot{y}, \; v_z=z, \tag{15'}$$

where the dot over the letter is a symbol of differentiation with respect to time. Thus, *the projections of the velocity on the coordinate axes are equal to the first derivatives of the corresponding coordinates of the particle with respect to time.*

Knowing the projections of the velocity, we can find the magnitude and direction (i.e., the angles α, β, γ which vector $\boldsymbol{v}$ makes with the coordinate axes) from the equation:

$$\left.\begin{aligned} &v=\sqrt{v_x^2+v_y^2+v_z^2};\\ &\cos\alpha=\frac{v_x}{v}, \quad \cos\beta=\frac{v_y}{v}, \quad \cos\gamma=\frac{v_z}{v}. \end{aligned}\right\} \tag{16}$$

(2) Determination of the Acceleration of a Particle. The acceleration vector of a particle is $\boldsymbol{w}=\frac{d\boldsymbol{v}}{dt}$. Hence, from the theorem of the projection of a derivative and from Eqs. (15), we obtain:

$$w_x=\frac{dv_x}{dt}=\frac{d^2x}{dt^2}, \quad w_y=\frac{dv_y}{dt}=\frac{d^2y}{dt^2}, \quad w_z=\frac{dv_z}{dt}=\frac{d^2z}{dt^2} \tag{17}$$

or

$$w_x = \dot{v}_x = \ddot{x}, \quad w_y = \dot{v}_y = \ddot{y}, \quad w_z = \dot{v}_z = \ddot{z}, \tag{17'}$$

i.e., *the projections of the acceleration on the coordinate axes are equal to the first derivatives of the projections of the velocities, or the second derivatives of the corresponding coordinates, of the particle with respect to time.* The magnitude and direction of the acceleration are given by the equations

$$\left.\begin{aligned} &w = \sqrt{w_x^2 + w_y^2 + w_z^2}\,; \\ &\cos\alpha_1 = \frac{w_x}{w}, \quad \cos\beta_1 = \frac{w_y}{w}, \quad \cos\gamma_1 = \frac{w_z}{w}, \end{aligned}\right\} \tag{18}$$

where α_1, β_1, and γ_1 are the angles made by the acceleration vector with the coordinate axes.

Thus, if the motion of a particle is described in rectangular Cartesian coordinates by Eqs. (3) or (4), the velocity of the particle is given by Eqs. (15) and (16) and the acceleration by Eqs. (17) and (18). In the case of plane motion, the z projection should be omitted in all the equations.

In the case of rectilinear motion, which is given by the single equation $x = f(t)$, we have:

$$v_x = \frac{dx}{dt}, \quad w_x = \frac{dv_x}{dt} = \frac{d^2x}{dt^2}. \tag{19}$$

As there are no projections on the other axes, in this case, consequently, $v_x = \pm v$, $w_x = \pm w$, i.e., in rectilinear motion Eqs. (19) immediately give a particle's velocity and acceleration.

§ 65. Solution of Problems of Particle Kinematics

Problems solved by the methods of particle kinematics may consist in determination of the particle's path, velocity or acceleration, determination of the time needed for the particle to travel through a given displacement, or of the displacement in a given period of time, etc.

Before proceeding with the solution of any of these problems it is necessary *to determine the law of motion of the particle.* One of two cases is possible:

(1) The law of motion is given in the conditions of the problem. In this case the solution consists in applying the respective equations (Problems 53 and 54).

(2) The law of motion of the particle is not given, but its motion is in some specific way dependent on the given motion of another particle (or another body). In this case *the solution of the problem*

begins with determining the equations describing the particle's motion (Problems 55 and 56).

Problem 53. The motion of a particle is described by the equations

$$x = 8t - 4t^2, \quad y = 6t - 3t^2,$$

where x and y are in metres and t is in seconds.

Determine the path, velocity and acceleration of the particle.

Solution. To determine the path, we first eliminate time t from the equations of motion. Multiplying both parts of the first equation by 3 and both parts of the second by 4, and subtracting the second from the first, we obtain $3x - 4y = 0$, or

$$y = \frac{3}{4}x.$$

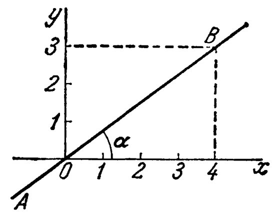

Fig. 143

Consequently, the path is a straight line making an angle α with axis x such that $\tan \alpha = \frac{3}{4}$ (Fig. 143).

Now let us determine the velocity of the particle. From Eqs. (15) and (16) we obtain:

$$v_x = \dot{x} = 8(1 - t), \quad v_y = \dot{y} = 6(1 - t)$$

$$v = \sqrt{v_x^2 + v_y^2} = 10\,|1 - t\,|.$$

To determine the acceleration of the particle, Eqs. (17) and (18) give us:

$$w_x = \ddot{x} = -8, \quad w_y = \ddot{y} = -6, \quad w = 10 \text{ m/s}^2.$$

Vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ are evidently directed along the path, i.e., along AB. The projections of the acceleration on the coordinate axes are always negative, consequently, the acceleration is always directed from B to A. The projections of the velocity at $0 < t < 1$ are positive, consequently, within this time interval, the velocity is directed from O to B. At time $t = 0$, $v = 10$ m/s, and at time $t = 1$ s, $v = 0$. When $t > 1$ s, both projections of the velocity are negative and, consequently, the velocity is directed from B to A, i.e., similarly to the acceleration.

Finally, note that, at $t = 0$, $x = 0$ and $y = 0$, at $t = 1$ s, $x = 4$ and $y = 3$ (point B), at $t = 2$ s, $x = 0$, $y = 0$; at $t > 2$ s, the absolute values of x and y continue to increase in the negative direction.

Thus, the equations given in the statement of the problem tell us everything about the motion of the particle. The motion begins at point O with an initial velocity $v_0 = 10$ m/s and is along a straight line AB inclined to axis x at an angle α such that $\tan \alpha = \frac{3}{4}$. On the portion OB the motion is uniformly retarded, and in one second the particle comes to rest at point B (4, 3). From there it moves back with uniform acceleration. At $t = 2$ s, the particle returns to the origin of coordinates and continues to move along OA. The acceleration is all the time 10 m/s^2.

Problem 54. The motion of a particle is described by the equations:

$$x = a \sin \omega t, \quad y = a \cos \omega t, \quad z = ut,$$

where a, ω, and u are constants. Determine the path, velocity and acceleration of the particle.

Solution. Squaring the first two equations and adding them, we obtain (since $\sin^2 wt + \cos^2 \omega t = 1$):

$$x^2 + y^2 = a^2.$$

Hence, the path lies on a circular cylinder of radius a, the axis of which is coincident with axis z (Fig. 144). Determining t from the third equation and substituting its value into the first, we find:

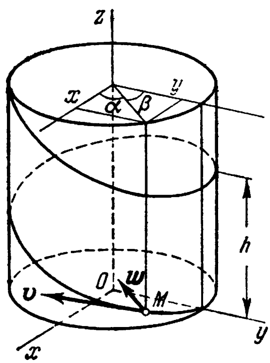

Fig. 144

$$x = a \sin \left(\frac{\omega}{u} z\right).$$

Thus, the path of the particle is the line of intersection of a sinusoidal surface, whose generators are parallel to axis y, with the cylindrical surface of radius a. This curve is called the *helical line*. It can be seen from the equations of motion that the particle makes one turn along the helical line in time t_1, determined by the equation $\omega t_1 = 2\pi$. The displacement of the particle parallel to axis z in that time is $h = ut_1 = \frac{2\pi u}{\omega}$ and is called the *pitch* of the helical line.

Let us now determine the velocity and acceleration of the particle. Differentiating the equations of motion with respect to time, we obtain:

$$v_x = a\omega \cos \omega t, \quad v_y = -a\omega \sin \omega t, \quad v_z = u,$$

whence

$$v = \sqrt{a^2\omega^2 (\cos^2 \omega t + \sin^2 \omega t) + u^2} = \sqrt{a^2\omega^2 + u^2}.$$

The quantities under the radical are constant. Consequently, the magnitude of the velocity is constant and is directed at a tangent to the path. From Eqs. (17) we calculate the projections of the acceleration:

$$w_x = -a\omega^2 \sin \omega t, \quad w_y = -a\omega^2 \cos \omega t, \quad w_z = 0,$$

whence

$$w=\sqrt{w_x^2+w_y^2}=a\omega^2.$$

Thus, the motion has an acceleration of constant magnitude. To determine the direction of acceleration, we have the equations

$$\cos\alpha_1=\frac{w_x}{w}=-\sin\omega t=-\frac{x}{a},$$

$$\cos\beta_1=\frac{w_y}{w}=-\cos\omega t=-\frac{y}{a},\quad \cos\gamma_1=\frac{w_z}{w}=0.$$

But evidently

$$\frac{x}{a}=\cos\alpha,\qquad \frac{y}{a}=\cos\beta,$$

where α and β are the angles made by the radius a, drawn from the axis of the cylinder to the moving particle, with the x and y axes. As the cosines of angles α_1 and β_1 differ from the cosines of the angles α and β only in sign, we conclude that the acceleration of the particle is continuously directed along the radius of the cylinder towards its axis.

It should be noted that although, in the present case, the motion has a velocity of constant magnitude (i.e., has a constant *speed*), the acceleration of the particle is not zero, as the direction of the velocity is continually changing.

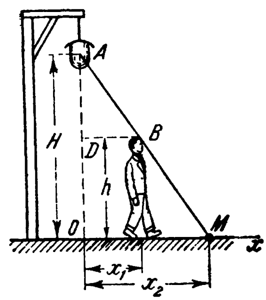

Fig. 145

Problem 55. A man of height h walks away from a lamp hanging at a height H with a velocity u. Determine the velocity of the tip of the man's shadow.

Solution. First let us establish the law of motion of the tip of the shadow. Choosing point O directly under the lamp as the origin of our frame of reference, draw axis x along the straight line where the end of the shadow moves, as shown in Fig. 145. Now, depicting the man at an arbitrary distance x_1 from O, we find that the tip of his shadow is at x_2.

By virtue of the similarity of triangles OAM and DAB, we have:

$$x_2=\frac{H}{H-h}x_1.$$

This is the equation of motion for the tip of the shadow M, provided the equation of motion for the man, i.e., $x_1=f(t)$, is known.

Differentiating both parts of the equation with respect to time and noting that according to formula (19) $\dot{x}_1=u\ \ =u$ and $\dot{x}_2=v_x=v$, where v is the required velocity, we obtain:

$$v = \frac{H}{H-h} u.$$

If the man moves with uniform velocity (u = const.), the velocity of the shadow v is also uniform, but it is $\frac{H}{H-h}$ times faster than that of the man.

It should be noted that *in developing the equations of motion, the moving body or mechanism* (see Problem 56) *should be drawn in an arbitrary position.* Only thus can we obtain an equation specifying the position of a moving particle (or body) at any moment of time.

Problem 56. Determine the path, velocity and acceleration of point M in the middle of the connecting rod of the crankgear in

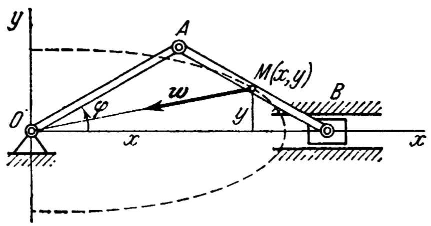

Fig. 146

Fig. 146, if $OA = AB = 2a$ and angle φ increases with time: $\varphi = \omega t$.

Solution. Let us first develop the equations of motion of point M. Drawing the coordinate axes as shown and denoting the coordinates of M by x and y, we obtain:

$$x = 2a \cos \varphi + a \cos \varphi, \quad y = a \sin \varphi.$$

Substituting the expression for φ, we obtain the equations of motion of point M:

$$x = 3a \cos \omega t, \quad y = a \sin \omega t.$$

To determine the path of M we write the equations of motion in the form

$$\frac{x}{3a} = \cos \omega t, \quad \frac{y}{a} = \sin \omega t.$$

Squaring these equations and adding them, we obtain:

$$\frac{x^2}{9a^2} + \frac{y^2}{a^2} = 1.$$

Thus, the path described by point M is an ellipse with semiaxes equal to $3a$ and a.

From Eqs. (15) and (16) we determine the velocity of point M:

$$v_x = -3a\omega \sin \omega t, \quad v_y = a\omega \cos \omega t;$$

$$v = a\omega \sqrt{9 \sin^2 \omega t + \cos^2 \omega t}.$$

We see that the velocity is a variable quantity, changing with time from $v_{min} = a\omega$ to $v_{max} = 3a\omega$.

Now, from Eqs. (17), we determine the projections of the acceleration of point B:

$$w_x = -3a\omega^2 \cos \omega t = -\omega^2 x, \quad w_y = -a\omega^2 \sin \omega t = -\omega^2 y,$$

whence

$$w = \sqrt{\omega^4 (x^2 + y^2)} = \omega^2 r,$$

where r is the radius vector from the origin to point M. Thus, the magnitude of the acceleration of the point changes in proportion to its distance from the centre of the ellipse.

To determine the direction of $\boldsymbol{w}$, we have, from Eqs. (18):

$$\cos \alpha_1 = \frac{w_x}{w} = -\frac{x}{r}, \qquad \cos \beta_1 = \frac{w_y}{w} = -\frac{y}{r}.$$

We see that, as in problem 54, the acceleration of point M is continually directed along MO towards the centre of the ellipse.

§ 66. Determination of the Velocity of a Particle When its Motion is Described by the Natural Method

Given (see § 59) the path of a particle and the law of motion along it in the form

$$s = f(t). \tag{20}$$

Let us see how the velocity of a particle can be determined. If in a time interval $\Delta t = t_1 - t$ a particle moves from position M to position M_1, the displacement along the arc of the path being $\Delta s = s_1 - s$ (Fig. 147), the numerical value of the average velocity will be *):

$$v_{av} = \frac{s_1 - s}{t_1 - t} = \frac{\Delta s}{\Delta t}. \tag{21}$$

*) It will be noticed that the values of v_{av} as obtained from Eqs. (8′) and (21) do not coincide with each other (the first specifying v_{av} for motion along an arc, the second for the shortest path from M to M_1). In the limit, however, when $\Delta t \to 0$, both equations give the same result for v, since in the limit the ratio of arc Δs to chord MM_1 is unity.

Passing to the limit, we obtain the numerical value of the instantaneous velocity for a given time t:

$$v=\lim_{\Delta t\to 0}\frac{\Delta s}{\Delta t} \quad \text{or} \quad v=\frac{ds}{dt}=\dot{s}. \tag{22}$$

Thus, *the numerical value of the instantaneous velocity of a particle is equal to the first derivative of the dispacement* (*of the arc coordinate*) *s of the particle with respect to time.*

The velocity vector is tangent to the path, the latter assumed to be known.

Eq. (22) (or 21) gives the *numerical* (*algebraic*) value of velocity, i.e., a quantity with a sign such that the sign of v is the same as the sign of Δs as always $\Delta t > 0$. As the numerical value of the velocity

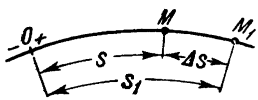

Fig. 147

vector differs from its magnitude only in sign, we shall denote both quantities by the same symbol v; this gives rise to practically no misunderstandings. Whenever it is necessary to stress that we are dealing with the magnitude of the velocity we shall denote it by the the symbol $|v|$.

As the sign of v is the same as that of Δs, it will be readily appreciated that if $v > 0$, the velocity vector $\boldsymbol{v}$ is in the positive direction of s, if $v < 0$, $\boldsymbol{v}$ is in the negative direction of s. Thus, the numerical value of the velocity defines simultaneously the modulus and the direction of the velocity vector.

Eq. (22) also shows that v can be calculated as the ratio of the infinitesimal displacement ds of a particle along the arc of its path to corresponding time interval dt.

§ 67. Tangential and Normal Accelerations of a Particle

In § 64 we computed the acceleration vector $\boldsymbol{w}$ according to its projections on the stationary coordinate axes $Oxyz$. In the natural method of describing motion, vector $\boldsymbol{w}$ is determined from its projections on a set of coordinate axes $M\tau nb$ whose origin is at M and who move together with the body (Fig. 148). These axes, called the *axes of a natural trihedron* (or velocity axes), are directed as follows: axis $M\tau$ along the tangent to the path in the direction of the positive displacements s, axis Mn along the normal in the osculating plane towards the inside of the path, and axis Mb perpendicular to the former two to form a right-hand set. The normal Mn, which lies in the osculating plane (or in the plane of the curve itself if the curve is two-dimensional), is called the *principal normal*, and the normal Mb perpendicular to it is called the *binormal*.

In § 62 it was shown that the acceleration $\boldsymbol{w}$ of a particle lies in the osculating plane, i.e., plane $M\tau n$, hence the projection of vector $\boldsymbol{w}$ on the binormal is zero ($w_b = 0$).

Let us calculate the projections of $\boldsymbol{w}$ on the other two axes. Let the particle occupy a position M and have a velocity $\boldsymbol{v}$ at any time

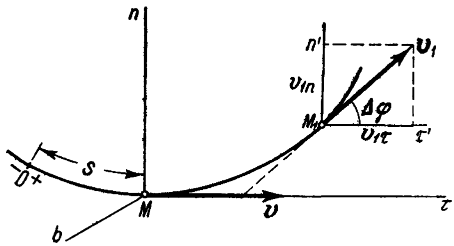

Fig. 148

t, and at time $t_1 = t + \Delta t$ let it occupy a position M_1 and have a velocity $\boldsymbol{v}_1$. Then, by virtue of the definition,

$$\boldsymbol{w} = \lim_{\Delta t \to 0} \frac{\Delta \boldsymbol{v}}{\Delta t} = \lim_{\Delta t \to 0} \frac{\boldsymbol{v}_1 - \boldsymbol{v}}{\Delta t}$$

Let us now express this equation in terms of the projections of the vectors on the axes $M\tau$ and Mn through point M (see Fig. 148). From the theorem of the projection of a vector sum (or difference) on an axis we obtain:

$$w_\tau = \lim_{\Delta t \to 0} \frac{v_{1\tau} - v_\tau}{\Delta t}, \quad w_n = \lim_{\Delta t \to 0} \frac{v_{1n} - v_n}{\Delta t}.$$

Noting that projections of a vector on parallel axes are equal, draw through point M_1 axes $M\tau'$ and Mn' parallel to $M\tau$ and Mn, respectively, and denote the angle between the direction of vector $\boldsymbol{v}_1$ and the tangent $M\tau$ by the symbol $\Delta\varphi$. This angle between the tangents to the curve at points M and M_1 is called the *angle of contiguity*.

It will be recalled that the limit of the ratio of the angle of contiguity $\Delta\varphi$ to the arc $\overset{\frown}{MM_1} = \Delta s$ defines the curvature k of the curve at point M. As the curvature is the inverse of the radius of curvature ρ at M, we have:

$$\lim_{\Delta s \to 0} \frac{\Delta\varphi}{\Delta s} = k = \frac{1}{\rho}.$$

From the diagram in Fig. 148, we see that the projections of vectors $\boldsymbol{v}$ and $\boldsymbol{v}_1$ on the axes $M\tau$ and Mn are*)

$$v_\tau = v, \qquad v_n = 0,$$
$$v_{1\tau} = v_1 \cos\Delta\varphi, \qquad v_{1n} = v_1 \sin\Delta\varphi,$$

where v and v_1 are the numerical values of the velocity of the particle at instants t and t_1. Hence,

$$w_\tau = \lim_{\Delta t\to 0} \frac{v_1 \cos\Delta\varphi - v}{\Delta t}, \qquad w_n = \lim_{\Delta t\to 0} \left(v_1 \frac{\sin\Delta\varphi}{\Delta t}\right).$$

It will be noted that when $\Delta t \to 0$, point M_1 approaches M indefinitely, and simultaneously $\Delta\varphi \to 0$, $\Delta s \to 0$, and $v_1 \to v$.

Hence, taking into account that $\lim\limits_{\Delta\varphi\to 0} (\cos\Delta\varphi) = 1$, we obtain for w_τ the expression

$$w_\tau = \lim_{\Delta t\to 0} \frac{v_1 - v}{\Delta t} = \frac{dv}{dt}.$$

We shall transform the right-hand side of the equation for w_n in such a way so that it includes ratios with known limits. For the purpose, multiplying the numerator and denominator of the fraction under the limit sign by $\Delta\varphi\Delta s$, we find:

$$w_n = \lim_{\Delta t\to 0} \left(v_1 \frac{\sin\Delta\varphi}{\Delta\varphi} \frac{\Delta\varphi}{\Delta s} \frac{\Delta s}{\Delta t}\right) = \frac{v^2}{\rho}, \tag{23}$$

since, when $\Delta t \to 0$, the limits of each of the cofactors inside the brackets are as follows:

$$\lim v_1 = v, \qquad \lim \frac{\sin\Delta\varphi}{\Delta\varphi} = 1, \quad \lim \frac{\Delta\varphi}{\Delta s} = \frac{1}{\rho}, \qquad \lim \frac{\Delta s}{\Delta t} = \frac{ds}{dt} = v.$$

Finally we obtain

$$w_\tau = \frac{dv}{dt} = \frac{d^2s}{dt^2}, \qquad w_n = \frac{v^2}{\rho}. \tag{24}$$

We have thus proved that *the projection of the acceleration of a particle on the tangent to the path is equal to the first derivative of the numerical value of the velocity, or the second derivative of the displacement (the arc coordinate) s, with respect to time; the projection of the acceleration on the principal normal is equal to the second power of the velocity divided by the radius of curvature of the path at the given point of the curve*, the projection of the acceleration on the binormal is zero ($w_b = 0$). This is an important theorem of particle kinematics.

*) If the path of particle M is not a plane curve, the equation $v_{1n} = v_1 \sin\Delta\varphi$ is approximate owing to the deviation of vector v_1 from plane $M\tau n$. The final result, however, will be exact, as in going over to the limit the deviation tends to zero.

When particle M is moving in one plane, the tangent $M\tau$ sweeps around the binormal Mb with an angular velocity $\omega = \lim_{\Delta t \to 0} \frac{\Delta\varphi}{\Delta t}$. By introducing this quantity into Eq. (23) we can obtain one more equation for calculating w_n that is frequently used in practice:

$$w_n = v\omega, \tag{24'}$$

i.e., normal acceleration equals the product of a particle's velocity by the angular velocity of the sweep of the tangent to the path.

Lay off vectors $\boldsymbol{w}_\tau$ and $\boldsymbol{w}_n$, i.e., the normal and tangential components of the acceleration, along the tangent $M\tau$ and the principal

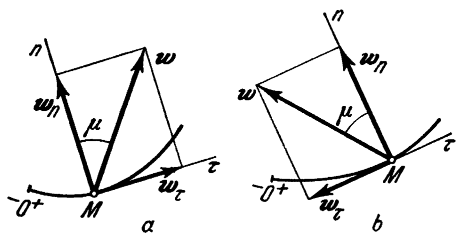

Fig. 149

normal Mn, respectively (Fig. 149). The component $\boldsymbol{w}_n$ is always directed along the inward normal, as $w_n > 0$, while the component $\boldsymbol{w}_\tau$ can be directed either in the positive or in the negative direction of the axis M_τ, depending on the sign of the projection w_τ (see Figs. 149*a* and *b*).

The acceleration vector $\boldsymbol{w}$ is the diagonal of a parallelogram constructed with the components $\boldsymbol{w}_\tau$ and $\boldsymbol{w}_n$ as its sides. As the components are mutually perpendicular, the magnitude of vector $\boldsymbol{w}$ and its angle μ to the normal Mn are given by the equations:

$$w = \sqrt{w_\tau^2 + w_n^2} = \sqrt{\left(\frac{dv}{dt}\right)^2 + \left(\frac{v^2}{\rho}\right)^2}, \quad \tan\mu = \frac{|w_\tau|}{w_n}. \tag{25}$$

Thus, if the motion of a particle is described by the natural method and the path (and, consequently, the radius of curvature at any point) and the equations of motion (20) are known, from Eqs. (22), (24), and (25) we can determine the magnitude and direction of the velocity and acceleration vectors of the particle for any instant*).

*) From Eq. (25) it is evident that in the general case $w \neq \frac{dv}{dt}$, which was already pointed out at the end of § 63.

§ 68. Some Special Cases of Particle Motion

Using the results obtained above, let us investigate some special cases of particle motion.

(1) Rectilinear Motion. If the path of a particle is a straight line, then $\rho = \infty$, and $w_n = \frac{v^2}{\rho} = 0$ and the total acceleration is equal to the tangential acceleration

$$w = w_\tau = \frac{dv}{dt}. \qquad (26)$$

As in this case the velocity changes only in magnitude, we con clude that the *tangential acceleration characterises the change of speed*

(2) Uniform Curvilinear Motion. Curvilinear motion is uniform when the speed is constant: $v = \text{const}$. Then $w_\tau = \frac{dv}{dt} = 0$, and the total acceleration of the particle is equal to the normal acceleration

$$w = w_n = \frac{v^2}{\rho}. \qquad (26')$$

The acceleration vector $\boldsymbol{w}$ is continuously directed along the nomal to the path of the particle.

As in this case acceleration is represented only by the change in the direction of the velocity, we conclude that the *normal acceleration characterises the change in direction of the velocity.*

Let us deduce the equation of uniform curvilinear motion. From Eq. (22) we have $\frac{ds}{dt} = v$, or $ds = v\,dt$.

Let a particle be at the initial time ($t = 0$) at a distance s_0 from the origin. Then, integrating both members of the equation over the respective intervals, we have

$$\int_{s_0}^{s} ds = \int_{0}^{t} v\,dt \quad \text{or} \quad s - s_0 = vt,$$

as $v = \text{const}$. Finally, we obtain the equation of uniform curvilinear motion in the form

$$s = s_0 + vt. \qquad (27)$$

Assuming $s_0 = 0$, Eq. (27) yields the particle's displacement in time t. Hence, in uniform motion displacement increases with time, and velocity is the ratio of displacement to time:

$$s = vt, \quad v = \frac{s}{t}. \qquad (27')$$

(3) **Uniform Rectilinear Motion.** In this case $w_n = w_\tau = 0$, therefore, $w = 0$. Note that *uniform rectilinear motion is the only case of motion in which the acceleration is continually zero.*

(4) **Uniformly Variable Curvilinear Motion.** Curvilinear motion is called uniformly variable if the *tangential acceleration* is constant: $w_\tau =$ const. Let us write the equation of this motion, assuming that, at $t = 0$, $s = s_0$ and $v = v_0$, where v_0 is the initial velocity of a particle. By Eq. (24), $\frac{dv}{dt} = w_\tau$, or $dv = w_\tau\, dt$.

As $w_\tau =$ const., integrating both members of the last equation over the corresponding intervals gives us:

$$v = v_0 + w_\tau t. \tag{28}$$

Let us write Eq. (28) in the form

$$\frac{ds}{dt} = v_0 + w_\tau t, \quad \text{or} \quad ds = v_0\, dt + w_\tau t\, dt.$$

Integrating again, we obtain the equation of the uniformly variable curvilinear motion of a particle in the form

$$s = s_0 + v_0 t + w_\tau \frac{t^2}{2}. \tag{29}$$

The velocity of this motion is given by Eq. (28).

If, in curvilinear motion, the magnitude of the velocity increases, the motion is said to be *accelerated*, if it decreases, the motion is said

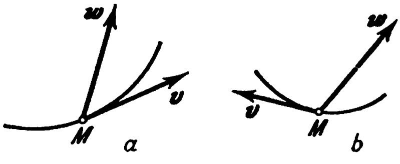

Fig. 150

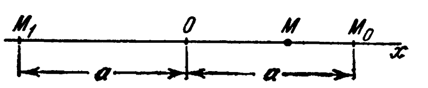

Fig. 151

to be *retarded*. As the change in magnitude of the velocity is characterized by the tangential acceleration, the motion is accelerated if v and w_τ have the same sign (the angle between vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ is acute, Fig. 150*a*) and retarded if the signs are different (the angle between $\boldsymbol{v}$ and $\boldsymbol{w}$ is obtuse, Fig. 150*b*).

In the particular case of uniformly variable motion, if v and w_τ in Eq. (28) are of the same sign the motion is *uniformly accelerated*, if they are of opposite sign the motion is *uniformly retarded*.

Assuming $s = x$, Eqs. (27)-(29) also give the laws of uniform or uniformly variable rectilinear motion. In this case, in Eqs. (28) and (29) $w_\tau = w$, where w is the magnitude of the particle's total acceleration [see Eq. (26)].

(5) Harmonic Motion. Consider the rectilinear motion of a particle, the distance x of which from the origin O of a coordinate system changes according to the equation

$$x = a \cos kt, \tag{30}$$

where a and k are constants.

In the motion defined by this equation, a particle M (Fig. 151) oscillates between positions M_0 ($+a$) and M_1 ($-a$). Motion of this type, which is known as *simple harmonic motion*, plays an important part in engineering. The quantity a, which is the greatest displacement of the particle from the centre O, is called the *amplitude of vibration.*

It will be noticed that if motion starts at time $t = 0$ from point M_0, the particle will return to that point at a time t_1, for which $\cos kt_1 = 1$, or $kt_1 = 2\pi$.

The time $T = t_1 = \frac{2\pi}{k}$ required for the point to make one complete cycle is called the *period of vibration.*

Differentiating x with respect to t, we obtain the numerical values of the velocity and acceleration of the particle:

$$v = v_x = -ak \sin kt, \quad w = w_x = -ak^2 \cos kt. \tag{30'}$$

Thus, in this type of motion the velocity and acceleration of a particle change with time according to the law of harmonic motion. The signs of v and w show that when the particle moves towards the centre of vibration, its motion is accelerated, when it moves away from the centre of vibration, it is retarded.

Motion of a similar type is described by the equation $x = a \sin kt$, only the motion starts at the centre O.

A particle oscillating according to the law $s = a \cos kt$ (or $s = a \sin kt$) may move along an arbitrary curve (see, for example, Problem 57 in § 70). In this case, everything said of the character of motion remains valid, with the exception that the last of Eqs. (30′) defines the particle's *tangential* acceleration; the particle's normal acceleration is $w_n = v^2/$.

§ 69. Graphs of Displacement, Velocity and Acceleration of a Particle

If we lay off the time t to scale on the axis of abscissas, and the displacement s on the ordinate axis, the plotted curve $s = f(t)$ gives us a *displacement-time graph* for a given particle. It shows graphically the displacement of the particle (the change of its x coordinate) with reference to time.

Similarly, we can plot curves to some scale to represent the *velocity-time* relationship $v(t)$ and the *acceleration-time* relationships $w_\tau(t)$, $w_n(t)$, $w(t)$ for the *tangential*, *normal* and *total accelerations*. In Figs. 152*a*, *b*, and *c* are given the graphs for the cases of motion described by Eqs. (27), (29), and (30). Also below on these graphs are the velocity-time curve and the tangential acceleration-time curve.

The displacement-time curve of uniform motion, we see, is a straight line inclined to the axis of abscissas, the velocity-time curve of such motion is a straight line parallel to the axis of abscissas (v =const.), and the tangential accelerationtime curve is a straight

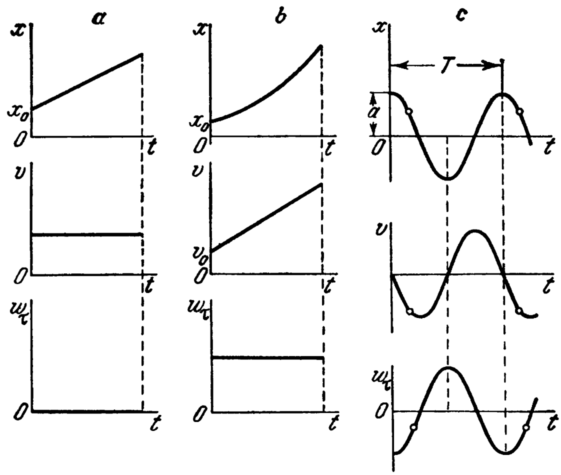

Fig. 152

line coincident with the axis of abscissas ($w_\tau = 0$). For uniformly variable motion (accelerated in the case shown in Fig. 152*b*), the displacement curve is a branch of a parabola, the velocity curve is a straight line inclined to the axis of abscissas, and the tangential acceleration curve is a straight line parallel to the axis of abscissas (w_τ = const.). Finally, for simple harmonic motion (Fig. 152*c*) the respective graphs are represented by cosine and sine curves.

A displacement graph for a particle should not be confused with its path, which in all the above cases must be given additionally. There are no graphs of normal and total acceleration in Fig. 152, because w_n and w depend not only on the law of motion but also on ρ, i.e., the type of path, being different for different paths for motion described by the same equations $s = f(t)$.

§ 70. Solution of Problems

As pointed out before, in order to solve problems of kinematics, we must know the equations of motion for a given particle. If the motion is described by the natural method (the path and the equation of motion along it are given) all its characteristics are found from the equations developed in §§ 66-68. The tangential and normal accelerations can also be determined when the motion is described by the *coordinate method*, i.e., by Eqs. (3) or (4). For this calculate v and w from Eqs. (15)-(18). Taking the derivative of v with respect to time, determine $w_\tau = \frac{dv}{dt}$. Now, knowing w and w_τ, determine w_n from the equation $w^2 = w_\tau^2 + w_n^2$. We can also determine the radius of curvature of the path ρ from the formula $w_n = v^2/\rho$. An example of this is given in Problem 59.

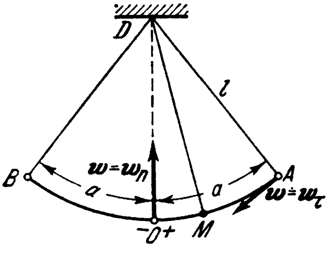

Fig. 153

Problem 57. Small oscillations of the pendulum shown in Fig. 153 are represented by the equation of motion $s = a \sin kt$ (the origin is at O, a and k are constants). Determine the velocity, tangential and normal accelerations of the bob and the positions, at which they become zero if the bob describes a circular arc of radius l.

Solution. From the respective equations we find:

$$v = \frac{ds}{dt} = ak \cos kt, \quad w_\tau = \frac{dv}{dt} = -ak^2 \sin kt,$$

$$w_n = \frac{v^2}{l} = \frac{a^2k^2}{l} \cos^2 kt.$$

The equation of motion is that of simple harmonic motion, the amplitude being a. In the extreme positions A and B, $\sin kt = \pm 1$ and consequently, $\cos kt = 0$. Hence, in positions A and B, velocity and normal acceleration are zero and the tangential acceleration has its maximum value $w_{\tau\,max} = ak^2$. At the origin O, where $s = 0$, the reverse is true, and $\sin kt = 0$ while $\cos kt = 1$. In this position, $w_\tau = 0$ and v and w_n have their maximum values:

$$v_{max} = ak, \quad w_{n\,max} = \frac{a^2k^2}{l}.$$

We observe from this example that in non-uniform curvilinear motion w_τ or w_n may become zero at different points of the path, specifically, $w_\tau = 0$ wherever

$$\frac{dv}{dt} = 0,$$

i.e., for example, where v is at its maximum or minimum; $w_n = 0$ at the points where $v = 0$ (as in the present case) or where $\rho = \infty$ (the points of inflection of the path).

Problem 58. A train starts moving from rest with uniform acceleration along a curve of radius $R = 800$ m and reaches a velocity $v_1 = 36$ km/h after travelling a distance $s_1 = 600$ m. Determine the velocity and acceleration of the train at the middle of this distance.

Solution. As the train moves with uniform acceleration and $v_0 = 0$, its equation of motion (assuming $s_0 = 0$) is

$$s = \frac{1}{2} w_\tau t^2,$$

and its velocity is

$$v = w_\tau t.$$

Eliminating time t from these equations, we obtain

$$v^2 = 2w_\tau s.$$

According to the conditions of the problem, at $s = s_1$, $v = v_1$, whence we find

$$w_\tau = \frac{v_1^2}{2s_1}.$$

At the middle of the path, where $s_2 = \frac{s_1}{2}$, the velocity v_2 is:

$$v_2 = \sqrt{2w_\tau s_2} = \sqrt{w_\tau s_1} = \frac{v_1}{\sqrt{2}}.$$

The normal acceleration at this point of the path is:

$$w_{n\,2} = \frac{v_2^2}{R} = \frac{v_1^2}{2R}.$$

Knowing w_τ and w_{n2} we find the total acceleration of the train at the middle of the curve:

$$w_2 = \sqrt{w_\tau^2 + w_{n2}^2} = \frac{v_1^2}{2}\sqrt{\frac{1}{s_1^2} + \frac{1}{R^2}}.$$

Substituting the numerical values, we obtain:

$$v_2 \approx 7.1 \text{ m/s}, \quad w_2 = \frac{5}{48} \approx 0.1 \text{ m/s}^2.$$

Problem 59. The equations of motion for a particle thrown with a horizontal velocity are:

$$x = v_0 t, \quad y = \frac{1}{2} g t^2,$$

where v_0 and g are constants.

Determine the path, velocity and acceleration of the particle, its tangential and normal accelerations and the radius of curvature of its path at any point, expressing them in terms of the velocity of the particle at the given point.

Solution. Determining t from the first equation and substituting its expression into the second, we obtain

$$y = \frac{g}{2v_0^2} x^2.$$

The path of the particle is a parabola (Fig. 154).

Differentiating the equations of motion with respect to time, we find:

$$v_x = \dot{x} = v_0, \quad v_y = \dot{y} = gt,$$

whence

$$v = \sqrt{v_0^2 + g^2t^2}. \tag{a}$$

Thus, at the initial moment ($t = 0$) the velocity of the particle $v = v_0$ and it continuously increases with time.

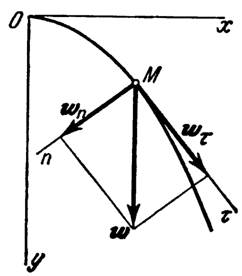

Fig. 154

Let us now determine the acceleration of the particle. From the respective equations we have:

$$w_x = \ddot{x} = 0, \quad w_y = \ddot{y} = g.$$

Consequently, the acceleration is

$$w = g.$$

In the present case the particle has an acceleration of constant magnitude and direction parallel to axis y (the acceleration of gravity). Note that, although w = const., the motion of the particle is not uniformly variable, since the condition for uniformly variable motion is not w = const., but w_τ = const. In this case, we shall find, w_τ is not constant.

Knowing the dependence of v on t [formula (a)], we can find w_τ:

$$w_\tau = \frac{dv}{dt} = \frac{g^2t}{\sqrt{v_0^2 + g^2t^2}} = \frac{g^2t}{v}.$$

But from equation (a) we have $v^2 = v_0^2 + g^2t^2$, and consequently,

$$t = \frac{1}{g}\sqrt{v^2 - v_0^2}.$$

Substituting this expression of t, we express w_τ in terms of the velocity v:

$$w_\tau = g\sqrt{1 - \frac{v_0^2}{v^2}}.$$

It follows that at the initial moment, when $v = v_0$, w_τ equals zero, then increases together with v, and, as $v \to \infty$, $w_\tau \to g$. Thus, in the limit the tangential acceleration approaches the total acceleration g.

To determine w_n, we refer to the equation

$$w^2 = w_\tau^2 + w_n^2,$$

whence

$$w_n^2 = w^2 - w_\tau^2 = g^2 - g^2\left(1 - \frac{v_0^2}{v^2}\right) = g^2 \frac{v_0^2}{v^2},$$

and

$$w_n = \frac{v_0 g}{v}.$$

Thus, at the initial moment ($v = v_0$), $w_n = g$, decreasing as v increases and in the limit approaching zero.

To determine the radius of curvature of the path, we use the equation

$$w_n = \frac{v^2}{\rho},$$

whence

$$\rho \frac{v^2}{w_n} = \frac{v^3}{v_0 g}.$$

At the initial moment the radius of curvature has its smallest value

$$\rho_{\min} = \frac{v_0^2}{g}.$$

As v increases, the radius of curvature increases, and consequently, the curvature of the path decreases. As $v \to \infty$, $\rho \to \infty$ and the curvature tends to zero.

§ 71*. Velocity in Polar Coordinates

If a particle moves in a plane, its position can be specified by its polar coordinates r and φ (Fig. 155). As the particle moves, these coordinates change with time, the motion of the particle in polar coordinates being given by the equations

$$r = f_1(t), \quad \varphi = f_2(t). \qquad (31)$$

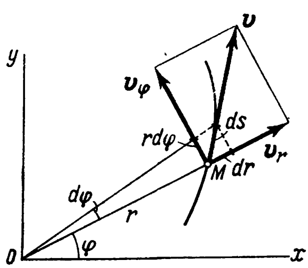

Fig. 155

The numerical value of the velocity of the particle is ds/dt, i.e., it is equal to the ratio of the infinitesimal displacement ds to the time interval dt. In this case the displacement ds is the geometrical sum of the radial displacement, equal in magnitude to dr, and the transverse displacement perpendicular to the radius and equal in magnitude to $r\,d\varphi$. The velocity $\boldsymbol{v}$ is thus the geometrical sum of the radial velocity $\boldsymbol{v}_r$ and the transverse velocity $\boldsymbol{v}_\varphi$, whose magnitudes are, respectively,

$$v_r = \frac{dr}{dt} = \dot{r}, \qquad v_\varphi = r\,\frac{d\varphi}{dt} = r\dot{\varphi}. \tag{32}$$

As v_r and v_φ are mutually perpendicular,

$$v = \sqrt{v_r^2 + v_\varphi^2} = \sqrt{\dot{r}^2 + r^2\dot{\varphi}^2}. \tag{33}$$

Eqs. (32) and (33) give the velocity of a particle moving in a plane in polar coordinates.

Equation (33) can also be derived by expressing the Cartesian coordinates of the particle in terms of r and φ in the form (see Fig. 155):

$$x = r\cos\varphi, \quad y = r\sin\varphi.$$

Then $\dot{x} = \dot{r}\cos\varphi - r\dot{\varphi}\sin\varphi$, $\dot{y} = \dot{r}\sin\varphi + r\dot{\varphi}\cos\varphi$, and from Eq. (16)

$$v = \sqrt{\dot{x}^2 + \dot{y}^2} = \sqrt{\dot{r}^2 + r^2\dot{\varphi}^2}.$$

In the same way, calculating $\ddot{x}$ and $\ddot{y}$, we can use Eq. (18) to determine the expression for the particle's acceleration in polar coordinates:

$$w = \sqrt{(\ddot{r} - r\dot{\varphi}^2)^2 + (r\ddot{\varphi} + 2\dot{r}\dot{\varphi})^2}\,. \tag{33'}$$

Here the quantity in the first brackets is equal to w_r, and in the second, to w_φ.

§ 72*. Graphical Analysis of Particle Motion

The graphical method of solving problems of particle kinematics is useful when the analytical expression of the relationship $s = f(t)$ [or, in rectilinear motion, $x = f(t)$] is too involved or when the motion is described by graphs obtained experimentally or plotted by recording instruments.

If we have a graph of motion expressing the displacement-time relationship (Fig. 156), we can plot a velocity curve by the method of graphical differentiation. It is apparent from the diagram that the average velocity of the given particle in the time interval $\Delta t = t_2 - t_1$ is given by the tangent of the angle which the secant K_1K_2 makes with the horizontal, as

$$v_{\text{av}} = \frac{s_2 - s_1}{t_2 - t_1} = \frac{\Delta s}{\Delta t} = \tan\alpha^*,$$

the accuracy depending on the scale factor.

The velocity of the particle at the given time t_1 is specified by the tangent of the angle made by the tangent to the curve at point K_1, as

$$v_1 = \left(\frac{ds}{dt}\right)_{t=t_1} = \tan \alpha_1, \tag{34}$$

also within the scale factor.

Thus, by drawing tangents to the displacement curve at points K_1, K_2, . . . we can find the corresponding velocities at instants t_1, t_2, . . . from the tangents of the respective angles to the horizontal

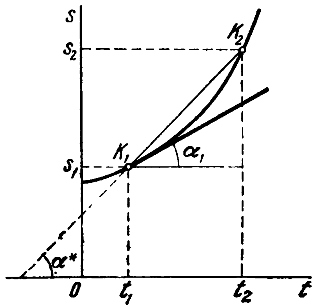

Fig. 156

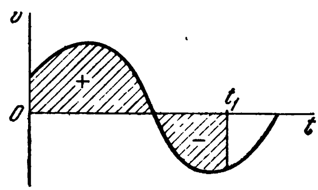

Fig. 157

and plot a velocity curve. Similarly we can plot a curve for the tangential acceleration $w_\tau = \frac{dv}{dt}$. The order to be followed in plotting the curves (with taking into account the scales to which t and s are drawn) is explained in the solution of Problem 60.

To plot the curves of the normal and total accelerations (in the case of curvilinear motion) the values of w_n and w at different moments are computed from the corresponding equations. The values of v and w_τ are taken from the plotted velocity and tangential acceleration curves, and ρ is determined from the given path.

If the velocity curve is known for a given motion, it is easy to plot a displacement curve by the method of graphical integration. As $ds = v\,dt$, then, assuming $s_0 = 0$, we have

$$s = \int_0^t v\,dt.$$

The integral in the right-hand side is computed as the respective area, taken with a plus for positive values of v and a minus for negative values. For greater convenience in computing areas, the curve should be plotted on millimeter graph paper: the area is then obtained by simply counting the squares. In the case in Fig. 157, the displacement s_1 at time t_1 is equal to the difference between the upper and lower shaded areas multiplied by a scale factor *). By determining s for different instants t, we can plot a displacement curve. A tangential acceleration curve can be used in the same way to plot a velocity curve

Problem 60. Investigate graphically the motion of a piston of a crankshaft mechanism (Fig. 158) if the crank is of length $O'A = 0.2$ m, the connecting rod is of length $AB = 0.4$ m, and the crank rotates uniformly making one revolution in $T = 1.6$ s.

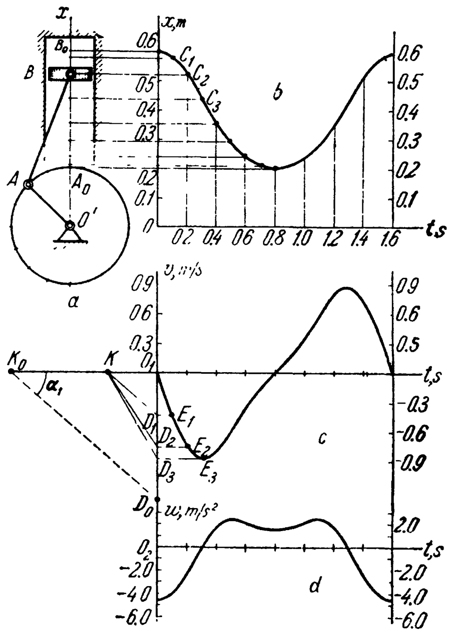

Fig. 158

Solution. **(1) Plotting the displacement curve for the piston.** The following operations should be performed:

(a) Choose scales for the displacement x and time t. If you are using graph paper, remember that the error in measurement can be as high as ± 0.25 to 0.5 mm (depending on the quality of the paper and the drawing). Let us take a scale of 0.1 m to 1 cm for x, and 0.2 s to 1 cm for t.

(b) Draw a schematic design of the mechanism to the scale chosen for x (Fig. 158a): we have $O'A_0 = 2$ cm and $A_0B_0 = 4$ cm. Draw the coordinate axis $O'x$ coincident with the path of the piston. Divide the semicircle along which point A travels into 8 equal parts (the greater the number of parts, the more accurate the graph). Point A travels along each part in 0.1 s. Set the legs of your compass 4 cm apart (the distance AB) and from the end of each arc of the semicircle make an intercept on axis $O'x$. These intercepts give the values of x for the instants t equal to 0, 0.1 s, 0.2 s, etc.

(c) The obtained values of x and t give us the points for plotting the displacement curve for the piston through one revolution of the crank (Fig. 158b); the right-hand branch of the curve in this case is symmetrical to the left-hand branch.

*) The sum of these areas defines the distance travelled by the particle in time t_1, if $s_0 = 0$.

(2) Plotting the velocity curve. (a) Draw axes v and t (Fig. 158*c*) to the same scale of t as in Fig. 158*b*. Lay off along axis t a segment O_1K_0, which denotes one second to the given scale ($O_1K_0 = 5$ cm).

(b) Determine the direction of the tangent to the curve $x = f(t)$ at point C_1 corresponding to time $t_1 = 0.1$ s *). Draw through point K_0 a line K_0D_0 parallel to the tangent (the dashed line in Fig. 158*c*). Segment O_1D_0 specifies the velocity at time $t_1 = 0.1$ s, since $O_1K_0 = 1$ s and, from Eq. (34),

$$(v)_{t=0.1\text{ s}} = \tan\alpha_1 = \frac{O_1D_0}{O_1K_0} = O_1D_0.$$

If the scale of the displacement x is 1 cm to 0.1 m, the scale of the velocity v will be 1 cm to 0.1 m/s. If the scale is inconvenient for drawing, being too large or too small, it can be altered as shown below.

(c) In our case the scale of v is too large. Let us reduce it by three. For this lay off from O_1 a segment $O_1K = \frac{1}{3} O_1K_0$. Draw through K a line KD_1 parallel to the tangent through point C_1 (or to line K_0D_0). Segment O_1D_1 now defines the velocity at time $t_1 = 0.1$ s to the scale 1 cm to 0.3 m/s (the scale of v).

Drawing D_1E_1 parallel to axis O_1t, we obtain point E_1 of the velocity curve.

Similarly, by drawing through K lines KD_2, KD_3, . . . parallel the tangents at points C_2, C_3, . . . we obtain points E_2, E_3, . . . joining which in a continuous curve, we obtain the velocity graph for the piston (Fig. 158*c*).

(3) Plotting the acceleration curve from the velocity curve (or the tangential acceleration curve if the motion is curvilinear). The procedure is analogous to that for plotting the velocity curve from the displacement curve. Fig. 158*d* gives the acceleration curve for the piston to the scale 1 cm to 2.5 m/s^2.

Problem 61. Determine by graphical construction the contour of the cam in Fig. 159*a*, such that when it rotates uniformly about its axle O', the motion of rod AB will be as described by the curve in Fig. 159*b*, where T is the period of one revolution of the cam. During the first quarter revolution the rod moves up 0.2 m, during the second quarter revolution it remains motionless, and during the

---

*) A convenient instrument for drawing a tangent through a given point of a curve is a ruler with a reflecting surface perpendicular to the plane of the ruler (you can make such an instrument yourself by pasting a strip of smoothed tinfoil to the side of your slide rule). Now place your ruler approximately perpendicular to the curve at the required point and turn it till the curve and its reflection make a continuous line. In this position the ruler gives the direction of the normal, the perpendicular to which through the given point is the tangent at that point. If you are using a strip of tinfoil instead of a mirror, better draw the curve in Indian ink.

second half revolution it returns to its initial position. Also plot the velocity and acceleration curves for the rod. The scale of the displacement curve is 1 cm to 0.1 m.

Solution. Let us plot the contour of the cam to the same scale as the given displacement curve. For this draw a circle of radius 3 cm with its centre at O' and divide it into 16 equal parts. Also divide

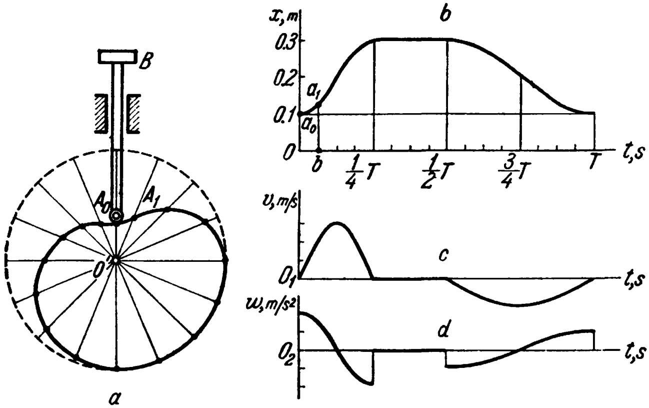

Fig. 159

segment OT on the t axis of the graph into 16 equal parts. Lay off the value of x for each instant of time along the corresponding radii from the centre O' ($O'A_0 = Oa_0$, $O'A_1 = ba_1$, etc.). Joining points A_0, A_1, etc., in a smooth curve, we obtain the required contour of the cam (Fig. 159a).

If the shape of the cam is given, we can plot the displacement curve for the rod AB by a similar construction.

By specifying any time T for one revolution of the cam, we can construct the velocity and acceleration curves for the rod AB as in the previous problem.

The curves are given in Figs. 159c and d. As stated in the conditions of the problem, the velocity and acceleration of the rod are zero during the second quarter revolution of the cam. The velocity of the rod changes continuously, but the acceleration jumps at instants $t_1 = 1/4\ T$ and $t_2 = 1/2\ T$.

Chapter 11
Translational and Rotational Motion of a Rigid Body

§ 73. Translational Motion

In kinematics, as in statics, we shall regard all solids as rigid bodies, i.e., we shall assume that the distance between any two points of a body remains the same during the whole period of motion.

Problems of kinematics of rigid bodies are basically of two types: (1) definition of the motion and analysis of the kinematic characteristics of the motion of a body as a whole; (2) analysis of the motion of every point of the body in particular.

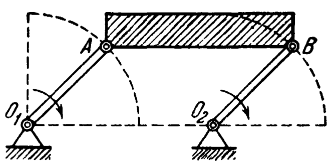

Fig. 160

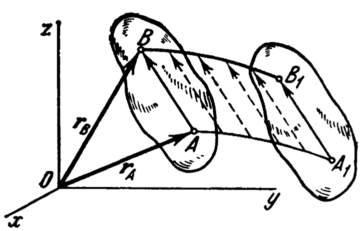

Fig. 161

We shall begin with the consideration of the motion of translation of a rigid body.

Translation of a rigid body is such a motion in which any straight line through the body remains continually parallel to itself.

Translation should not be confused with rectilinear motion. In translation the particles of a body may move on any curved paths. Here are some examples of translation.

(1) When a motor car travels along a horizontal road, the motion of its body is that of translation, since every point of the body moves on a straight-line path.

(2) The motion of the connecting rod AB in Fig. 160 is also that of translation, since, when the cranks O_1A and O_2B ($O_1A = O_2B$)

rotate, any straight line through the rod remains parallel to itself. The particles of the connecting rod travel in circles.

The properties of translational motion are defined by the following theorem: *In translational motion, all the particles of a body move along similar paths (which will coincide if superimposed) and have at any instant the same velocity and acceleration.*

To prove the theorem, consider a rigid body translated with reference to a system of axes *Oxyz*. Take two arbitrary points *A* and *B* on the body whose positions at time *t* are specified by radius vectors $\boldsymbol{r}_A$ and $\boldsymbol{r}_B$ (Fig. 161). Draw a vector $\overline{AB}$ joining the two points. It is evident that

$$\boldsymbol{r}_B = \boldsymbol{r}_A + \overline{AB}. \tag{35}$$

The length of *AB* is constant, being the distance between two points of a rigid body, and the direction of $\overline{AB}$ is constant by virtue of the translational motion of the body. Thus, the vector $\overline{AB}$ is constant throughout the motion of the body ($\overline{AB} = \text{const.}$). It follows then from Eq. (35) (and the diagram) that the path of particle *B* can be obtained by a parallel displacement of all the points of the path of particle *A* through a constant vector $\overline{AB}$. Hence, the paths of particles *A* and *B* are identical curves which will coincide if superimposed.

To determine the velocities of points *A* and *B*, we differentiate both parts of Eq. (35) with respect to time. We have

$$\frac{d\boldsymbol{r}_B}{dt} = \frac{d\boldsymbol{r}_A}{dt} + \frac{d(\overline{AB})}{dt}.$$

But the derivative of the constant vector $\overline{AB}$ is zero while the derivatives of vectors $\boldsymbol{r}_A$ and $\boldsymbol{r}_B$ with respect to time give the velocities of points *A* and *B*. Thus we find that

$$\boldsymbol{v}_A = \boldsymbol{v}_B,$$

i.e., at any instant the velocities of points *A* and *B* are equal in magnitude and direction

Again, differentiating both sides of the equation with respect to time, we obtain

$$\frac{d\boldsymbol{v}_A}{dt} = \frac{d\boldsymbol{v}_B}{dt} \quad \text{or} \quad \boldsymbol{w}_A = \boldsymbol{w}_B.$$

Hence, at any instant the accelerations of *A* and *B* are equal in magnitude and direction.

As points *A* and *B* are arbitrary, it follows that the paths and the velocities and accelerations of all the points of a body at any instant are the same, which proves the theorem.

It follows from the theorem that the translational motion of a rigid body is fully described by the motion of any point belonging to it. Thus, the analysis of translational motion of a rigid body is reduced to the methods of particle kinematics examined before

The common velocity v of all the points of a body in translational motion is called the *velocity of translation*, and the common acceleration w is called the *acceleration of translation*. Vectors v and w can, obviously, be shown as applied at any point of the body.

It should be noted that the notions of velocity and acceleration of a body make sense only when translation is considered. In all other cases, as we shall see later, the points of a body have different velocities and accelerations and the expressions "velocity of a body" or "acceleration of a body" are meaningless.

§ 74. Rotational Motion of a Rigid Body. Angular Velocity and Angular Acceleration

Rotation of a rigid body is such a motion in which there are always two points of the body (or body extended) which remain motionless (see Fig. 162). The line AB through these fixed points is called the *axis of rotation*

Since the distance between the points of a rigid body does not change, it is evident that in rotational motion all points of the body on the axis of rotation are motionless, while all the other points of the body describe circular paths the planes of which are perpendicular to the axis of rotation and the centres of which lie on it *).

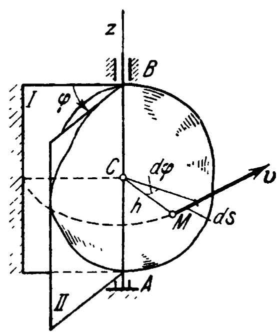

Fig. 162

To determine the position of a rotating body, let us pass two planes through the axis of rotation Az: plane I, which is fixed, and plane II through the rotating body and rotating with it (Fig. 162). The position of the body at any instant will be fully specified by the angle φ between the two planes, taken with the appropriate sign, which we shall call the *angle of rotation* of the body. We shall consider the angle positive if it is laid off counterclockwise from the fixed plane by an observer looking from the positive end of axis Az, and negative if it is laid off clockwise. Angle φ is always measured in *radians*.

*) A body may rotate about an axis without any point of it belonging to that axis, e.g., the rotation of a wheel on an axle or the rotation of a person riding a merry-go-round.

The position of a body at any instant is completely specified if we know the angle φ as a function of time t, i.e.,

$$\varphi = f(t). \tag{36}$$

Eq. (36) describes the *rotational motion of a rigid body.*

The principal kinematic characteristics of the rotation of a rigid body are its *angular velocity* ω and *angular acceleration* ε.

If in an interval of time $\Delta t = t_1 - t$ a body turns through an angle $\Delta\varphi = \varphi_1 - \varphi$, the average angular velocity of the body in the given time interval is

$$\omega_{av} = \frac{\Delta\varphi}{\Delta t}.$$

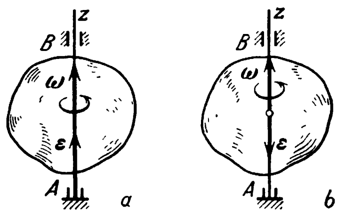

Fig. 163

The angular velocity of a body at a given time t is the value towards which ω_{av} tends when the time interval Δt tends to zero:

$$\omega = \lim_{\Delta t \to 0} \frac{\Delta\varphi}{\Delta t} \quad \text{or} \quad \omega = \frac{d\varphi}{dt}. \tag{37}$$

Thus, *the angular velocity of a body at a given time is equal in magnitude to the first derivative of the angle of rotation with respect to time.* Eq. (37) also shows that the value of ω is equal to the ratio of the infinitesimal angle of rotation $d\varphi$ to the corresponding time interval dt. The sign of ω specifies the direction of the rotation. It will be noticed that $\omega > 0$ when the rotation is counterclockwise, and $\omega < 0$ when the rotation is clockwise. The dimension of angular velocity, if the time is measured in seconds, is

$$[\omega] = \frac{\text{radian}}{\text{sec}} = \text{s}^{-1},$$

as the radian is a dimensionless unit.

The angular velocity of a body can be denoted by a vector $\boldsymbol{\omega}$ of magnitude $\omega = \frac{d\varphi}{dt}$ along the axis of rotation of the body in the direction from which the rotation is seen as counterclockwise (see Fig. 163). Such a vector simultaneously gives the magnitude of the angular velocity, the axis of rotation, and the sense of rotation about that axis.

Angular acceleration characterises the time rate of change of the angular velocity of a rotating body.

If in a time interval $\Delta t = t_1 - t$ the change of angular velocity of a body is $\Delta\omega = \omega_1 - \omega$, the average angular acceleration of the body in that interval of time is

$$\varepsilon_{av} = \frac{\Delta\omega}{\Delta t}.$$

The angular acceleration at a given time t is the value towards which ε_{av} tends when the time interval Δt tends to zero. Thus,

$$\varepsilon = \lim_{\Delta t \to 0} \frac{\Delta\omega}{\Delta t} = \frac{d\omega}{dt},$$

or, taking into account Eq. (37)

$$\varepsilon = \frac{d\omega}{dt} = \frac{d^2\varphi}{dt^2}. \tag{38}$$

Thus, *the angular acceleration of a body at a given time is equal in magnitude to the first derivative of the angular velocity, or the second derivative of the angular displacement, of the body with respect to time.* The dimension of angular acceleration is $[\varepsilon] = s^{-2}$.

If the angular velocity increases in magnitude, the rotation is *accelerated*, if it decreases, the rotation is *retarded*. It will be readily noticed that the rotation is accelerated when ω and ε are of the same sign, and retarded when they are of different sign.

By analogy with angular velocity, the angular acceleration of a body can be denoted by a vector $\boldsymbol{\varepsilon}$ along the axis of rotation. The direction of $\boldsymbol{\varepsilon}$ coincides with that of $\boldsymbol{\omega}$ when the rotation is accelerated (Fig. 163*a*), and is of opposite sense when the rotation is retarded (Fig. 163*b*).

§ 75. Uniform and Uniformly Variable Rotations

If the angular velocity of a rotating body does not change ($\omega =$ = const.), the rotation is said to be *uniform*. Let us develop the equation of uniform rotation. We have from Eq. (37) $d\varphi = \omega\, dt$. Hence, assuming that at the initial moment ($t = 0$) angle $\varphi = 0$ and integrating the left-hand member from 0 to φ and the right-hand member from 0 to t, we obtain:

$$\varphi = \omega t. \tag{39}$$

It follows from Eq. (39) that in uniform rotation

$$\omega = \frac{\varphi}{t}. \tag{40}$$

In engineering, the velocity of uniform rotation is often expressed as the number of revolutions per minute, viz. n rpm *).

Let us establish the relation between n rpm and ω s^{-1}. A complete revolution turns a body through an angle of 2π and n revolutions take it through an angle $2\pi n$. If the duration of this rotation is

*) It should be noted that n is an angular velocity and not an angle.

$t = 1$ min $= 60$ s, then from Eq. (40) we have

$$\omega = \frac{\pi n}{30} \approx 0.1\, n. \tag{41}$$

If the angular acceleration of a body does not change during the rotation (ε = const.), the rotation is said to be *uniformly variable.* Let us develop the equation of uniformly variable rotation assuming that at the initial instant ($t = 0$) angle $\varphi = 0$ and that the angular velocity $\omega = \omega_0$ (where ω_0 is the initial angular velocity).

From Eq. (38) we have $d\omega = \varepsilon\, dt$. Integrating the left-hand member over the interval ω_0 to ω and the right-hand member from 0 to t, we obtain

$$\omega = \omega_0 + \varepsilon t. \tag{42}$$

Let us write Eq. (42) in the form

$$\frac{d\varphi}{dt} = \omega_0 + \varepsilon t \quad \text{or} \quad d\varphi = \omega_0\, dt + \varepsilon t\, dt.$$

Integrating again, we obtain the equation of uniformly variable rotation

$$\varphi = \omega_0 t + \varepsilon \frac{t^2}{2}. \tag{43}$$

The angular velocity ω of this rotation is given by Eq. (42). If ω and ε have the same sign, the rotation is uniformly accelerated, if they have opposite signs, it is uniformly retarded.

§ 76. Velocities and Accelerations of the Points of a Rotating Body

Having established in the previous sections the characteristics of the motion of bodies as a whole, let us now investigate the motion of the individual points of a body.

Consider a point M of a rigid body at a distance h from the axis of rotation Az (Fig. 162). When the body rotates, point M describes a circle of radius h in a plane perpendicular to the axis of rotation with its centre C on that axis. If in time dt the body makes an infinitesimal displacement through an angle $d\varphi$, point M will have made a very small displacement $ds = h\, d\varphi$ along its path. The velocity of the point is the ratio of ds to dt, i.e.,

$$v = \frac{ds}{dt} = h \frac{d\varphi}{dt},$$

or

$$v = h\omega \tag{44}$$

This velocity v is called the *linear*, or *circular*, velocity of the point M (not to be confused with its angular velocity).

Thus, *the linear velocity of a point belonging to a rotating body is equal to the product of the angular velocity of that body and the distance*

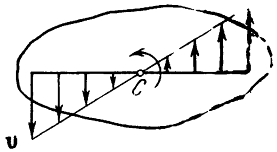

Fig. 164

of the point from the axis of rotation. The linear velocity is tangent to the circle described by point M, or perpendicular to the plane through the axis of rotation and the point M.

As the value of ω at any given instant is the same for all points of the body, it follows from Eq. (44) that the linear velocity of any

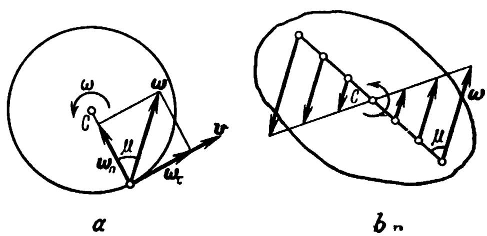

Fig. 165

point of a rotating body is proportional to its distance from the axis of rotation (Fig. 164).

In order to determine the acceleration of point M, we apply equations

$$w_\tau = \frac{dv}{dt}, \qquad w_n = \frac{v^2}{\rho}.$$

In our case, $\rho = h$. Substituting the expression for v from Eq. (44), we obtain

$$w_\tau = h\frac{d\omega}{dt}, \qquad w_n = \frac{h^2\omega^2}{h},$$

and finally

$$w_\tau = h\varepsilon, \ w_n = h\omega^2. \tag{45}$$

The tangential acceleration w_τ is tangent to the path (in the direction of the rotation if it is accelerated and in the reverse direction if it is retarded); the normal acceleration w_n is always directed along the radius h towards the axis of rotation (Fig. 165*a*).

The total acceleration of point M is

$$w=\sqrt{w_\tau^2+w_n^2}=\sqrt{h^2\varepsilon^2+h^2\omega^4},$$

or

$$w=h\sqrt{\varepsilon^2+\omega^4}. \tag{46}$$

The inclination of the vector of total acceleration to the radius of the circle described by the point is specified by the angle μ given by the equation

$$\tan\mu=\frac{|w_\tau|}{w_n}.$$

Substituting the expressions of w_τ and w_n from Eqs. (45), we obtain

$$\tan\mu=\frac{|\varepsilon|}{\omega^2}. \tag{47}$$

Since at any given instant ε and ω are each the same for all the points of the body, it follows from Eqs. (46) and (47) that the accelerations of all the points of a rotating rigid body are proportional to their distances from the axis of rotation and make the same angle μ with the radii of the circles described by them (Fig. 165*b*).

Eqs. (44)-(47) make it possible to determine the velocity and acceleration of any point of a body if the equation of rotation of the body and the distance of the given point from the axis of rotation are known. With these formulas, knowing the motion of any single point of a body, it is possible to determine the motion of any other point and the characteristics of the motion of the body as a whole.

Problem 62. A shaft rotating with a speed of $n = 90$ rpm decelerates uniformly when the motor is switched off and stops in $t_1 = 40$ s. Determine the number of revolutions made by the shaft in this time.

Solution. As the rotation is uniformly retarded,

$$\varphi=\omega_0 t+\varepsilon\frac{t^2}{2}, \tag{a}$$

$$\omega=\omega_0+\varepsilon t. \tag{b}$$

The initial angular velocity of the uniformly retarded rotation is that which the shaft had before the motor was switched off. Hence,

$$\omega_0=\frac{\pi n}{30}.$$

At the instant $t = t_1$, when the shaft stopped, its angular velocity was $\omega_1 = 0$. Substituting these values into equation (b), we obtain:

$$0=\frac{\pi n}{30}+\varepsilon t_1, \quad \varepsilon=-\frac{\pi n}{30t_1}.$$

If we denote as N the number of revolutions of the shaft in time t_1 (not to be confused with n which is the angular velocity!), the angle of rotation in that time will be $\varphi_1 = 2\pi N$. Substituting the values of ε and φ_1 in equation (a), we obtain

$$2\pi N = \frac{\pi n}{30} t_1 - \frac{\pi n}{60} t_1 = \frac{\pi n}{60} t_1,$$

whence

$$N = \frac{n t_1}{120} = 30 \text{ revolutions}.$$

Problem 63. A flywheel of radius $R = 1.2$ m rotates uniformly making $n = 90$ rpm. Determine the linear velocity and acceleration of a point on the rim of the flywheel.

Solution. The linear velocity of such a point is $v = R\omega$, where the angular velocity ω must be expressed in radians per second. In our case

$$\omega = \frac{\pi n}{30} = 3\pi \text{ s}^{-1}.$$

Hence,

$$v = \frac{\pi n}{30} R \approx 11.3 \text{ m/s}.$$

As $\omega = \text{const}$, $\varepsilon = 0$ and, consequently,

$$w = w_n = R\omega^2 = \frac{\pi^2 n^2}{900} R \approx 106.6 \text{ m/s}^2.$$

The acceleration is directed towards the axis of rotation.

Problem 64. The equation of motion of an accelerated flywheel is

$$\varphi = \frac{9}{32} t^3.$$

Determine the linear velocity and acceleration of a point lying at a distance $h = 0.8$ m from the axis of rotation at the instant when its tangential and normal accelerations are equal.

Solution. We determine the angular velocity and angular acceleration of the flywheel:

$$\omega = \frac{d\varphi}{dt} = \frac{27}{32} t^2, \quad \varepsilon = \frac{d\omega}{dt} = \frac{27}{16} t.$$

The formulas for the tangential and normal accelerations of the point are $w_\tau = h\varepsilon$, and $w_n = h\omega^2$.

Denote the instant when $w_\tau = w_n$ by the symbol t_1. Obviously, at that instant $\varepsilon_1 = \omega_1^2$ or

$$\frac{27}{16} t_1 = \left(\frac{27}{32}\right)^2 t_1^4,$$

whence

$$t_1^3 = \frac{64}{27}, \quad t_1 = \frac{4}{3} \text{ s}.$$

Substituting this value of t_1 in the expressions for ω and ε, we find that at time t_1

$$\omega_1 = \frac{3}{2}\ \text{s}^{-1}, \quad \varepsilon_1 = \frac{9}{4}\ \text{s}^{-2}.$$

The required values are thus

$$v_1 = h\omega_1 = 1.2\ \text{m/s}; \quad w_1 = h\sqrt{\varepsilon_1^2 + \omega_1^4} = 1.8\sqrt{2} \approx 2.54\ \text{m/s}^2.$$

Vector w_1 makes an angle of 45° with the vector of the point's velocity.

Problem 65. The weight B in Fig. 166 rotates a shaft of radius r with gear *1* of radius r_1 mounted on it. The weight starts moving from rest with a constant acceleration a. Develop the equation of rotation of the gear *2* of radius r_2 which is meshed with gear *1*.

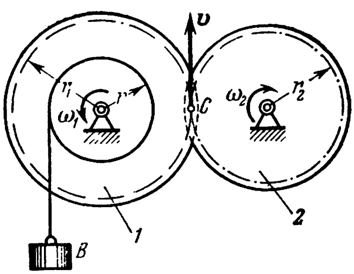

Fig. 166

Solution. As the initial velocity of the weight is zero, its velocity v_B at any instant t is at ($v_B = at$). All the points on the surface of the shaft have the same velocity. At the same time, their velocity is $\omega_1 r$, where ω_1 is the angular velocity of both the shaft and gear *1*. Consequently,

$$\omega_1 r = at, \quad \omega_1 = \frac{at}{r}.$$

Now let us determine ω_2. As at point C, where the gears mesh, the linear velocity of both gears must be the same, we have $v_C = \omega_1 r_1 = \omega_2 r_2$, whence

$$\omega_2 = \frac{r_1}{r_2}\omega_1 = \frac{r_1 a}{r_2 r}t.$$

Thus, the angular velocity of gear *2* increases in proportion to time. Since $\omega_2 = \frac{d\varphi_2}{dt}$, where φ_2 is the angle of rotation of gear *2*, we have

$$d\varphi_2 = \frac{r_1 a}{r_2 r}t\,dt,$$

from which, integrating both sides and assuming angle $\varphi_2 = 0$ at time $t = 0$, we obtain the equation of uniformly accelerated rotation of gear *2* in the form

$$\varphi_2 = \frac{r_1 a}{2rr_2}t^2.$$

Chapter 12
Plane Motion of a Rigid Body

§ 77. Equations of Plane Motion. Resolution of Motion Into Translation and Rotation

Plane motion of a rigid body is such motion in which all its points move parallel to a fixed plane P (Fig. 167). Many machine parts have plane motion, for example, a wheel running on a straight track or the connecting rod of a reciprocating engine. Rotation is, in fact, a special case of plane motion.

Let as consider the section S of a body produced by passing any plane Oxy parallel to a fixed plane P (see Fig. 167). All the points of the body belonging to line MM' normal to plane P move in the same way. Therefore, *in investigating plane motion it is sufficient to investigate the motion of section S of that body* in the plane Oxy. In this book we shall always take the plane Oxy parallel to the page and represent a body by its section S.

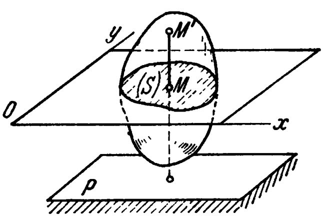

Fig. 167

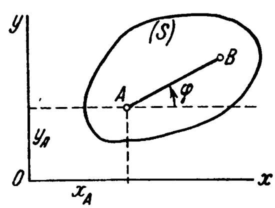

Fig. 168

The position of section S in plane Oxy is completely specified by the position of any line AB in this section (Fig. 168). The position of the line AB may be specified by the coordinates x_A and y_A of point A and the angle φ between an arbitrary line AB in section S and axis x.

The point A chosen to define the position of section S is called the *pole*. As the body moves, the quantities x_A, y_A, and φ will change and the motion of the body, i.e., its position in space at any moment of time, will be completely specified if we know

$$x_A = f_1(t), \ y_A = f_2(t), \ \varphi = f_3(t). \tag{48}$$

Eqs. (48) are the *equations of plane motion of a rigid body*. Let us show that plane motion is a combination of translation and rotation. Consider the successive positions I and II of the section S of a moving body at instants t_1 and $t_2 = t_1 + \Delta t$ (Fig. 169). It will be observed that the following method can be employed to move section S, and with it the whole body, from position I to position II.

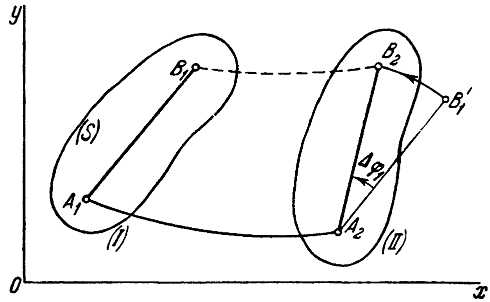

Fig. 169

Let us first translate the body so that pole A occupies position A^2 (line A_1B_1 occupies position A_2B_1') and than turn the section about pole A_2 through angle $\Delta\varphi_1$. In the same way we can move the body from position II to some new position III, etc. We conclude that *the plane motion of a rigid body is a combination of a translation, in which all the points move in the same way as the pole A, and of a rotation about that pole* *).

The translational component of plane motion can, evidently, be described by the first two of Eqs. (48), and the rotational component by the third.

The *principal kinematic characteristics* of this type of motion are the velocity and acceleration of translation, each equal to the velocity and acceleration of the pole ($v_{\text{trans}} = v_A$, $w_{\text{trans}} = w_A$), and the angular velocity ω and angular acceleration ε of the rotation about the pole. The values of these characteristics can be found for any time t from Eqs. (48).

In analysing plane motion, we are free to choose any point of the body as the pole. Let us consider a point C as a pole instead of A and determine the position of the line CD making an angle φ_1 with axis x (Fig. 170). The characteristics of the translatory com-

*) The rotation takes place about an axis perpendicular to the plane P through the pole A. For the sake of brevity, however, we shall speak simply of rotation about the pole A.

ponent of the motion would have been different, for in the general case $v_C \neq v_A$ and $w_C \neq w_A$ (otherwise the motion would be that of pure translation). The characteristics of the rotational component of the motion ω and ε remain, however, the same. For, drawing CB_1 parallel to AB, we find that at any instant of time angle $\varphi_1 = \varphi - \alpha$, where $\alpha =$ const. Hence,

$$\frac{d\varphi_1}{dt} = \frac{d\varphi}{dt}, \quad \frac{d^2\varphi_1}{dt^2} = \frac{d^2\varphi}{dt^2},$$

or

$$\omega_1 = \omega, \quad \varepsilon_1 = \varepsilon.$$

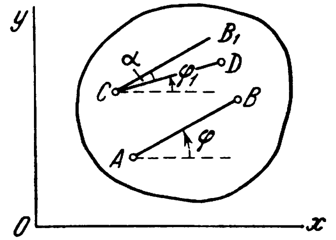

Fig. 170

This result can also be obtained from an examination of Fig. 169: whatever point is taken as the pole, to carry section S from position I to position II line A_1B_1 must be made parallel to A_2B_2, i.e., the section must be rotated around any pole through the same angle $\Delta\varphi_1$ equal to the angle between the two lines. Hence, *the rotational component of motion does not depend on the position of the pole.*

§ 78. Determination of the Path of a Point of a Body

Let us now investigate the motion of individual points of a rigid body, i.e., determine their paths, velocities and accelerations. For this, as has been shown, it is sufficient to analyse the motion of the points lying in section S. We shall begin with the determination of the paths.

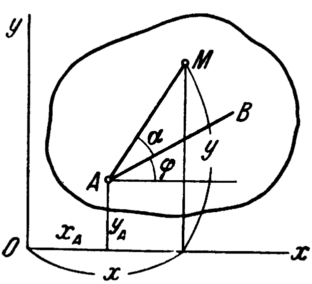

Fig. 171

Consider a point M of a body whose position in section S is specified by its distance $b = AM$ from the pole A and the angle $BAM = \alpha$ (Fig. 171). If the motion of the body is described by Eqs. (48), the x and y coordinates of point M in the system Oxy will be

$$\left.\begin{aligned} x &= x_A + b\cos(\varphi + \alpha), \\ y &= y_A + b\sin(\varphi + \alpha), \end{aligned}\right\} \quad (49)$$

where x_A, y_A, φ are the functions of time t given by Eqs. (48).

Eqs. (49) describe the motion of point M in plane Oxy and at the same time give the equation of the point's path in parametric form. The usual equation of the path can be obtained by eliminating time t from Eqs. (49).

If the body under consideration is part of a mechanism, the path of any point M of the body can be determined by expressing the coordinates of the point in terms of a parameter specifying the position of the mechanism and then eliminating that parameter. In this case the equations of motion (48) are not necessary.

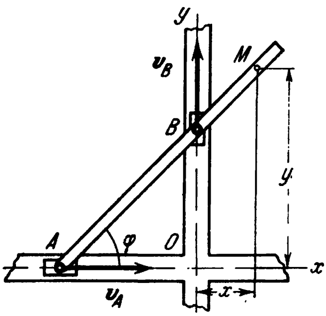

Fig. 172

Problem 66. Blocks A and B, to which the rule of an ellipsograph is attached, slide in mutually perpendicular slots (Fig. 172). Assuming $AB = l$, determine the path of point M of the rule.

Solution. Taking point A as the pole, let us specify the position of M on the rule in terms of segment $AM = b$. The position of the rule itself is specified by angle φ. Hence, for coordinates x and y of point M we obtain:

$$x = (b - l) \cos \varphi, \quad y = b \sin \varphi.$$

Eliminating parameter φ, we find that the path of the point (independent of the equation of motion of the rule) is an ellipse

$$\frac{x^2}{(b-l)^2} + \frac{y^2}{b^2} = 1$$

with semiaxes $a = |b - l|$ and b and centre at O.

By adjusting the distances l and b, we can trace with a pencil at M an ellipse with any given semiaxes not longer than the rule, which is why the instrument is called an ellipsograph.

§ 79. Determination of the Velocity of a Point of a Body

Plane motion of a rigid body is a combination of a translation in which all points of the body move with the velocity of the pole $\boldsymbol{v}_A$ and a rotation about that pole. Let us show that the velocity of any point M of the body is the geometrical sum of its velocities for each component of the motion.

The position of a point M in section S of the body is specified with reference to the coordinate axes Oxy by the radius vector $\boldsymbol{r} = \boldsymbol{r}_A + \boldsymbol{r}'$ (Fig. 173), where $\boldsymbol{r}_A$ is the radius vector of the pole A, $\boldsymbol{r}' = \overline{AM}$ is the vector which specifies the position of point M with reference to the axes $Ax'y'$ that perform translational motion together with A (the motion of section S with reference to those axes is the motion about pole A). Then, $v_M = \frac{dr}{dt} = \frac{dr_A}{dt} + \frac{dr'}{dt}$.

In this equation $\frac{dr_A}{dt} = v_A$ is equal to the velocity of pole A; the quantity $\frac{dr'}{dt}$ is equal to the velocity v_{mA} of point M at $r_A =$

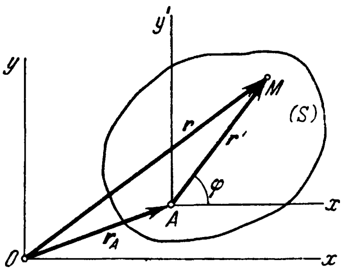

Fig. 173

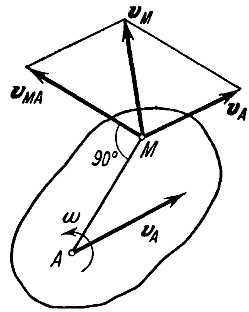

Fig. 174

= const., i.e., when A is fixed or, in other words, *when the body* (or, strictly speaking, its section S) *rotates about pole* A. It thus follows from the preceding equation that

$$v_M = v_A + v_{MA}. \tag{50}$$

The velocity of rotation v_{mA} of point M about pole A is (§ 76)

$$v_{MA} = \omega \cdot MA \quad (v_{MA} \perp \overline{MA}), \tag{51}$$

where ω is the angular velocity of the rotation of the body.

Thus, *the velocity of any point M of a body is the geometrical sum of the velocity of any other point A taken as the pole and the velocity of rotation of point M about the pole.* The magnitude and direction of the velocity v_m are found by constructing a parallelogram (Fig. 174).

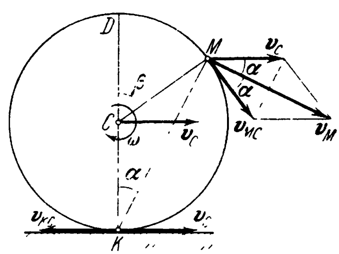

Fig. 175

Problem 67. Determine the velocity of a point M on the rim of a wheel (Fig. 175) rolling without slipping along a straight rail if the velocity of the centre C of the wheel is v_C and angle $DKM = \alpha$.

Solution. Taking point C whose velocity is known as the pole, we find that $v_M = v_C + v_{MC}$, where $v_{MC} \perp \overline{CM}$ and in magnitude $v_{MC} = \omega \cdot MC = \omega R$ (R is the radius of the wheel). The magnitude of the angular velocity ω is determined from the condition that point K of the wheel does not slip and, consequently, at the given moment $v_K = 0$. On the other hand, just as for point M, $\boldsymbol{v}_K = \boldsymbol{v}_C + \boldsymbol{v}_{KC}$, where $v_{KC} = \omega \cdot KC = \omega R$. For point K, $\boldsymbol{v}_{KC}$ and $\boldsymbol{v}_C$ are collinear. Therefore, at $v_K = 0$, $v_{KC} = v_C$, whence $\omega = \frac{v_C}{R}$. Thus we we find that

$$v_{MC} = \omega R = v_C.$$

The parallelogram constructed with vectors $\boldsymbol{v}_{MC}$ and $\boldsymbol{v}_C$ as its sides is a rhombus. But the angle between $\boldsymbol{v}_C$ and $\boldsymbol{v}_{MC}$ is equal to angle β as the sides of these angles are mutually perpendicular. On the other hand, angle $\beta = 2\alpha$, as a central angle subtended by the same arc as the inscribed angle α. Hence, by virtue of the properties of a rhombus, the angles between $\boldsymbol{v}_C$ and $\boldsymbol{v}_M$ and between $\boldsymbol{v}_{MC}$ and $\boldsymbol{v}_M$ are also equal to α. Finally, as the diagonals of a rhombus are mutually perpendicular, we obtain

$$v_M = 2v_C \cos \alpha \quad \text{and} \quad \boldsymbol{v}_M \perp \overline{KM}.$$

The computations, we see, were rather cumbersome. Further on we shall discuss methods which make it possible to solve such problems much simpler (see Problem 69 in § 82.).

§ 80. Theorem of the Projections of the Velocities of Two Points of a Body

The use of Eq. (50) to determine the velocities of the points of a body usually leads to involved computations (cf. Problem 67). However, we can evolve from Eq. (50) several simpler and more

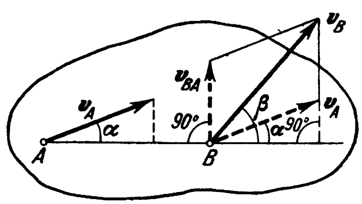

Fig. 176

convenient methods of determining the velocity of any point of a body.

One of these methods is given by the theorem: *The projections of the velocities of two points of a rigid body on the straight line joining those points are equal.*

Consider any two points A and B of a body. Taking point A as the pole (Fig. 176) we have from Eq. (50) $\boldsymbol{v}_B = \boldsymbol{v}_A + \boldsymbol{v}_{BA}$. Projecting both members of the equation on AB and taking into account that vector $\boldsymbol{v}_{BA}$ is perpendicular to AB, we obtain:

$$v_B \cos \beta = v_A \cos \alpha, \tag{52}$$

and the theorem is proved. This result offers a simple method of determining the velocity of any point of a body if the direction of motion of that point and the velocity of any other point of the same body are known.

Problem 68. Determine the relation between the velocities of points A and B of the ellipsograph in Fig. 172 if angle φ is given.

Solution. The directions of the velocities of points A and B are known. Hence, projecting vectors $\boldsymbol{v}_A$ and $\boldsymbol{v}_B$ on AB and applying the above theorem, we obtain:

$$v_A \cos \varphi = v_B \cos (90^\circ - \varphi),$$

whence

$$v_A = v_B \tan \varphi.$$

§ 81. Determination of the Velocity of a Point of a Body Using the Instantaneous Centre of Zero Velocity. Centrodes

Another simple and visual method of determining the velocity of any point of a body performing plane motion is based on the concept of instantaneous centre of zero velocity. *The instantaneous centre of zero velocity is a point belonging to the section S of a body or its extension which at the given instant is momentarily at rest.*

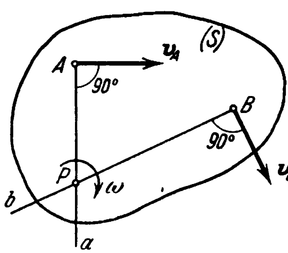

Fig. 177

It will be readily noticed that if a body is in non-translational motion, such one and only one point always exists at any instant t. Let points A and B in section S of a body (Fig. 177) have, at time t, non-parallel velocities $\boldsymbol{v}_A$ and $\boldsymbol{v}_B$. Then point P of intersection of perpendiculars Aa to vector $\boldsymbol{v}_A$ and Bb to vector $\boldsymbol{v}_B$ will be the instantaneous centre of zero velocity, as $\boldsymbol{v}_P = 0$ *). For, if we assumed that $\boldsymbol{v}_P \neq 0$, then, by the theorem of the projections of the velocities of the points of a body, vector $\boldsymbol{v}_P$ would have to be simultaneously perpendicular to AP (as $\boldsymbol{v}_A \perp AP$) and to BP (as $\boldsymbol{v}_B \perp BP$), which is impossible. It also follows from the theorem that, at the given instant, no other point of section S can have zero velocity

*) The section S can always be assumed to include point P (see the example and Fig. 178 further on).

(e.g., for point a, the projection of $\boldsymbol{v}_B$ on Ba is not zero and consequently $\boldsymbol{v}_a \neq 0$).

If, now, we take a point P as the pole at time t, the velocity of point A will, by Eq. (50), be

$$\boldsymbol{v}_A = \boldsymbol{v}_P + \boldsymbol{v}_{AP} = \boldsymbol{v}_{AP},$$

as $v_P = 0$. The same result can be obtained for any other point of the body. Thus, *the velocity of any point of a body lying in section S is equal to the velocity of its rotation about the instantaneous centre of zero velocity P*. From Eqs. (51) we have

$$\begin{aligned} v_A &= \omega \cdot PA \ (\boldsymbol{v}_A \perp \overline{PA}), \\ v_B &= \omega \cdot PB \ (\boldsymbol{v}_B \perp \overline{PB}), \text{ etc.} \end{aligned} \tag{53}$$

It also follows from Eqs. (53) that

$$\frac{v_A}{PA} = \frac{v_B}{PB}, \tag{54}$$

i.e., that *the velocity of any point of a body is proportional to its distance from the instantaneous centre of zero velocity.*

These results lead to the following conclusions:

(1) *To determine the instantaneous centre of zero velocity, it is sufficient to know the directions of the velocities $\boldsymbol{v}_A$ and $\boldsymbol{v}_B$ of any two points A and B of a section of a body* (or their paths); the instantaneous centre of zero velocity lies at the intersection of the perpendiculars erected from points A and B to their respective velocities, or to the tangents to their paths.

(2) *To determine the velocity of any point of a body, it is necessary to know the magnitude and direction of the velocity of any point A of that body and the direction of the velocity of another point B of the same body*. Then, by erecting from points A and B perpendiculars to $\boldsymbol{v}_A$ and $\boldsymbol{v}_B$, we obtain the instantaneous centre of zero velocity P and, from the direction of $\boldsymbol{v}_A$, the sense of rotation of the body. Next, knowing v_A, we can find from Eq. (54) the velocity v_M of any point M of the body. Vector $\boldsymbol{v}_M$ is perpendicular to $\overline{PM}$ in the direction of the rotation.

(3) *The angular velocity of a body*, as can be seen from Eqs. (53), *is at any given instant equal to the ratio of the velocity of any point belonging to the section S to its distance from the instantaneous centre of zero velocity P*:

$$\omega = \frac{v_B}{PB}. \tag{55}$$

Let us evolve another expression for ω. It follows from Eqs. (50) and (51) that $v_{BA} = |\boldsymbol{v}_B - \boldsymbol{v}_A|$ and $v_{BA} = \omega \cdot AB$ whence

$$\omega = \frac{|\boldsymbol{v}_B - \boldsymbol{v}_A|}{AB} = \frac{|\boldsymbol{v}_B + (-\boldsymbol{v}_A)|}{AB}. \tag{56}$$

When $v_A = 0$ (point A is the instantaneous centre of zero velocity), Eq. (56) transforms into Eq. (55).

Eqs. (55) and (56) give the same quantity, since from § 77 it follows that the rotation of the section S about either point A or point P takes place with the same angular velocity ω.

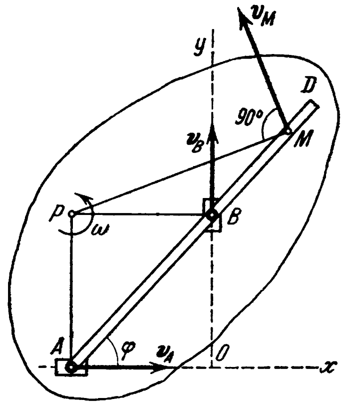

Fig. 178

Example. The directions of the velocities of points A and B of the ellipsograph in Fig. 178 are known. Erecting perpendiculars to them, we obtain the instantaneous centre of zero velocity P of the rule (the ellipsograph can be visualised as a plastic backing hinged to the slides A and B, and the rule AD as drawn on it, point P, which belongs to the backing, has a velocity $v_P = 0$).

Knowing P, we obtain from the proportion $\frac{v_A}{PA} = \frac{v_B}{PB}$: $v_A = v_B \frac{PA}{PB} = v_B \tan\varphi$, i.e., the same result as in Problem 68. Similarly, we obtain for point M: $v_M = v_B \frac{PM}{PB}$. The length of PM can be calculated if we know AB, AM, and angle φ. The direction of vector $\boldsymbol{v}_M$ is shown in the diagram ($\boldsymbol{v}_M \perp PM$).

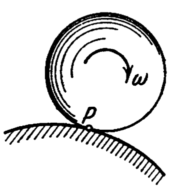

Fig. 179

From Eqs. (55) and (56) we find the angular velocity of the rule:

$$\omega = \frac{v_B}{PB} \quad \text{or} \quad \omega = \frac{|\boldsymbol{v}_B - \boldsymbol{v}_A|}{AB}.$$

It is easy to verify that both equations give the same answer.

Let us consider some special cases of the instantaneous centre of zero velocity.

(a) If plane motion is performed by a cylinder rolling without slipping along a fixed cylindrical surface, the point of contact P (for the section shown in Fig. 179) is momentarily at rest and, consequently, is the instantaneous centre of zero velocity ($\boldsymbol{v}_P = 0$ because if there is no slipping, the contacting points of both bodies must have the same velocity, and the second body is motionless). An example of such motion is that of a wheel running on a rail.

(b) If the velocities of points A and B of the body are parallel to each other, and AB is not perpendicular to $\boldsymbol{v}_A$ (Fig. 180a) the instantaneous centre of zero velocity lies in infinity, and the velocities of all points are parallel to $\boldsymbol{v}_A$. From the theorem of the projections of velocities it follows that $v_A \cos\alpha = v_B \cos\beta$, i.e., $v_B = v_A$;

the result is the same for all other points of the body. Consequently, in this case the velocities of all points of the body are equal in magnitude and direction at every instant, i.e., *the instantaneous distribution of the velocities of the body is that of translation* (this state of motion is also called *instantaneous translation*). It will be found from Eq. (56) that the angular velocity ω of the body at the given instant is zero.

(c) If the velocities of points A and B are parallel and AB is perpendicular to $\boldsymbol{v}_A$, the instantaneous centre of zero velocity P can

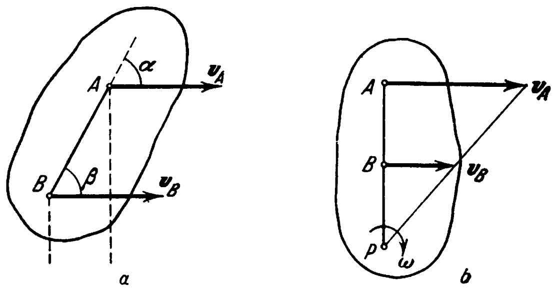

Fig. 180

be located by the construction shown in Fig. 180*b*. The validity of this construction follows from the proportion (54). In this case, unlike the previous ones, we have to know the magnitudes of velocities v_A and v_B to locate the instantaneous centre of zero velocity P.

(d) If the velocity vector $\boldsymbol{v}_B$ of a point in section S and the angular velocity ω are known, the position of the instantaneous centre of zero velocity P, lying on the perpendicular to $\boldsymbol{v}_B$ (see Fig. 177), can be immediately found from Eq. (55), which yields $BP = v_B/\omega$.

Instantaneous Centre of Rotation and Centrodes. It is apparent from Figs. 177 and 180*b* and from formulas (53) and (54) that at every instant the velocities of the points in section S are distributed as though the section's movement represents a rotation around a centre P. That is why the point of a stationary plane coincident with the instantaneous centre of zero velocity, which we shall also denote by the symbol P, is called the *instantaneous centre of rotation*; the axis Pz perpendicular to the section S through point P is called the *instantaneous axis of rotation* of the body in plane motion. Unlike a fixed axis (or centre) of rotation, the instantaneous axis (or centre) continually changes its position. In § 77 we established that plane motion is a combination of translational motion together with a fixed pole and rotation about that pole. The result makes possible another geometrical picture of plane motion, namely, *plane motion is compounded of a series of consecutive elemental rota-*

tions about constantly changing instantaneous axes (or centres) of rotation.

For example, the rolling of the wheel in Fig. 183 can be pictured either as a combination of translational motion together with the pole C and rotation about that pole or as a series of elemental rotations around a constantly changing point of contact P between the wheel rim and the rail.

As the section S moves, the instantaneous centre P continuously changes its position both in the fixed plane Oxy and in the section S

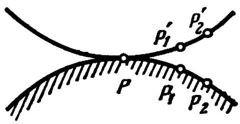

Fig. 181

(and the body to which it belongs). The locus of instantaneous centres of rotation, i.e., of the positions of point P on a fixed plane, is called a *fixed centrode*, the locus of the instantaneous centres of zero velocity, i.e., the positions of point P in a plane moving together with the section S, is the *moving centrode* (Fig. 181). At the given instant both centrodes touch at point P which is the instantaneous centre of rotation (or zero velocity) at the given time; the centrodes cannot intersect as otherwise there would be more than one instantaneous centre at a given moment, which is impossible. At the next instant points P_1' of the moving centrode and P_1 of the fixed centrode will be in contact, the point of contact being the instantaneous centre of rotation for that moment, etc. Hence, as the position of the instantaneous centre P changes continuously, and at each given moment $v_P = 0$, we may conclude that in plane motion the moving centrode rolls without slipping along the fixed one. Conversely, if both centrodes are represented as physical bodies, the geometrical picture of the motion of section S can be obtained by attaching it to the moving centrode and rolling it without slipping along the fixed centrode.

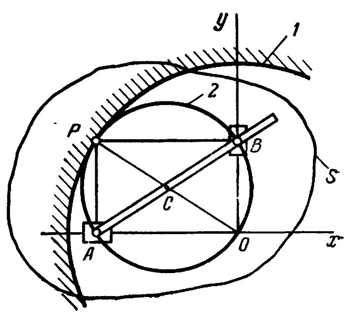

Fig. 182

It will be readily observed that for the wheel in Fig. 183, axis Ox is the fixed centrode and circle $PEDK$ is the moving centrode. The motion of the wheel is achieved by the rolling of the moving centrode along the fixed without slipping.

Example. The instantaneous centre of rotation of the rule AB of the ellipsograph in Fig. 182 is at P. Since at any instant the distance $PO = AB = l$ the locus of points P in plane Oxy, i.e., the fixed centrode, is a circle of radius l whose centre is at O. But at the same time, if the rule AB is imagined as drawn on a piece of plastic S, the distance $PC = l/2$ of the centre P from point C of the rule will also be constant. Hence, the locus of points P on S, i.e., the stationary centrode, is a circle of radius $l/2$ with centre at point C. When the ellipsograph moves, circle *2* rolls without slipping along circle *1*, and their point of contact at every instant is the instantaneous centre of rotation. Conversely, if circles *1* and *2* are represented as physical bodies (gears) with one rolling along the other (stationary) without slipping, the diameter AB of circle *2* will reproduce the motion of the ellipsograph rule.

§ 82. Solution of Problems*)

In order to determine the kinematic characteristics of motion (the angular velocity of a body or the velocities of its points), we must know the magnitude and direction of the velocity of an arbitrary point of the body and the direction of the velocity of any other point of that body [with the exception of cases (a) and (b) discussed in § 81]. The solution of any problem begins with the determination of these characteristics from the statement of the problem.

The mechanism whose motion is being investigated should be drawn in the position for which the corresponding characteristics are being determined. In solving a problem one should remember that the concept of instantaneous centre of zero velocity applies to a single rigid body. *In mechanisms consisting of several bodies, every body performing non-translational motion has at any instant its own instantaneous centre of zero velocity and its own angular velocity.*

Problem 69. Determine the velocity of point M on the rim of the rolling wheel in Problem 67 by introducing the instantaneous centre of zero velocity.

Solution. The point of contact P of the wheel (Fig. 183) is the instantaneous centre of zero velocity, as $v_P = 0$. Consequently, $v_M \perp PM$. As the right angle PMD rests on the diameter, the velocity vector $\boldsymbol{v}_M$ of any point of the rim passes through point D. Writing the proportion

$$\frac{v_M}{PM} = \frac{v_C}{PC}$$

*) An example of the methods of problem solution discussed here can be found in § 95, Problem 90.

and noting that $PC = R$ and $PM = 2R \cos\alpha$, we find:

$$v_M = 2v_C \cos\alpha.$$

The further point M is from P, the greater its velocity is. The upper end D of the vertical diameter has the maximum velocity $v_D = 2v_C$. The angular velocity of the wheel, from Eq. (55), is

$$\omega = \frac{v_C}{PC} = \frac{v_C}{R}.$$

The velocities are similarly distributed for all cases of a wheel or gear rolling along a cylindrical surface (see Fig. 179).

Problem 70. Determine the velocity of the centre C of the free pulley of radius r in Fig. 184 and its angular velocity ω if load A

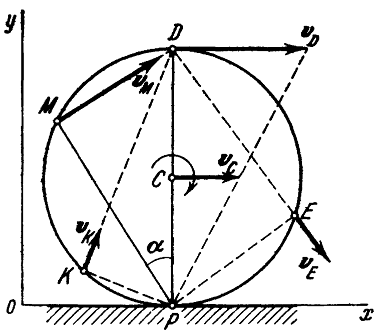

Fig. 183

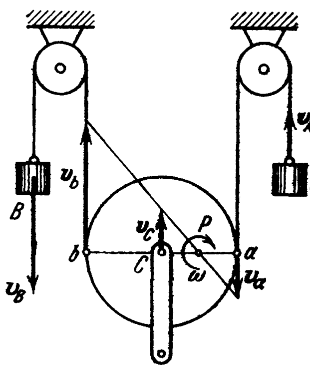

Fig. 184

is moving up with a velocity v_A and load B is moving down with a velocity v_B. The thread does not slip and all its sections are vertical.

Solution. As the thread does not slip on the free pulley, the velocities of points a and b of the pulley are equal in magnitude to the velocities of the loads, i.e., $v_a = v_A$ and $v_b = v_B$. Knowing the velocities of points a and b and assuming for convenience that $v_B > v_A$, we can determine the position of the instantaneous centre P of zero velocity of the free pulley by the same method as in Fig. 180*b*. The velocity of the centre C of the pulley is denoted by the vector v_C. From Eq. (56) we develop the equations

$$\omega = \frac{|v_b + (-v_a)|}{ab}, \quad \omega = \frac{|v_b - v_C|}{bC},$$

whence, as $ab = 2r$ and $bC = r$, we obtain

$$\omega = \frac{v_B + v_A}{2r}, \quad v_C = \frac{v_B - v_A}{2}.$$

At $v_B > v_A$ the centre moves up; if $v_B < v_A$ it moves down; at $v_B = v_A$, $v_C = 0$.

The values of ω and v_C for the case of both loads lowering can be found by substituting $-v_A$ for v_A in the equations.

Problem 71. Crank OA of length r of the mechanism in Fig. 185 rotates with an angular velocity ω_{OA}. The length of the connecting rod $AB = l$. If angle φ is given, determine: (1) the velocity of block B, (2) the position of the point M on the connecting rod which has the least velocity, (3) the angular velocity ω_{AB} of the connecting rod. Also analyse the positions of the mechanism for $\varphi = 0$ and $\varphi = 90°$.

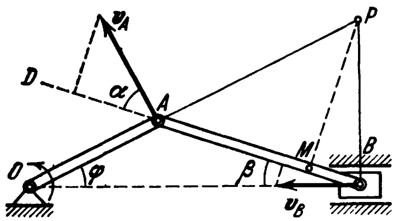

Fig. 185

Solution. It follows from the statement of the problem that the velocity of point A is $v_A = \omega_{OA} r$ perpendicular to OA, and the velocity of B is directed along BO. These data are sufficient to determine all the kinematic characteristics of the connecting rod.

(1) From the theorem of the projections of velocities we have $v_A \cos \alpha = v_B \cos \beta$. Angle OAD, as a supplementary angle of triangle OAB, is equal to $\varphi + \beta$. Hence, $\alpha = 90° - (\varphi + \beta)$, and

$$v_B = \omega_{OA} r \frac{\sin(\varphi + \beta)}{\sin \beta} = \omega_{OA} r (\sin \varphi + \cos \varphi \tan \beta).$$

Now eliminate angle β from the equation. From triangle OAB,

$$\frac{\sin \beta}{r} = \frac{\sin \varphi}{l}.$$

Furthermore,

$$\tan \beta = \frac{\sin \beta}{\sqrt{1 - \sin^2 \beta}},$$

and finally

$$v_B = \omega_{OA} r \left(1 + \frac{r \cos \varphi}{\sqrt{l^2 - r^2 \sin^2 \varphi}}\right) \sin \varphi.$$

(2) The instantaneous centre of zero velocity P of the connecting rod (AP is an extension of OA) is located by erecting perpendiculars from points A and B. The point with the least velocity M is that which is closest to the centre P, i.e., on the perpendicular PM to

AB. Its velocity is

$$v_M = v_A \cos\alpha = \omega_{OA} r \sin(\varphi+\beta).$$

(3) From Eq. (55), the angular velocity of the rod is

$$\omega_{AB} = \frac{v_A}{PA} \quad \text{or} \quad \omega_{AB} = \frac{v_B}{PB}.$$

The length PB (or PA) is calculated from the data given in the statement of the problem.

(4) At angle $\varphi = 0$ (Fig. 186a), the perpendicular AB to velocity $\boldsymbol{v}_A$ and the perpendicular Bb to $\boldsymbol{v}_B$ intersect at B. Consequently, for

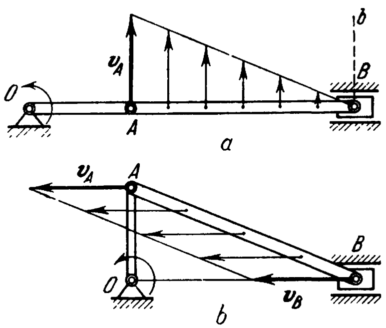

Fig. 186

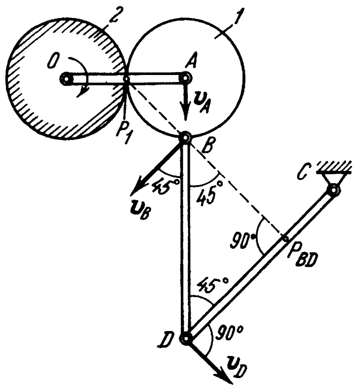

Fig. 187

this position point B is the instantaneous centre of zero velocity, and $v_B = 0$ (the "dead" position of the mechanism). For this position

$$\omega_{AB} = \frac{v_A}{AB} = \frac{r}{l}\,\omega_{OA}.$$

The distribution of velocities along the connecting rod is shown in the diagram.

(5) At angle $\varphi = 90°$ (Fig. 186b), velocities $\boldsymbol{v}_A$ and $\boldsymbol{v}_B$ are parallel, and the perpendiculars to them intersect at infinity. Consequently, at that instant all the points of the rod have the same velocity $\boldsymbol{v}_A$; $\omega_{AB} = 0$.

Problem 72. Link OA in Fig. 187 rotates about axis O with an angular velocity ω_{OA}; with it moves gear *1* which rolls around the fixed gear *2*. The radii of the two gears are both equal to r. Hinged to gear *1* is a connecting rod BD of length l, attached to which is a rock-shaft DC. Determine the angular velocity ω_{BD} of the connecting rod for the instant when it is perpendicular to link OA if at that instant angle $BDC = 45°$.

Solution. To determine ω_{BD} we must know the velocity of an arbitrary point on the connecting rod BD and the position of its instantaneous centre of zero velocity. Let us determine the velocity of point B from the fact that it also belongs to gear *1*, for which the velocity is known to be $v_A = \omega_{OA}\, 2r$ ($v_A \perp OA$), and the instantaneous centre of zero velocity is at P_1. Consequently, $v_B \perp P_1B$, and from the theorem of the projections of velocities $v_B \cos 45° = v_A$, whence $v_B = v_A\sqrt{2} = 2r\omega_{OA}\sqrt{2}$.

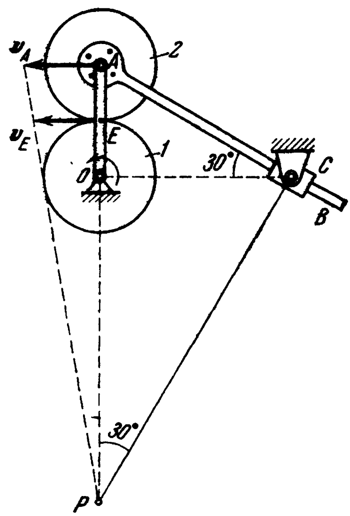

Fig. 188

Now we know the velocity v_B of the connecting rod and the direction of velocity $\boldsymbol{v}_D$ ($\boldsymbol{v}_D \perp DC$). Erecting perpendiculars to v_B and v_D, we obtain the instantaneous centre of zero velocity P_{BD} of the connecting rod. It will be readily noticed that the segment

$$BP_{BD} = l\,\frac{\sqrt{2}}{2},$$

whence

$$\omega_{BD} = \frac{v_B}{BP_{BD}} = 4\,\frac{r}{l}\,\omega_{OA}.$$

It should be noted that it would be wrong to attempt to locate the instantaneous centre of zero velocity by drawing perpendiculars to vectors $\boldsymbol{v}_A$ and $\boldsymbol{v}_D$. Points A and D belong to different bodies, and the intersection of these perpendiculars does not locate the centre of zero velocity (compare with Problem 73).

Problem 73. A gear *1* and crank OA are mounted independently of each other on an axle O (Fig. 188). The crank rotates with an angular velocity ω_{OA}. Fixed to the connecting rod AB with its centre at A is gear *2*. The crank OA carries an axis A of gear *2*, which is fixed to a connecting rod AB passing through a rocker slide C. The radii of gears and *2* are both equal. Determine the angular velocity ω_1 of gear *1* at the instant when $OA \perp OC$, if $\angle ACO = 30°$.

Solution. To determine the angular velocity of gear *1*, we have to find the linear velocity of its point E. For this we make use of the fact that point E of gear *2* has the same velocity. For gear *2* we know the direction and magnitude of the velocity of point A:

$$v_A \perp \overline{OA}, \quad v_A = \omega_{OA} 2r,$$

where r is the radius of the gear. Besides, we know the direction of velocity $\boldsymbol{v}_E$, but in this case this is not sufficient, as $\boldsymbol{v}_E \parallel v_A$. Neither can the value of v_E be found from the theorem of the projections of velocities, since v_A and v_E are perpendicular to AE.

Therefore we shall exploit the fact that, being riveted together, gear *2* and the connecting rod are actually one body for which we know the direction of the velocity of point C: vector v_C (not shown in the figure) is directed along CA, as at point C the rod can only slide in the rocker. By erecting perpendiculars to v_A and v_C, we obtain the instantaneous centre of zero velocity P of the body BAE.

From the statement of the problem, $\angle ACO = 30°$, hence $\angle CPA = 30°$. Therefore $AC = 2AO = 4r$, $PA = 2AC = 8r$, $PE = 7r$, and from the proportion

$$\frac{v_E}{PE} = \frac{v_A}{PA}$$

we find that

$$v_E = \frac{7}{8} v_A = \frac{7}{4} r\omega_{OA},$$

whence

$$\omega_1 = \frac{v_E}{OE} = \frac{7}{4} \omega_{OA}.$$

§ 83*. Velocity Diagram

The velocity of any point of a body can be determined graphically by constructing a velocity diagram. A *velocity diagram* is a drawing in which from some centre are laid off the velocity vectors of points of a body.

Let v_A, v_B, v_C be the velocities of points A, B, C of a given body (Fig. 189*a*). Then the velocity diagram can be drawn by laying off

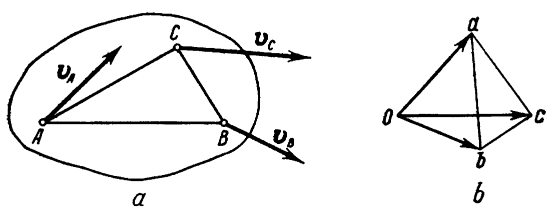

Fig. 189

to some scale from an arbitrary point O (Fig. 189*b*) line segments

$$\overline{Oa} = v_A,\ \overline{Ob} = v_B,\ \overline{Oc} = v_C.$$

Let us establish the properties of, and the procedure for constructing, a velocity diagram. From Eqs. (50) and (51) (see § 79) we have

$$v_B = v_A + v_{BA}, \tag{57}$$

where

$$\boldsymbol{v}_{BA} \perp AB \quad \text{and} \quad v_{BA} = \omega \cdot AB. \tag{57'}$$

But from triangle Oab we have $\overline{Ob} = \overline{Oa} + \overline{ab}$, or $\boldsymbol{v}_B = \boldsymbol{v}_A + \overline{ab}$. Comparing this result with Eq. (57), we find that $\overline{ab} = \boldsymbol{v}_{BA}$. Similarly we find that $\overline{ac} = \boldsymbol{v}_{CA}$, etc. Then, from formulas (57'),

$$ab \perp AB,\ ac \perp AC, \text{ etc.} \tag{58}$$

Furthermore, it also follows from those formulas that $ab = \omega \cdot AB$, $ac = \omega \cdot AC$, etc., whence

$$\frac{ab}{AB} = \frac{ac}{AC} = \frac{bc}{BC} = \ldots = \omega. \tag{58'}$$

Thus, *the lines joining the tips of the velocity vectors in the velocity diagram are perpendicular to the lines joining the respective points of the body and are proportional to them in magnitude*; the figures denoted by the same letters in the velocity diagram and in section S of the body are similar and lie at right angles to each other.

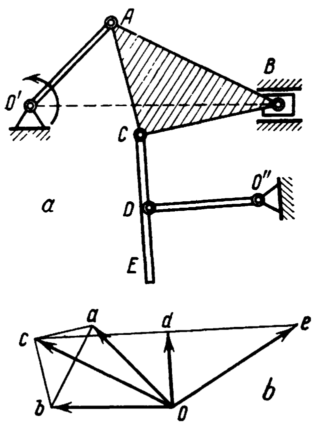

Fig. 190

The relations in (58) and (58') make it possible to construct a velocity diagram and to determine the velocity of any point of a body if the magnitude and direction of the velocity of any one point and the direction of the velocity of any other point of the body are known.

If a velocity diagram is available, the angular velocity is determined by formula (58').

The velocity diagram of a mechanism is constructed as a combination of the velocity diagrams of all its parts (bodies), all the vectors being laid off from a common centre O. An example of such a construction is given in Problem 74.

Problem 74. Construct a velocity diagram for the mechanism in the position shown in Fig. 190*a*, if velocity $\boldsymbol{v}_A$ of the end of crank $O'A$ is known. The connecting link ABC is a rigid triangular lamina. Rod $O''D$ is hinged at D to the centre of rod CE ($CD = DE$).

Solution. (1) Choose a scale (e.g., 1 cm to 0.1 m) and draw the mechanism in the required position (Fig. 190*a*).

(2) Determination of $\boldsymbol{v}_B$. Choose a velocity scale (e.g., 1 cm to 0.5 m/s) and lay off from an arbitrary centre O a vector $\overline{Oa} = \boldsymbol{v}_A$ perpendicular to $O'A$ (Fig. 190*b*). From the same centre lay off the line Ob parallel to $\boldsymbol{v}_B$ (velocity $\boldsymbol{v}_B$ is directed along BO'), and from point

a the line $ab \perp AB$ to its intersection with Ob. Then, from Eq. (58), point b gives the tip of vector $\overline{Ob} = \boldsymbol{v}_B$.

(3) Determination of $\boldsymbol{v}_C$. From point a draw a line perpendicular to AC, and from point b a line perpendicular to BC. By Eq. (58), their intersection gives point c. Joining points O and c, we obtain vector $\overline{Oc} = \boldsymbol{v}_C$.

(4) Determination of $\boldsymbol{v}_D$. The direction of $\boldsymbol{v}_D$ is known ($\boldsymbol{v}_D \perp O''D$). Drawing from O line Od parallel to $\boldsymbol{v}_D$, and from point c a line perpendicular to CD, we obtain at their intersection point d. Joining O and d, we obtain vector $\overline{Od} = \boldsymbol{v}_D$.

(5) Determination of $\boldsymbol{v}_E$. Point E lies on line CDE, hence, by virtue of similarity, point e on the velocity diagram must lie on cde and from Eq. (58′) we obtain $cd : de = CD : DE$. As $DE = CD$, by laying off $de = cd$ along the extension of cd, we obtain point e. Joining points O and e, we obtain vector $\overline{Oe} = \boldsymbol{v}_E$.

Note: The relations in (58) are valid *only* for a given rigid body. Therefore line be in the velocity diagram, for instance, will not be perpendicular to BE, as points B and E of the mechanism belong to different bodies.

From Eq. (58′) we obtain the formulas for calculating the angular velocities of links ABC and CE for the given instant:

$$\omega_{ABC} = \frac{ab}{AB}, \qquad \omega_{CE} = \frac{cd}{CD}.$$

In calculating the values, the different scales should be taken into account.

§ 84. Determination of the Acceleration of a Point of a Body

We shall demonstrate that, like velocity, the acceleration of any point M of a body in plane motion is composed of its accelerations of translation and rotation. The location of point M with respect to axes Oxy (see Fig. 173) is specified by the radius vector $\boldsymbol{r} = \boldsymbol{r}_A + \boldsymbol{r}'$, where $\boldsymbol{r}' = \overline{AM}$. Hence,

$$\boldsymbol{w}_M = \frac{d^2\boldsymbol{r}}{dt^2} = \frac{d^2\boldsymbol{r}_A}{dt^2} + \frac{d^2\boldsymbol{r}'}{dt^2}.$$

In this equation the quantity $\frac{d^2\boldsymbol{r}_A}{dt^2} = \boldsymbol{w}_A$ is the acceleration of the pole A, and the quantity $\frac{d^2\boldsymbol{r}'}{dt^2} = \boldsymbol{w}_{MA}$ is the acceleration of point M in its rotation with the body round A (see § 79). Hence,

$$\boldsymbol{w}_M = \boldsymbol{w}_A + \boldsymbol{w}_{MA}. \tag{59}$$

From Eqs. (46) and (47) (see § 76), the acceleration of point M in its rotation about A is

$$w_{MA}=MA\sqrt{\varepsilon^2+\omega^4}, \quad \tan\ \mu=\frac{|\varepsilon|}{\omega^2}, \tag{60}$$

where ω and ε are the angular velocity and angular acceleration of the body *), and μ is the angle between the direction of w_{MA} and line MA.

Thus, *the acceleration of any point M of a body is composed of the acceleration of any other point taken for the pole and the acceleration of the point M in its rotation together with the body about that pole.* The magnitude and direction of the acceleration w_M are determined by constructing a parallelogram (Fig. 191).

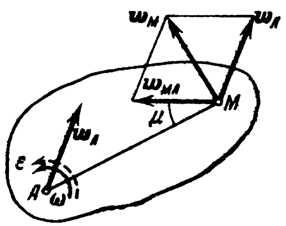

Fig. 191

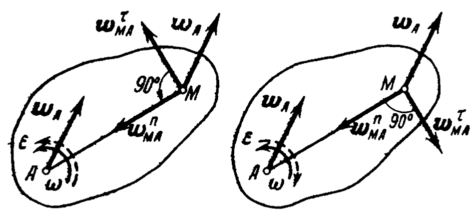

Fig. 192

However, the computation of w_M by means of the parallelogram in Fig. 191 makes the solution more difficult, as it becomes necessary first to calculate the angle μ and then the angle between vectors w_{MA} and w_A. Therefore, in problem solutions it is more convenient to replace vector w_{MA} by its tangental and normal components w^τ_{MA} and w^n_{MA}, where

$$w^\tau_{MA}=AM\cdot\varepsilon, \quad w^n_{AM}=AM\cdot\omega^2. \tag{61}$$

Vector w^τ_{MA} is perpendicular to AM in the direction of the rotation if it is accelerated, and opposite the rotation if it is retarded; vector w^n_{MA} is always directed from point M to the pole A (Fig. 192).

Instead of Eq. (59) we obtain

$$w_M=w_A+w^\tau_{MA}+w^n_{MA}. \tag{62}$$

If pole A is in non-rectilinear motion, its acceleration is also composed of the tangential and normal accelerations, hence

$$w_M=w_{A\tau}+w_{An}+w^\tau_{MA}+w^n_{MA}, \tag{62'}$$

*) In the diagram the solid circular arrow indicates the direction of ω (the direction of rotation), and the dashed arrow, the direction (the sign) of ε. If the motion is accelerated, both arrows point in the same direction, if the motion is retarded, they are of opposite sense.

the magnitudes of the latter two components being obtained from Eq. (61). Eqs. (61) and (62) should be used in solving problems, first computing the vectors in the right-hand part of the equation and then finding their geometrical sum or making a graphic construction.

Solution of Problems. The acceleration of any point of a body at any time can be determined if the following data are known: (1) the vectors of the velocity $\boldsymbol{v}_A$ and acceleration $\boldsymbol{w}_A$ of any point A of the body at the prescribed time; (2) the path of some other point B of the body. In some cases, instead of the path of the second point of the body, it is sufficient to know the location of the instantaneous centre of zero velocity.

In solving problems, the body (or mechanism) should be drawn in the position for which the acceleration of the required point has to

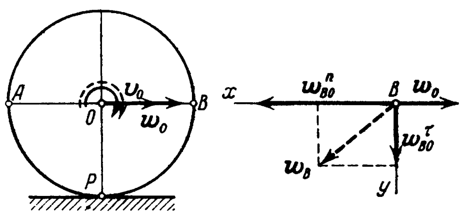

Fig. 193

be found. The computation starts with the determination, from the conditions of the problem, of the velocity and acceleration of the point chosen as the pole. The subsequent stages of the computation are examined in detail in the sample problems below, together with necessary additional suggestions.

Problem 77 gives the graphical method of solution.

Problem 75. The centre O of a wheel of radius $R = 0.2$ m rolling along a straight rail (Fig. 193) has at a given instant a velocity $v_O = 1$ m/s and an acceleration $w_O = 2$ m/s^2. Determine the acceleration of point B lying at the end of diameter AB perpendicular to OP and the acceleration of point P coincident with the instantaneous centre of zero velocity.

Solution. (1) As $\boldsymbol{v}_O$ and $\boldsymbol{w}_O$ are known, we take point O for the pole.

(2) Determination of ω. The point of contact P is the instantaneous centre of zero velocity; hence the angular velocity of the wheel is

$$\omega = \frac{v_O}{PO} = \frac{v_O}{R}. \qquad \text{(a)}$$

The direction of ω is determined by the direction of $\boldsymbol{v}_O$ and is shown in the diagram by the solid arrow.

(3) Determination of ε. As in equation (a) the quantity $PO = R$ is constant for any position of the wheel, by differentiating the equation with respect to time we obtain:

$$\frac{d\omega}{dt}=\frac{1}{R}\frac{dv_O}{dt} \quad \text{or} \quad \varepsilon=\frac{w_O}{R}. \tag{b}$$

The signs of ε and ω are the same, therefore the rotation of the wheel is accelerated.

It is important to remember that *the value of* ε *is determined from the equation* (b) *only when the distance PO in equation* (a) *is constant.*

Note: (a) It should not be assumed that v_O is constant only because the given value of $v_O = 1$ m/s. This value, stated in the conditions of the problem, is for the given instant, and it changes with time, since $w_O \neq 0$.

(b) In this case $\frac{dv_O}{dt} = w_O$, as the motion of point O is rectilinear. In the general case $\frac{dv_O}{dt} = w_{O\tau}$.

(4) Determination of $\boldsymbol{w}_{BO}^{\tau}$ and $\boldsymbol{w}_{BO}^{n}$. As point O is the pole, from Eq. (62) we have

$$\boldsymbol{w}_B=\boldsymbol{w}_O+\boldsymbol{w}_{BO}^{\tau}+\boldsymbol{w}_{BO}^{n}. \tag{c}$$

In our case $BO = R$, and taking into account (a) and (b), we have

$$w_{BO}^{\tau}=BO\cdot\varepsilon=w_O=2\ \text{m/s}^2, \qquad w_{BO}^{n}=BO\cdot\omega^2=\frac{v_O^2}{R}=5\ \text{m/s}^2. \tag{d}$$

Now draw a separate diagram for point B, showing (not necessarily to scale) the component vectors of the acceleration $\boldsymbol{w}_B$, namely vector $\boldsymbol{w}_O$ (transferred from point O), vector $\boldsymbol{w}_{BO}^{\tau}$ (in the direction of the rotation as it is accelerated), and vector $\boldsymbol{w}_{BO}^{n}$ (always from B towards the pole O).

(5) Determination of $\boldsymbol{w}_B$. Drawing axes Bx and By, we find that

$$w_{Bx}=w_{BO}^{n}-w_O=3\ \text{m/s}^2, \quad w_{By}=w_{BO}^{\tau}=2\ \text{m/s}^2,$$

whence

$$w_B=\sqrt{w_{Bx}^2+w_{By}^2}=\sqrt{13}\approx 3.6\ \text{m/s}^2.$$

Similarly we can easily find that the acceleration of point P is $w_P = w_{PO}^{n} = 5$ m/s² and is directed along PO. Thus, the acceleration of point P, whose velocity at the given instant is zero, is not zero.

Problem 76. Gear *1* of radius $r_1 = 0.3$ m in Fig. 194*a* is fixed; rolling around it is gear *2* of radius $r_2 = 0.2$ m mounted on link OA. The link turns about axis O and has at the given instant an angular velocity $\omega = 1\ \text{s}^{-1}$ and an angular acceleration $\varepsilon = -4\ \text{s}^{-2}$. Determine the acceleration of point D on the rim of the moving gear at the given instant (radius AD is perpendicular to the link).

Solution. (1) To solve the problem, consider the motion of gear *2*. From the statement of the problem it is easy to determine the velocity $\boldsymbol{v}_A$ and acceleration $\boldsymbol{w}_A$ of point A of the gear, which we take as the pole.

(2) Determination of $\boldsymbol{v}_A$ and $\boldsymbol{w}_A$. Knowing ω and ε of the link, we obtain

$$v_A = OA \cdot \omega = 0.5 \text{ m/s},$$
$$w_{A\tau} = OA \cdot \varepsilon = -2 \text{ m/s}^2, \quad \text{(a)}$$
$$w_{An} = OA \cdot \omega^2 = 0.5 \text{ m/s}^2.$$

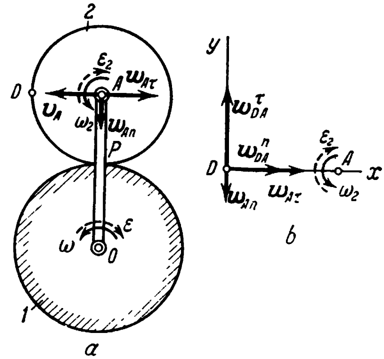

Fig. 194

As the signs of v_A and $\omega_{A\tau}$ are different, the motion of point A from the given position is retarded. Vectors $\boldsymbol{w}_{A\tau}$ and $\boldsymbol{w}_{An}$ are directed as shown in the diagram.

(3) Determination of ω_2. The point of contact P is the instantaneous centre of zero velocity of gear *2*; consequently, the angular velocity of gear *2* is

$$\omega_2 = \frac{v_A}{AP} = \frac{v_A}{r_2}, \quad \omega_2 = 2.5 \text{ s}^{-1}. \quad \text{(b)}$$

The direction of ω_2 (the direction of rotation of the gear) is determined by the direction of $\boldsymbol{v}_A$ and is indicated by the solid arrow.

(4) Determination of ε_2. As in the previous problem, the quantity $AP = r_2$ is constant, and consequently

$$\varepsilon_2 = \frac{d\omega_2}{dt} = \frac{1}{r_2}\frac{dv_A}{dt} = \frac{w_{A\tau}}{r_2}, \quad \varepsilon_2 = -10 \text{ s}^{-2}. \quad \text{(c)}$$

As ω_2 and ε_2 are of different sign, the rotation of gear *2* is retarded.

(5) Determination of $\boldsymbol{w}^\tau_{DA}$ and $\boldsymbol{w}^n_{DA}$. The acceleration of point D is found from Eq. (62′):

$$\boldsymbol{w}_D = \boldsymbol{w}_{A\tau} + \boldsymbol{w}_{An} + \boldsymbol{w}^\tau_{DA} + \boldsymbol{w}^n_{DA}.$$

In our case $DA = r_2$, and

$$w^\tau_{DA} = DA \cdot \varepsilon_2 = -2 \text{ m/s}^2, \quad w^n_{DA} = DA \cdot \omega_2^2 = 1.25 \text{ m/s}^2.$$

Now draw the component vectors of acceleration $\boldsymbol{w}_D$ (Fig. 194*b*), namely $\boldsymbol{w}_{A\tau}$, $\boldsymbol{w}_{An}$ (transferred from point A), w^τ_{DA} (directed against the rotation since it is retarded), and $\boldsymbol{w}^n_{DA}$ (from D towards the pole A).

(6) Calculation of w_D. Drawing axes Dx and Dy, we find that

$$w_{Dx} = |w_{A\tau}| + w^n_{DA} = 3.25 \text{ m/s}^2, \quad w_{Dy} = |w^\tau_{DA}| - w_{An} = 1.5 \text{ m/s}^2,$$

whence

$$w_D = \sqrt{w_{Dx}^2 + w_{Dy}^2} \approx 3.58 \text{ m/s}^2.$$

The magnitude of w_D can also be obtained graphically by constructing a vector polygon with vectors $\boldsymbol{w}_{A\tau}$, $\boldsymbol{w}_{An}$, $\boldsymbol{w}_{DA}^{\tau}$, and $\boldsymbol{w}_{DA}^{n}$ to scale.

Problem 77. Attached to a crank OA (Fig. 195) rotating uniformly about axis O with an angular velocity $\omega_{OA} = 4\ \text{s}^{-1}$ is connecting rod AB hinged to a rockshaft BC. The given dimensions are: $OA = r = 0.5$ m, $AB = 2r$, $BC = r\sqrt{2}$. In the position shown in the diagram, $\angle OAB = 90°$ and $\angle ABC = 45°$. Determine for this

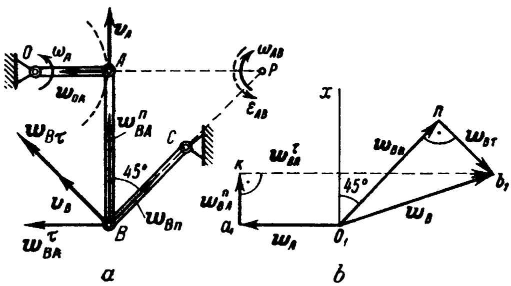

Fig. 195

position the acceleration of point B of the connecting rod and the angular velocity and angular acceleration of the rockshaft BC and connecting rod AB.

Solution. The problem can be solved either graphically or analytically.

(A) Analytical Solution. (1) Considering the motion of the connecting rod AB, we take point A as the pole. As ω_{OA} =const., we obtain

$$v_A = r\omega_{OA} = 2 \text{ m/s}, \quad w_A = w_{An} = r\omega_{OA}^2 = 8 \text{ m/s}^2. \tag{a}$$

Depict vectors $\boldsymbol{v}_A$ and $\boldsymbol{w}_A$ in the diagram.

(2) Determination of ω_{AB}. We know the path of point B of the connecting rod (a circle of radius BC). Hence, knowing the direction of $\boldsymbol{v}_B$ ($\boldsymbol{v}_B \perp BC$), we can locate the instantaneous centre of zero velocity P of the rod. It is evident that $AP = AB = 2r$. Therefore,

$$\omega_{AB} = \frac{v_A}{AP}, \quad \text{or} \quad \omega_{AB} = \frac{\omega_{OA}}{2} = 2\ \text{s}^{-1}. \tag{b}$$

The direction of rotation is shown in the diagram.

In this case the distance AP changes with the motion of the mechanism and we cannot apply the method used in the previous two problems to determine ε_{AB}. Let us consider, for this reason, another method.

(3) Analysis of the vector equation (62). Taking into account that $\boldsymbol{w}_B = \boldsymbol{w}_{B\tau} + \boldsymbol{w}_{Bn}$, we can write Eq. (62) in the form

$$\underline{\boldsymbol{w}_{B\tau}} + \boldsymbol{w}_{Bn} = \boldsymbol{w}_A + \underline{\boldsymbol{w}_{BA}^{\tau}} + \boldsymbol{w}_{BA}^{n}. \tag{c}$$

We depict all the vectors in Fig. 195*a* (directing $\boldsymbol{w}_{B\tau}$ and $\boldsymbol{w}_{BA}^{\tau}$ in such a way as if the corresponding rotations were accelerated). Let us see what members in (c) are known or can be calculated using the conditions of the problem. We know the acceleration w_A of pole A. Furthermore, knowing ω_{AB} we can find w_{BA}^n, and knowing v_A we can find v_B and calculate w_{Bn}. Thus, in the vector equation (c) only the values of the underlined vectors, i.e., $w_{B\tau}$ and w_{BA}^{τ}, are not known. But projecting Eq. (c) on the axes we obtain two scalar equations, which can be used to determine the unknown quantities.

First determine w_{BA}^n and w_{Bn}.

(4) Determination of $\boldsymbol{w}_{BA}^n$. Knowing ω_{AB}, from Eq. (61) we have

$$w_{BA}^n = AB \cdot \omega_{AB}^2 = 4 \text{ m/s}^2. \tag{d}$$

(5) Determination of $\boldsymbol{w}_{Bn}$. Knowing the path of point B, we can determine its normal acceleration w_{Bn}. For this, applying the theorem of projections (or the instantaneous centre of zero velocity P), we first determine the velocity v_B. We have $v_B \cos 45° = v_A$, whence $v_B = v_A\sqrt{2}$. Therefore,

$$w_{Bn} = \frac{v_B^2}{BC} = \frac{2v_A^2}{r\sqrt{2}} = 8\sqrt{2} \text{ m/s}^2. \tag{e}$$

(6) Determination of $\boldsymbol{w}_{B\tau}$ and $\boldsymbol{w}_B$. To obtain $\boldsymbol{w}_{B\tau}$ let us project both sides of the vector equation (c) on axis BA, which is perpendicular to another unknown vector $\boldsymbol{w}_{BA}^{\tau}$. We obtain:

$$w_{B\tau} \cos 45° + w_{Bn} \cos 45° = w_{BA}^n.$$

Substituting the calculated values of w_{Bn} and w_{BA}^n, we find:

$$w_{B\tau} = w_{BA}^n \sqrt{2} - w_{Bn} = -4\sqrt{2}. \tag{f}$$

The "minus" shows that vector $\boldsymbol{w}_{B\tau}$ is directed opposite to $\boldsymbol{v}_B$ (the rotation of the rock CB from the considered position is retarded).

Finally,

$$w_B = \sqrt{w_{B\tau}^2 + w_{Bn}^2} = 4\sqrt{10} = 12.65 \text{ m/s}^2.$$

(7) Determination of ω_{BC} and ε_{BC}. Knowing $v_B = v_A\sqrt{2}$ and $w_{B\tau}$, we find:

$$\omega_{BC} = \frac{v_B}{BC} = 4 \text{ s}^{-1}, \quad \varepsilon_{BC} = \frac{w_{B\tau}}{BC} = -8 \text{ s}^{-2}$$

(8) Determination of ε_{AB}. To find ε_{AB}, we must find w_{AB}^{τ}. We project both sides of equation (c) on axis BP, which is perpendicular

to $\boldsymbol{w}_{B\tau}$, and obtain:

$$w_{Bn} = -w_A \cos 45° - w_{BA}^{\tau} \cos 45° + w_{BA}^{n} \cos 45°,$$

whence

$$w_{BA}^{\tau} = -w_A + w_{BA}^{n} - w_{Bn}\sqrt{2} = -20 \text{ m/s}^2.$$

Finally, by formula (61),

$$\varepsilon_{AB} = \frac{w_{BA}^{\tau}}{AB} = -20 \text{ s}^{-2}.$$

In both cases the "minus" shows that the rotation of the connecting rod AB from the considered position is retarded.

(B) Graphical Solution. Perform the computations of sections (1), (2), (4), and (5) and, thus, find w_A, ω_{AB}, w_{BA}^{n}, and w_{Bn}.

Note: If a velocity diagram has been constructed for the mechanism, then

(a) the value of ω_{AB} can be found from Eq. (58'): $\omega_{AB} = \frac{ab}{AB}$. Then the computations in sections (2) and (4) can be omitted and w_{BA}^{n} found directly from the equation

$$w_{BA}^{n} = AB \cdot \omega_{AB}^{2} = \frac{(ab)^2}{AB};$$

(b) the value of v_B for determining w_{Bn} can also be obtained directly from the velocity diagram.

Determination of $\boldsymbol{w}_B$. Let us write Eq. (62), as in the analytical solution, in the following form

$$\boldsymbol{w}_A + \boldsymbol{w}_{BA}^{n} + \boldsymbol{w}_{BA}^{\tau} = \boldsymbol{w}_{Bn} + \boldsymbol{w}_{B\tau}.$$

Let us express this equation graphically. From an arbitrary centre O_1 (Fig. 195*b*) lay off to some scale vector $\overline{O_1a_1} = \boldsymbol{w}_A$, from point a_1 lay off vector $\overline{a_1k} = \boldsymbol{w}_{BA}$ ($\boldsymbol{w}_{BA}^{n} \uparrow\uparrow \overline{BA}$), and from point k draw kb_1 perpendicular to a_1k. This line gives the direction of $\boldsymbol{w}_{BA}^{\tau}$, and somewhere on it lies the tip of the required vector $\boldsymbol{w}_B$.

Now from point O_1 lay off vector $\overline{O_1n} = \boldsymbol{w}_{Bn}$ ($\boldsymbol{w}_{Bn} \uparrow\uparrow \overline{BC}$) and draw perpendicular to it nb_1, which gives the direction of $\boldsymbol{w}_{B\tau}$. The tip of vector $\boldsymbol{w}_B$ must lie on this line as well. Consequently, point b_1, where kb_1 and nb_1 intersect, gives us the tip of vector $\boldsymbol{w}_B$ *). Thus, $\boldsymbol{w}_B = \overline{O_1b_1}$. Scaling O_1b_1, we find that $w_B \approx 13$ m/s².

At the same time it follows from the construction that $\overline{kb_1} = \boldsymbol{w}_{BA}^{\tau}$ and $\overline{nb_1} = \boldsymbol{w}_{B\tau}$.

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*) If the direction of $\boldsymbol{w}_B$ were known, the vector $\boldsymbol{w}_B = \overline{O_1b_1}$ could be obtained at once as the intersection of kb_1 and $O_1b_1 \parallel \boldsymbol{w}_B$.

By constructing the accelerations of the other points of the mechanism in the same way (and from the same centre O), we obtain an *acceleration diagram.*

Determination of ε_{AB}. Measuring the length of kb_1, we find from Eq. (61)

$$|\varepsilon_{AB}| = \frac{|w^{\tau}_{BA}|}{AB} = \frac{kb_1}{AB}.$$

Applying the scale, we find that $|\varepsilon_{AB}| = 20\ \text{s}^{-2}$. It can be seen in the diagram that vector $v_{BA} = v_B - v_A$ is directed opposite to w^{τ}_{BA}; consequently, the rotation of rod AB is retarded and $\varepsilon_{AB} = -20\ \text{s}^{-2}$.

§ 85*. Instantaneous Centre of Zero Acceleration

If a rigid body is in nontranslatory motion, at every moment there is in its section S or in a plane rigidly connected with it a point Q whose acceleration is zero. This point is called the *instantaneous centre of zero acceleration.* If we know the acceleration w_A of an arbitrary point A of the body and the values of ω and ε, we can locate the instantaneous centre of zero acceleration by the following method:

(1) Compute angle μ (Fig. 196) by the formula $\tan\mu = \frac{|\varepsilon|}{\omega^2}$.

(2) From point A draw AE at an angle μ to vector w_A, the line being turned from w_A in the direction of the rotation of the body if the rotation is accelerated, and opposite to that direction if the rotation is retarded.

(3) Along AE lay off a segment

$$AQ = \frac{w_A}{\sqrt{\varepsilon^2 + \omega^4}}. \tag{63}$$

Point Q thus obtained is the instantaneous centre of zero acceleration. For, from Eqs. (59) and (60),

$$w_Q = w_A + w_{QA},$$

where $w_{QA} = QA\sqrt{\varepsilon^2 + \omega^4}$. Substituting the value of QA from Eq. (63), we find that $w_{QA} = w_A$. Besides, vector w_{QA} must make an angle μ with QA and, consequently, vector w_{QA} is parallel to w_A but opposite in sense. Therefore, $w_{QA} = -w_A$ and $w_Q = 0$.

If point Q is taken as the pole, then, as $w_Q = 0$, the acceleration of any point M of the body is, by Eqs. (59) and (60),

$$w_M = w_Q + w_{MQ} = w_{MQ} \quad \text{and} \quad w_M = QM\sqrt{\varepsilon^2 + \omega^4}. \tag{64}$$

Thus, *the acceleration of any point of a body is equal to its acceleration of rotation about the instantaneous centre of zero acceleration Q.* And from (64), we have

$$\frac{w_M}{QM} = \frac{w_A}{QA} = \ldots, \text{ etc.,} \tag{64'}$$

i.e., *the acceleration of any point of a body is proportional to its distance from the instantaneous centre of zero acceleration.* A diagram of the distribution of accelerations is given in Fig. 197.

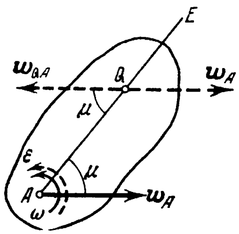

Fig. 196

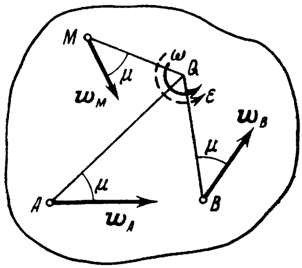

Fig. 197

It should be borne in mind that the positions of the instantaneous centre of zero velocity P and the instantaneous centre of zero acceleration Q at any given time do not coincide. For example, if a wheel rolls along a straight rail (see Fig. 198), the velocity of its centre C being constant ($v_C = \text{const.}$), the instantaneous centre of zero velocity is at point P ($v_P = 0$), but, as was shown in Problem 75, $w_P \neq 0$, and P is not the instantaneous centre of zero acceleration. In this case it is obviously located at C, since it is in uniform rectilinear motion and $w_C = 0$. The instantaneous centres of zero velocity and zero acceleration coincide only when a body rotates about a fixed axis.

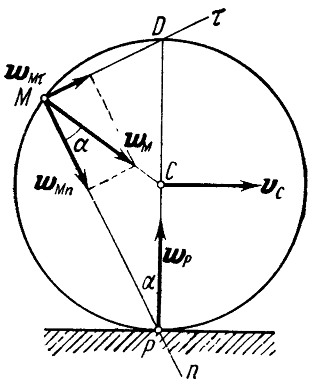

Fig. 198

The concept of instantaneous centre of zero acceleration is convenient in solving certain problems.

Problem 78. A wheel rolls along a straight rail so that the velocity v_C of its centre C is constant. Determine the acceleration of a point M on the rim of the wheel (Fig. 198).

Solution. As $v_C = \text{const.}$, then, as shown above, point C is the instantaneous centre of zero acceleration. The instantaneous centre of zero velocity is at P. Consequently,

$$\omega = \frac{v_C}{PC} = \frac{v_C}{R} = \text{const.}, \quad \varepsilon = \frac{d\omega}{dt} = 0, \quad \tan\mu = \frac{\varepsilon}{\omega^2} = 0, \quad \mu = 0,$$

and from Eq. (64) we obtain

$$w_M = CM \cdot \omega^2 = \frac{v_C^2}{R}.$$

Thus, the acceleration of any point M on the rim (including P) is equal to v_C^2/R and is directed towards the centre of the wheel, since angle $\mu = 0$. Note that for point M this is not the normal acceleration, for the velocity of M is perpendicular to PM (see Problem 69)

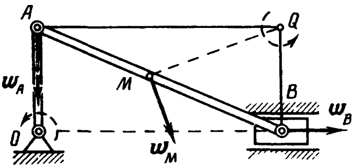

Fig. 199

and, consequently, the tangent $M\tau$ to the path of point M is directed along MD, while the principal normal Mn is directed along MP. Therefore, $w_{Mn} = w_M \cos \alpha$ and $w_{M\tau} = w_M \sin \alpha$.

Problem 79. Crank OA rotates with a constant angular velocity ω_{OA} (Fig. 199). Determine the acceleration of the slide block B and the angular acceleration of the connecting rod AB at the instant when $\angle BOA = 90°$ if $OA = r$ and $AB = l$.

Solution. At the given instant the velocity of all the points of the connecting rod AB is v_A (see Problem 71, Fig. 186*b*), the instantaneous centre of zero velocity is at infinity, and $\omega_{AB} = 0$. Therefore, $\tan \mu = \frac{\varepsilon_{AB}}{\omega_{AB}^2} = \infty$ and $\mu = 90°$ ($\varepsilon_{AB} \neq 0$, as otherwise, according to Eqs. (59) and (60), $w_{BA} = 0$ and $\boldsymbol{w}_B = \boldsymbol{w}_A$, which is impossible since these vectors are perpendicular to each other).

The acceleration of point A is $w_A = w_{An} = r\omega_{OA}^2$ and is directed along AO. The acceleration of point B, which is in rectilinear motion, is directed along OB. It will be noticed from Fig. 197 that the acceleration of any point M of the body makes an angle μ with QM. In the present case $\mu = 90°$; consequently, QA and QB are perpendicular to $\boldsymbol{w}_A$ and $\boldsymbol{w}_B$. Erecting these perpendiculars, we locate point Q. Writing the proportion (64′),

$$\frac{w_B}{QB} = \frac{w_A}{QA},$$

where $QB = r$ and $QA = \sqrt{l^2 - r^2}$, we obtain:

$$w_B = \frac{r^2}{\sqrt{l^2 - r^2}} \omega_{OA}^2.$$

The acceleration w_M of any other point M of the connecting rod is perpendicular to QM ($\mu = 90°$); the magnitude of w_M is found from Eq. (64').

The angular acceleration ε_{AB} of the connecting rod is found from the equation $w_A = QA \cdot \varepsilon_{AB}$ obtained from Eq. (64) at $\omega_{AB} = 0$. Hence

$$\varepsilon_{AB} = \frac{r}{\sqrt{l^2 - r^2}} \omega_{OA}^2.$$

Chapter 13
Motion of a Rigid Body Having One Fixed Point and Motion of a Free Rigid Body

§ 86. Motion of a Rigid Body Having One Fixed Point

Let us investigate the motion of a body having a fixed point O with respect to a reference system $Ox_1y_1z_1$. A top, whose point of contact with the plane on which it spins is fixed relative to it, or any body with a ball-and-socket joint are illustrations of such motion.

Let us find the parameters that determine the position of a body having one fixed point. For this, assume the body rigidly attached to the trihedron $Oxyz$ whose position defines the position of the body (Fig. 200). Line OK, along which planes Oxy and Ox_1y_1 intersect, is called the *line of nodes.* The position of the trihedron $Oxyz$, and hence of the body, with respect to the axes $Ox_1y_1z_1$ is given by the angles:

$$\varphi = \angle KOx, \quad \psi = \angle x_1OK, \quad \theta = \angle z_1Oz.$$

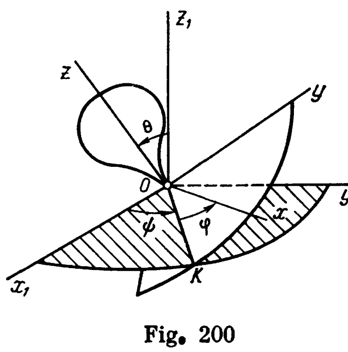

Fig. 200

These angles, called the *Euler angles*, are known by the following names, taken from celestial mechanics: φ—the *angle of proper rotation*, ψ—*angle of precession*, θ—*angle of nutation.* The positive directions are shown in Fig. 200 by arrows. A change in angle φ indicates a rotation of the body around axis Oz (proper rotation), a change in angle ψ indicates a rotation around axis Oz_1 (precession), and a change in angle θ indicates a rotation around the line of nodes OK (nutation).

To describe the body's motion, its position with respect to axes $Ox_1y_1z_1$ must be known for any instant, i.e.,

$$\varphi = f_1(t), \quad \psi = f_2(t), \quad \theta = f_3(t). \tag{65}$$

Equations (65) define the law of motion and are called the *equations of motion of a rigid body about a fixed point.*

To get a full picture of this type of motion, let us demonstrate the following **theorem of Euler-d'Alembert:** *Any elementary displacement of a body having a fixed point represents an elementary rotation about an instantaneous axis of rotation through that point.*

Let the body's position be given by the angles φ, ψ, θ. Then its displacement in an elementary time interval dt can be represented as the resultant of a series of rotations through angles $d\varphi$, $d\psi$, and $d\theta$ about axes Oz, Oz_1, and OK, respectively. Combined, the three rotations yield the true elementary displacement of the body. Consider first the resultant displacement of the rotations about axes Oz and Oz_1 (Fig. 201). A rotation through angle $d\varphi$ about axis Oz imparts to any point of the body in the plane zOz_1 (inside angle zOz_1) an elementary displacement perpendicular to the plane and equal in magnitude to $h_1 d\varphi$, where h_1 is the distance of the point from axis Oz. Simultaneously, in the rotation about axis Oz_1, the point will receive an oppositely directed displacement of magnitude $h_2 d\psi$. Accordingly, within angle zOz_1 there will always be a point B for which $h_1 d\varphi = h_2 d\psi$ and whose displacement is zero (if the direction of rotation is opposite to that shown in Fig. 201, the point lies outside angle zOz_1). Hence we conclude that the body's elementary displacement resulting from the rotations about axes Oz and Oz_1 is the same as the displacement of a body with two stationary points O and B, i.e., it is an elementary rotation about axis OB through point O. By the same reasoning, the elementary rotations about axes OB and OK are equivalent to an elementary rotation about an axis OP through point O (Fig. 201), and the theorem is proved.

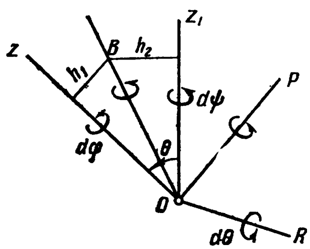

Fig. 201

Axis OP is called the *instantaneous axis of rotation*; an elementary rotation of the body about it brings the body into a neighbouring position infinitesimally close to the given position; the velocities of all points of the body lying on the instantaneous axis of rotation are zero. Unlike a fixed axis, the instantaneous axis of rotation continuously changes its direction in space and in the body. The rotation about axis OP, which brought the body into the neighbouring position, is followed by a rotation about the new instantaneous axis of rotation OP_1, etc. Thus, *the motion of a rigid body about a fixed point is compounded of a series of consecutive elementary rotations about instantaneous axes of rotation through that fixed point* (Fig. 202). Let us examine the kinematic characteristics of this motion.

(1) The angular velocity ω with which a body makes an elementary rotation about the instantaneous axis of rotation is called the

instantaneous angular velocity of the body. It can be denoted by a vector $\boldsymbol{\omega}$ directed along axis OP (see § 74). As the direction of axis OP changes continuously, vector $\boldsymbol{\omega}$ changes with time both in magnitude and direction, and its tip A describes a curve AD in space, which is the hodograph of vector $\boldsymbol{\omega}$ (Fig. 202).

(2)* The instantaneous angular acceleration $\boldsymbol{\varepsilon}$ of a body, which characterises the time rate of change of the angular velocity $\boldsymbol{\omega}$ both in magnitude and direction, is a vector quantity

$$\boldsymbol{\varepsilon} = \frac{d\boldsymbol{\omega}}{dt}.$$

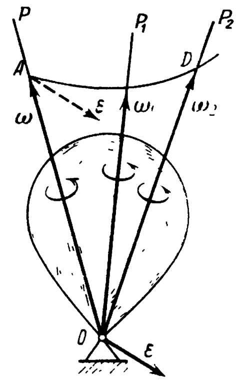

Fig. 202

Comparing this expression with the equation $\boldsymbol{v} = d\boldsymbol{r}/dt$ (§ 61), we conclude that the angular acceleration $\boldsymbol{\varepsilon}$ can be computed as the velocity with which the tip of vector $\boldsymbol{\omega}$ moves along curve AD (see Fig. 202). Specifically, the direction of $\boldsymbol{\varepsilon}$ coincides with the direction of the tangent to curve AD at the respective point. Consequently, unlike the case of rotation about a fixed axis, the direction of vector $\boldsymbol{\varepsilon}$ does not coincide with that of vector $\boldsymbol{\omega}$.

Vectors $\boldsymbol{\omega}$ and $\boldsymbol{\varepsilon}$ are the basic kinematic characteristics of the motion of a body having a fixed point. They can be computed when the equations of motion (65) are known. An example (determination of vector $\boldsymbol{\omega}$) is examined in § 97.

§ 87*. Velocity and Acceleration of a Point of a Body

Since at any moment of time a body moving about a fixed point has an instantaneous axis of rotation OP, the magnitude of the velocity of any point M of the body (Fig. 203) at that moment will be specified (by analogy with § 76) by the equation

$$v = \omega h, \tag{66}$$

where ω is the angular velocity of the body and h is the distance of point M from the instantaneous axis of rotation. The velocity vector $\boldsymbol{v}$ is normal to plane MOP through the instantaneous axis and point M in the direction of the rotation of the body.

Eq. (66) is not always convenient for determining v, as (unlike the case in § 76) the quantity h changes with time. For the same reason we cannot obtain from Eq. (66) an expression for the acceleration of point M, as was done in § 76, where h = const.

Let us therefore develop another formula which would enable us to obtain directly the velocity vector $\boldsymbol{v}$ for point M.

Consider the vector product $\boldsymbol{\omega} \times \boldsymbol{r}$, where $\boldsymbol{r}$ is the radius vector from the fixed point O to the point M. The absolute value of the product is

$$|\boldsymbol{\omega} \times \boldsymbol{r}| = \omega r \sin \alpha = \omega h.$$

Vectors $\boldsymbol{\omega} \times \boldsymbol{r}$ and $\boldsymbol{v}$, it will be readily observed, have the same direction (the direction of a vector product was discussed in § 42) and dimension. Consequenty,

$$\boldsymbol{v} = \boldsymbol{\omega} \times \boldsymbol{r}, \tag{67}$$

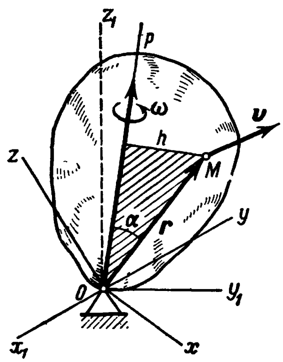

Fig. 203

i.e., *the velocity vector for any point M of a body is equal to the vector product of the angular velocity of that body and the radius vector of the point.*

Vector $\boldsymbol{v}$ can be calculated analytically from its projections on a set of coordinate axes. Let us determine the projections of $\boldsymbol{v}$ on the set of axes $Oxyz$ rigidly attached to the body and moving with it (see Fig. 203); the advantage of such a coordinate system is that the x, y, z coordinates of point M are constant quantities. Remembering that $r_x = x$, $r_y = y$, $r_z = z$, from the known formula of vector algebra we have:

$$\boldsymbol{v} = \boldsymbol{\omega} \times \boldsymbol{r} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \omega_x & \omega_y & \omega_z \\ x & y & z \end{vmatrix}.$$

Hence, reasoning as in the deduction of Eqs. (50) in § 42, we obtain:

$$\left.\begin{aligned} v_x &= \omega_y z - \omega_z y, \\ v_y &= \omega_z x - \omega_x z, \\ v_z &= \omega_x y - \omega_y x. \end{aligned}\right\} \tag{67'}$$

Eqs. (67) and (67′) are known as the *Euler equations.*

Now determine the acceleration of point M. From Eq. (67), differentiating with respect to time, we have:

$$\boldsymbol{w} = \frac{d\boldsymbol{v}}{dt} \left(\frac{d\boldsymbol{\omega}}{dt} \times \boldsymbol{r} \right) + \left(\boldsymbol{\omega} \times \frac{d\boldsymbol{r}}{dt} \right).$$

But $\frac{d\boldsymbol{\omega}}{dt} = \boldsymbol{\varepsilon}$ and $\frac{d\boldsymbol{r}}{dt} = \boldsymbol{v}$, therefore,

$$\boldsymbol{w} = (\boldsymbol{\varepsilon} \times \boldsymbol{r}) + (\boldsymbol{\omega} \times \boldsymbol{v}). \tag{68}$$

The acceleration $\boldsymbol{w}_1 = \boldsymbol{\varepsilon} \times \boldsymbol{r}$ is the rotational component, and $\boldsymbol{w}_2 = \boldsymbol{\omega} \times \boldsymbol{v}$ is the component of the acceleration of point M directed towards the axis of rotation. Vector $\boldsymbol{w}_1$ is normal to the plane through point M and vector $\boldsymbol{\varepsilon}$ (Fig. 204); in magnitude $w_1 = \varepsilon r \sin \beta = \varepsilon h_1$, where h_1 is the distance from point M to vector $\boldsymbol{\varepsilon}$. Vector $\boldsymbol{w}_2$, which is perpendicular to both $\boldsymbol{v}$ and $\boldsymbol{\omega}$, is directed along MC (see Figs. 203 and 204), and in magnitude $w_2 = \omega v \sin 90° =$

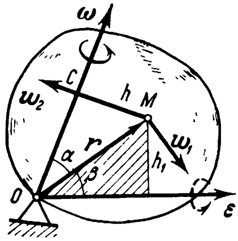

Fig. 204

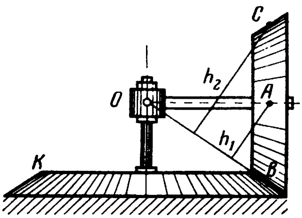

Fig. 205

$= \omega^2 h$, as $v = \omega h$. Eqs. (67) and (68) are, of course, valid for a body rotating about a fixed axis, in which case vectors $\boldsymbol{\omega}$ and $\boldsymbol{\varepsilon}$ are both directed along the axis of rotation.

Problem 80. Determine the velocities of points B and C of the bevel wheel in Fig. 205 if the velocity v_A of the wheel centre A along its path is known. The wheel runs without slipping on the fixed conic surface K.

Solution. The wheel rotates about a fixed point O. As it runs without slipping, the points of the wheel on line OB must have the same velocity as the points of surface K, i. e., zero, and OB is the instantaneous axis of rotation of the wheel. Therefore $v_A = \omega h_1$, where ω is the angular velocity of the wheel in its motion about axis OB, and h_1 is the distance of A from that axis. Hence, $\omega = v_A/h_1$.

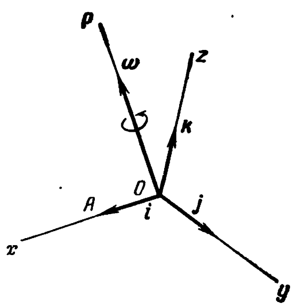

Fig. 206

The velocity v_C of point C is ωh_2, where h_2 is the distance of C from OB. As in this case $h_2 = 2h_1$, we have $v_C = 2v_A$. For point B, which is on the instantaneous axis of rotation, $v_B = 0$.

Derivatives of Unit Vectors of Moving Axes. In some problems of mechanics moving axes $Oxyz$ are used (see, for example, Chapter 14). When the axes are in translational motion, the unit vectors $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$ remain constant. However, if the trihedron $Oxyz$ in Fig. 206 rotates about an axis OP, the unit vectors cease to be constants as their directions change with time. In this case, to calculate the derivative of a vector $\boldsymbol{u} = u_x\boldsymbol{i} + u_y\boldsymbol{j} + u_z\boldsymbol{k}$, one must know the derivatives of the unit vectors $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$. Unit vector $\boldsymbol{i}$ can be treated as the radius vector $\boldsymbol{r}_A = \boldsymbol{i}$ of a point A on the axis x at unit distance from the origin O. Then

$$\frac{d\boldsymbol{i}}{dt} = \frac{d\boldsymbol{r}_A}{dt} = \boldsymbol{v}_A.$$

But according to Eq. (67), $\boldsymbol{v}_A = \boldsymbol{\omega} \times \boldsymbol{r}_A = \boldsymbol{\omega} \times \boldsymbol{i}$, where $\boldsymbol{\omega}$ is the angular velocity of the rotation about axis OP. Similar relationships are obtained for the derivatives of $\boldsymbol{j}$ and $\boldsymbol{k}$, and finally we obtain:

$$\frac{d\boldsymbol{i}}{dt} = \boldsymbol{\omega} \times \boldsymbol{i}, \quad \frac{d\boldsymbol{j}}{dt} = \boldsymbol{\omega} \times \boldsymbol{j}, \quad \frac{d\boldsymbol{k}}{dt} = \boldsymbol{\omega} \times \boldsymbol{k}. \tag{69}$$

Equations (69) are known as the *Poisson equations.*

§ 88. The General Motion of a Free Rigid Body

Let us now examine the most general motion of a rigid body free to move in any direction with respect to a reference system $Ox_1y_1z_1$ (Fig. 207). Let us establish the form of the equations that define the law of motion. Take an arbitrary point A in the body as a pole and draw through it the coordinate axes $Ax_1'y_1'z_1'$; they will move together with the pole translationally. The position of the body in the reference system $Ox_1y_1z_1$ will be known if we know the position of the pole A, i.e., its coordinates x_{1A}, y_{1A}, z_{1A}, and the body's position with respect to the reference system $Ax_1'y_1'z_1'$, which is given, as in the case examined in Sec. 86, by the Euler angles φ, ψ, θ (see Fig. 200; the Euler angles have not been shown in Fig. 207 so as not to clutter up the drawing). Consequently, the equations of motion of a free rigid body, which can be used to determine its position with respect to a reference system $Ox_1y_1z_1$ at any instant, have the form:

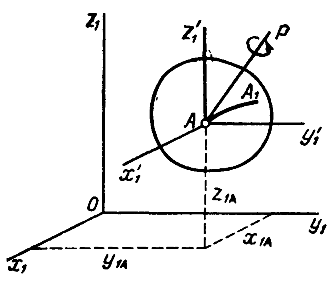

Fig. 207

$$\left.\begin{array}{lll} x_{1A} = f_1(t), & y_{1A} = f_2(t), & z_{1A} = f_3(t), \\ \varphi = f_4(t), & \psi = f_5(t), & \theta = f_6(t). \end{array}\right\} \tag{70}$$

Let us now establish the geometrical picture of the motion. It will be readily observed that the elementary displacement of a free rigid body is compounded of a translation, together with the pole A, carrying the latter into a neighbouring position A_1, and a displacement with respect to axes $Ax_1'y_1'z_1'$, i.e., about A as a fixed point.

But the latter, according to the Euler-d'Alembert theorem (§ 86), represents a rotation about an instantaneous axis of rotation through point A. Consequently, any elementary displacement of a free rigid body is compounded of an elementary translation together with a pole A and an elementary rotation about the instantaneous axis of rotation AP through that pole. Since a body's motion represents a sum of elementary displacements, we finally conclude that *the most general motion of a free rigid body is composed of a translation of the body, in which all its points move with a velocity* v_A *in the same way as an arbitrary pole* A, *and a series of infinitesimal rotations with an angular velocity* ω *about the instantaneous axes of rotation through the pole* A (Fig. 208). This, for example, is the picture of motion in

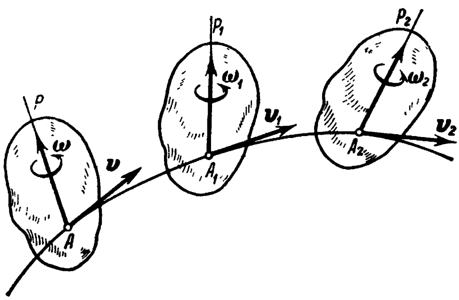

Fig. 208

any nontranslatory displacement of a body in air: a thrown stone, an aircraft engaged in stunt flying, a gun-shell, etc. Finally, a similar picture of motion con be obtained for a constrained body with appropriate constraints (see, for example, § 98, Fig. 235).

The translational component of motion of a free rigid body is described by the first three Eqs. (70), and the rotation about the pole, by the latter three. The basic kinematic characteristics of the motion are the velocity v_A and acceleration w_A of the pole, which determine the velocity and acceleration of the translational component, and the angular velocity ω and angular acceleration ε of the rotation about the pole. The values of these quantities at any instant can be determined from Eqs. (70).

Plane motion of a free body (Chapter 12) can be treated as a special case in which the vector ω remains continually normal to the plane of motion. It should be noted that both in the general case and in plane motion the rotational component of the motion (for instance, the value of ω) does not depend on the choice of the pole.

Let us now use the results of § 87* to calculate the velocities and accelerations of the points of a body in this type of motion. As in plane motion (see § 79, Fig. 174), the velocity v_M of any point M

of a body is the resultant of the velocity $\boldsymbol{v}_A$ of the pole A and the velocity $\boldsymbol{v}_{MA}$ of point M in its rotation together with the body about A, i.e.,

$$\boldsymbol{v}_M = \boldsymbol{v}_A + \boldsymbol{v}_{MA}. \tag{71}$$

The validity of this result is proved as in § 79. From Eq. (67), $\boldsymbol{v}_{MA} = \boldsymbol{\omega} \times \overline{AM}$, whence finally:

$$\boldsymbol{v}_M = \boldsymbol{v}_A + (\boldsymbol{\omega} \times \overline{AM}). \tag{71'}$$

Similarly, we obtain for the acceleration of any point M (see § 84)

$$\boldsymbol{w}_M = \boldsymbol{w}_A + \boldsymbol{w}_{MA}. \tag{72}$$

The quantity $\boldsymbol{w}_{MA}$ is given by Eq. (68) assuming $\boldsymbol{r} = \overline{AM}$ and $\boldsymbol{v} = \boldsymbol{v}_{MA} = \boldsymbol{\omega} \times \overline{AM}$.

Chapter 14
Resultant Motion of a Particle

§ 89. Relative, Transport, and Absolute Motion

So far we have considered the displacement of a particle or body with respect to one given frame of reference. But in solving problems of mechanics it is often more expedient (and sometimes necessary) to consider the motion of a particle (or body) simultaneously with respect to two frames of reference, one of which is assumed to be fixed and the other moving in some specified way with reference to the first. The motion performed in this case by the particle (or body) is called resultant, or combined motion.

For example, when a sphere rolls on the deck of a moving boat, its motion with respect to the shore is the resultant of its rolling relative to the deck (the moving frame of reference) and its motion together with the deck with respect to the shore (the fixed frame of reference). Thus, the resultant motion of the sphere can be resolved into two simpler, and easier analysed, motions. The method of resolving a motion into simpler motions by introducing a supplementary moving frame of reference is widely employed in kinematic calculations, thereby underlining the practical value of the theory of resultant motion considered in this and the following chapters. Furthermore, the conclusions of this theory are used in dynamics to investigate the relative equilibrium and motion of bodies subjected to the action of forces.

Consider the resultant motion of a particle M moving with respect to a frame of reference $Oxyz$ which is in turn moving with relation to another frame of reference $O_1x_1y_1z_1$, which we assume to be fixed (Fig. 209). (Each of these frames of reference is, of course, associated with a definite body, not shown in the diagram.) We employ the following definitions.

(1) The motion performed by the particle M with respect to the moving coordinate system is called *relative motion* (this is the motion seen by an observer moving together with the moving axes $Oxyz$). The path AB described by the particle in relative motion is called

the *relative path*. The velocity of the motion of particle M relative to the axes $Oxyz$ (i.e., along the curve AB) is called the *relative velocity* (denoted by the symbol v_{rel}), and the particle's acceleration in that motion is the *relative acceleration* (denoted w_{rel}). It follows from the definition that in computing v_{rel} and w_{rel} axes $Oxyz$ can be assumed to be fixed.

(2) The motion performed by the moving frame of reference $Oxyz$, together with all the points of space fixed with respect to it, relative to the fixed system $O_1x_1y_1z_1$ is, for the particle M, the *motion of transport*.

The velocity of the point fixed in the moving axes $Oxyz$ with which the particle M coincides at a given instant is called the *transport velocity* of the particle M at that instant (denoted by v_{tr}), and the acceleration of that point is called the *transport acceleration* of the particle M (denoted by w_{tr}). Thus,

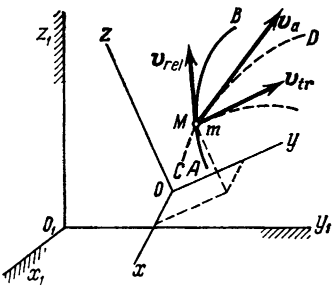

Fig. 209

$$v_{\text{tr}} = v_m, \quad w_{\text{tr}} = w_m, \tag{73}$$

where m is a point fixed with respect to the x, y, z axes, with which the moving particle M is coincident at the given instant. If we imagine the relative motion of particle M to be taking place on the surface of (or inside) a rigid body in which the moving coordinates $Oxyz$ are fixed, then the transport velocity (or acceleration) of particle M at any given instant is the velocity (or acceleration) of the point of the body which coincides with M at that instant.

(3) The motion of the particle with respect to the fixed frame of reference $O_1x_1y_1z_1$ is called the *absolute*, or *resultant*, motion. The path CD described in this motion is called the *absolute path*, the velocity is the *absolute velocity* (denoted v_{a}), and the acceleration, the *absolute acceleration* (denoted w_{a}).

In the example cited in the beginning of this article, the motion of the sphere with respect to the deck is relative motion, and the velocity of this motion is the relative velocity of the sphere; the motion of the ship with respect to the shore is, for the sphere, the motion of transport, and the velocity of the point of the deck with which the sphere coincides at the given time is, for the sphere, the transport velocity; finally, the motion of the sphere with respect to the shore is the absolute motion of the sphere, and the velocity of that motion is the absolute velocity of the sphere.

In order to solve the relevant problems of kinematics, it is necessary to establish the relationships between the velocities and accelerations of the relative, transport, and absolute motions.

§ 90. Composition of Velocities

Consider a particle M performing a resultant motion. Let the relative displacement of the particle along its path AB in the time interval $\Delta t = t_1 - t$ be specified by the vector $\overline{MM'}$ (Fig. 210*a*). In the same time interval, the curve AB, moving together with the moving axes $Oxyz$ (not shown in Fig. 210*a*), occupies a new position A_1B_1. Simultaneously, the point m on curve AB, with which the particle M is coincident at time t, performs a transport displacement $\overline{mm_1} = \overline{Mm_1}$. As a result of these displacements particle M will

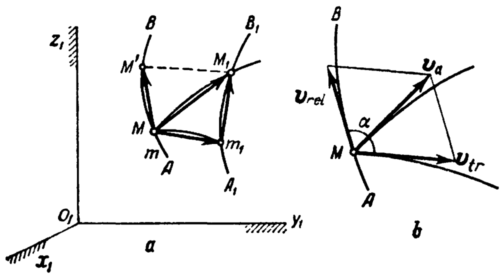

Fig. 210

occupy a position M_1, its absolute displacement in the time interval Δt being $\overline{MM_1}$. From the vector triangle Mm_1M_1 we have:

$$\overline{MM_1} = \overline{Mm_1} + \overline{m_1M_1}.$$

Dividing the equation by Δt and passing to the limit, we obtain:

$$\lim_{\Delta t \to 0} \frac{\overline{MM_1}}{\Delta t} = \lim_{\Delta t \to 0} \frac{\overline{Mm_1}}{\Delta t} + \lim_{\Delta t \to 0} \frac{\overline{m_1M_1}}{\Delta t}.$$

But by definition

$$\lim_{\Delta t \to 0} \frac{\overline{MM_1}}{\Delta t} = \boldsymbol{v}_a \quad \text{and} \quad \lim_{\Delta t \to 0} \frac{\overline{Mm_1}}{\Delta t} = \boldsymbol{v}_{\text{tr}}.$$

As for the last component, since at $\Delta t \to 0$ curve A_1B_1 tends to coincide with curve AB, in the limit we have

$$\lim_{\Delta t \to 0} \frac{\overline{m_1M_1}}{\Delta t} = \lim_{\Delta t \to 0} \frac{\overline{MM'}}{\Delta t} = \boldsymbol{v}_{\text{rel}}.$$

As a result we obtain:

$$\boldsymbol{v}_a = \boldsymbol{v}_{\text{rel}} + \boldsymbol{v}_{\text{tr}}. \tag{74}$$

Vectors $\boldsymbol{v}_{\text{a}}$, $\boldsymbol{v}_{\text{rel}}$, and $\boldsymbol{v}_{\text{tr}}$ are tangential to the respective paths (see Fig. 210*b*).

We have thus proved the following **theorem of the composition of velocities:** *In resultant motion the absolute velocity of a point is the geometrical sum of the relative velocity and the transport velocity.* The construction in Fig. 210*b* is called the *parallelogram of velocities*.

If the angle between the directions of velocities v_{rel} and v_{tr} is α, then in magnitude

$$v_a = \sqrt{v_{rel}^2 + v_{tr}^2 + 2v_{rel}v_{tr}\cos\alpha}. \tag{74'}$$

The parallelogram of velocities is used in solving the following problems of particle kinematics: (a) given $\boldsymbol{v}_{rel}$, $\boldsymbol{v}_{tr}$, determine $\boldsymbol{v}_a$;

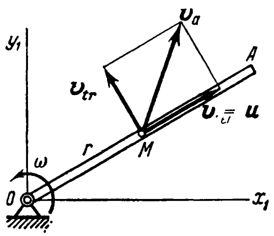

Fig. 211

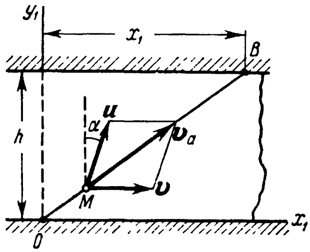

Fig. 212

(b) given $\boldsymbol{v}_a$ and the directions of $\boldsymbol{v}_{rel}$ and $\boldsymbol{v}_{tr}$, determine the magnitudes of $\boldsymbol{v}_{rel}$ and $\boldsymbol{v}_{tr}$; (c) given $\boldsymbol{v}_a$ and $\boldsymbol{v}_{tr}$, determine $\boldsymbol{v}_{rel}$ from the equation

$$\boldsymbol{v}_{rel} = \boldsymbol{v}_a + (-\boldsymbol{v}_{tr}),$$

i.e., as the geometric sum of $\boldsymbol{v}_a$ and a vector equal in magnitude and opposite in sense to $\boldsymbol{v}_{tr}$ (see Problem 83).

Problem 81. Point M moves in a straight line along OA (Fig. 211) with a velocity u, while OA itself turns in the plane Ox_1y_1 round O with an angular velocity ω. Find the velocity of point M relative to the axes Ox_1y_1 expressed as a function of the distance $OM = r$.

Solution. Consider the motion of point M as a resultant motion consisting of its relative motion along OA and its motion together with OA. Then the velocity $\boldsymbol{u}$ along OA is the relative velocity of the point. The rotational motion of OA about O is, for the point M, the motion of transport, and the velocity of the point of OA with which M coincides at the given instant is the latter's transport velocity $\boldsymbol{v}_{tr}$. As this point of OA moves along a circle of radius $OM = r$, $v_{tr} = \omega r$ and is perpendicular to OM. Constructing a parallelogram with vectors $\boldsymbol{u}$ and $\boldsymbol{v}_{tr}$ as its sides, we obtain the absolute velocity $\boldsymbol{v}_a$ of M relative to the axes Ox_1y_1. As $\boldsymbol{u}$ and $\boldsymbol{v}$ are mutually perpendicular, in magnitude

$$v_a = \sqrt{u^2 + \omega^2 r^2}.$$

Problem 82. The current of a river of width h has a constant velocity $\boldsymbol{v}$. A man can row a boat in motionless water with a velocity $\boldsymbol{u}$. Determine the direction he should take in order to cross the river in the least possible time and the point where he will reach the opposite bank.

Solution. Assume that the boat has started from point O (Fig. 212). Draw the coordinate axes Ox_1y_1 and depict the boat in an arbitrary position M. Assume further that the rower steers his boat at a constant angle α to axis Oy_1. Then the absolute velocity v_a of the boat is compounded of the relative velocity $\boldsymbol{v}_{rel}$ imparted to it by the rower ($\boldsymbol{v}_{rel} = \boldsymbol{u}$) and the transport velocity $\boldsymbol{v}_{tr}$, which is the velocity of the stream ($\boldsymbol{v}_{tr} = \boldsymbol{v}$):

$$\boldsymbol{v}_a = \boldsymbol{v}_{rel} + \boldsymbol{v}_{tr} = \boldsymbol{u} + \boldsymbol{v}.$$

The projections of the absolute velocity on the coordinate axes are (according to the theorem of the projection of a vector sum)

$$v_{ax_1} = u \sin \alpha + v, \quad v_{ay_1} = u \cos \alpha.$$

As both projections are constant, the displacements of the boat along the coordinate axes are

$$x_1 = (u \sin \alpha + v)\, t, \quad y_1 = (u \cos \alpha)\, t.$$

When the boat reaches the opposite bank, $y_1 = h$, whence the duration of the crossing is

$$t_1 = \frac{h}{u \cos \alpha}.$$

Obviously t_1 will have the least value when $\cos \alpha = 1$, i.e., when $\alpha = 0$. Consequently, in order to cross the river in the shortest time, the rower should steer his boat perpendicular to the bank. This time is

$$t_{min} = \frac{h}{u}.$$

Assuming $\alpha = 0$ and $t = t_{min}$ in the expression for x_1, we have:

$$x_1 = \frac{v}{u} h.$$

Thus, the boat will reach the other bank at a point B at a distance x_1 downstream from Oy_1 directly proportional to v and h and inversely proportional to u.

Problem 83. At a given instant, the arm OM of a recording mechanism makes an angle α with the horizontal and the pencil M has a velocity v directed perpendicular to OM (Fig. 213). The drum with the paper rotates about a vertical axis with an angular velocity ω. Determine the velocity $\boldsymbol{u}$ of the pencil on the paper if the radius of the drum is a.

Solution. The absolute velocity of the pencil is $v_a = v$. Velocity v can be regarded as the geometrical sum of the velocity of the pencil relative to the paper (i.e., the required velocity u) and the transport velocity v_{tr}, which is equal to the velocity of the point of the paper with which the pencil coincides at the given moment; its magnitude is $v_{tr} = \omega a$.

From the theorem on the composition of velocities we have $v = u + v_{tr}$, whence $u = v + (-v_{tr})$. Constructing a parallelogram with vectors v and $(-v_{tr})$ as its sides, we obtain, the required velocity u. As the angle between v and $(-v_{tr})$ is $90° - \alpha$, in magnitude

$$u = \sqrt{v^2 + \omega^2 a^2 + 2v\omega a \sin \alpha}.$$

The angle between u and the direction of v_{tr} can now be determined by the law of sines.

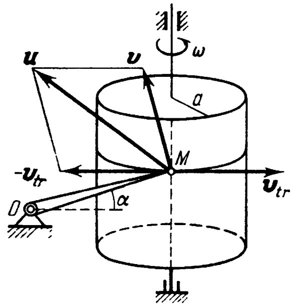

Fig. 213

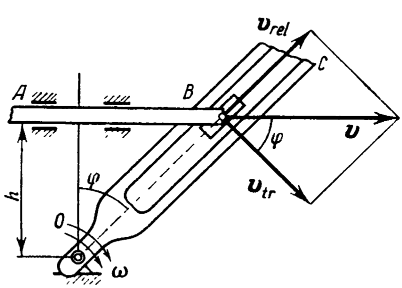

Fig. 214

Problem 84. The end B of a horizontal rod AB is hinged to a block sliding along the slot of a rocker OC and turns the latter round axis O (Fig. 214). The distance from O to AB is h. Find the dependence of the angular velocity of the rocker on the velocity v of the rod and angle φ.

Solution. The absolute velocity of the slide block equals the velocity v of the rod. It can be regarded as compounded of the relative velocity v_{rel} of the block in its motion in the slot of the rocker and the transport velocity v_{tr}, which is the velocity of the point of the rocker with which the block coincides at the given time. The directions of these velocities are along OB and perpendicular to OB, respectively. We obtain v_{rel} and v_{tr} by resolving velocity v along them. From the parallelogram we find that in magnitude $v_{tr} = v \cos \varphi$.

But, on the other hand, the transport velocity $v_{tr} = \omega \cdot OB = \omega h/\cos \varphi$, where ω is the angular velocity of the rocker. Equating these two expressions of v_{tr}, we obtain the angular velocity:

$$\omega = \frac{v}{h} \cos^2 \varphi.$$

§ 91*. Composition of Accelerations

Let us determine the dependence between the relative and transport accelerations of a particle. From Eq. (74) we obtain:

$$w_a = \frac{dv_a}{dt} = \frac{dv_{rel}}{dt} + \frac{dv_{tr}}{dt}. \tag{75}$$

Let us compute the derivatives in the right-hand side of the equation which, as we shall see, are in the most general case not equal to w_{rel} and w_{tr}. For this we need the expressions of vectors v_{rel}, w_{rel}, v_{tr}, and w_{tr}.

Let the position of particle M with respect to the moving coordinate axes $Oxyz$ in Fig. 215 be given by its x, y, z coordinates. Since in computing v_{rel} and w_{rel} the motion of the moving axes is disregarded

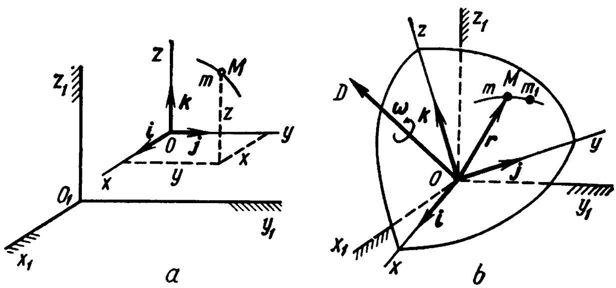

Fig. 215

(they can be assumed fixed), the projections of vectors v_{rel} and w_{rel} on axes $Oxyz$ in any transport motion are then given by the Eqs. (15′) and (17′) from § 64. Consequently,

$$v_{rel} = \dot{x}i + \dot{y}j + \dot{z}k, \quad w_{rel} = \ddot{x}i + \ddot{y}j + \ddot{z}k, \tag{76}$$

where i, j and k are the unit vectors of axes $Oxyz$.

The further calculation depends on the nature of the transport motion. First, let us consider the case when it is translational.

Composition of Accelerations in Translational Motion of Transport. If a moving reference system $Oxyz$ is in translational motion relative to a fixed system $O_1x_1y_1z_1$ (Fig. 215*a*), then, obviously, at any position of particle M,

$$v_{tr} = v_O, \quad w_{tr} = w_O, \tag{77}$$

where v_O and w_O are the velocity and acceleration of the origin O.

Besides, as the axes $Oxyz$ are in translational motion, the unit vectors remain continually parallel to themselves and, hence, are constant. Then from Eqs. (76) and (77) we obtain:

$$\frac{dv_{rel}}{dt} = \ddot{x}i + \ddot{y}j + \ddot{z}k = w_{rel}, \quad \frac{dv_{tr}}{dt} = \frac{dv_O}{dt} = w_O = w_{tr}.$$

Consequently, from Eq. (75) we obtain:

$$\boldsymbol{w}_{\mathrm{a}} = \boldsymbol{w}_{\mathrm{rel}} + \boldsymbol{w}_{\mathrm{tr}}. \tag{78}$$

Thus, *in translational motion of transport the absolute acceleration of a particle is equal to the geometrical sum of its relative and transport accelerations.* The result is analogous to that of the theorem of the composition of velocities.

Composition of Accelerations in Non-translational Motion of Transport. Coriolis Theorem. Assume first that the motion of transport (i.e., the motion of the moving reference system *Oxyz*) is rotational with an angular velocity ω (Fig. 215*b*). In this case axis *OD* may either be fixed (§ 74) or it may be the instantaneous axis of rotation (when point *O* is fixed; see § 86). In both cases the unit vectors $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ are no longer constant as, rotating together with the coordinate axes, they change their directions, which had not been taken into account in computing $\boldsymbol{w}_{\mathrm{rel}}$. Therefore we obtain from Eqs. (76), which are valid for any motion of transport:

$$\frac{d\boldsymbol{v}_{\mathrm{rel}}}{dt} = (\ddot{x}\boldsymbol{i} + \ddot{y}\boldsymbol{j} + \ddot{z}\boldsymbol{k}) + \left(\dot{x}\frac{d\boldsymbol{i}}{dt} + \dot{y}\frac{d\boldsymbol{j}}{dt} + \dot{z}\frac{d\boldsymbol{k}}{dt}\right) = \boldsymbol{w}_{\mathrm{rel}} + \boldsymbol{w}_1,$$

where $\boldsymbol{w}_1$ denotes the second bracket in the right-hand side of the equation. Calculating $\boldsymbol{w}_1$ with the help of the Poisson equations (69), we obtain:

$$\boldsymbol{w}_1 = \dot{x}(\omega + \boldsymbol{i}) + \dot{y}(\omega \times \boldsymbol{j}) + \dot{z}(\omega \times \boldsymbol{k}) = \boldsymbol{\omega} \times (\dot{x}\boldsymbol{i} + \dot{y}\boldsymbol{j} + \dot{z}\boldsymbol{k}) = \boldsymbol{\omega} \times \boldsymbol{v}_{\mathrm{rel}},$$

and finally,

$$\frac{d\boldsymbol{v}_{\mathrm{rel}}}{dt} = \boldsymbol{w}_{\mathrm{rel}} + \boldsymbol{w}_1, \quad \text{where} \quad \boldsymbol{w}_1 = \boldsymbol{\omega} \times \boldsymbol{v}_{\mathrm{rel}}. \tag{79}$$

In this equation the quantity $\boldsymbol{w}_{\mathrm{rel}}$ takes into account the change in vector $\boldsymbol{v}_{\mathrm{rel}}$ only in the relative motion of the particle *M*, and the new member $\boldsymbol{w}_1$ takes into account the change of vector $\boldsymbol{v}_{\mathrm{rel}}$ in its rotation together with the trihedron *Oxyz* around the axis *OD*, i.e., in the motion of transport (see also Fig. 221*a*).

Furthermore, in rotational motion the velocity and acceleration of any point *m* fixed with respect to axes *Oxyz* are determined by Eqs. (67) and (68), the same as for the points of a rigid body. But according to Eqs. (73), $\boldsymbol{v}_{\mathrm{rel}} = \boldsymbol{v}_m$ and $\boldsymbol{w}_{\mathrm{tr}} = \boldsymbol{w}_m$, hence Eqs. (67) and (68) yield

$$\boldsymbol{v}_{\mathrm{tr}} = \boldsymbol{\omega} \times \boldsymbol{r}, \quad \boldsymbol{w}_{\mathrm{tr}} = (\boldsymbol{\varepsilon} \times \boldsymbol{r}) + (\boldsymbol{\omega} \times \boldsymbol{v}_{\mathrm{tr}}), \tag{80}$$

where $\boldsymbol{r}$ is the radius vector of point *m* coincident at the given instant with the radius vector of the moving particle *M*. Hence,

$$\frac{d\boldsymbol{v}_{\mathrm{tr}}}{dt} = \left(\frac{d\boldsymbol{\omega}}{dt} \times \boldsymbol{r}\right) + \left(\boldsymbol{\omega} \times \frac{d\boldsymbol{r}}{dt}\right).$$

As the derivative in the left-hand side of the equation is a member of the right-hand side of Eq. (75), which gives the absolute acceleration of particle M, i.e., its acceleration in the coordinate system $Ox_1y_1z_1$, the derivative of the radius vector $\boldsymbol{r}$ in the right-hand side yields the velocity of M in the same coordinate system, i.e., its absolute velocity. Consequently, here $\frac{d\boldsymbol{r}}{dt}=\boldsymbol{v}_a=\boldsymbol{v}_{rel}+\boldsymbol{v}_{tr}$, and besides, $\frac{d\boldsymbol{\omega}}{dt}=\boldsymbol{\varepsilon}$; therefore

$$\frac{d\boldsymbol{v}_{tr}}{dt}=(\boldsymbol{\varepsilon}\times\boldsymbol{r})+(\boldsymbol{\omega}\times\boldsymbol{v}_{tr})+(\boldsymbol{\omega}\times\boldsymbol{v}_{rel}).$$

From this, taking into account the second of Eqs. (80), we obtain:

$$\frac{d\boldsymbol{v}_{tr}}{dt}=\boldsymbol{w}_{tr}+\boldsymbol{w}_2, \quad \text{where} \quad \boldsymbol{w}_2=\boldsymbol{\omega}\times\boldsymbol{v}_{rel}. \tag{81}$$

The quantity $\boldsymbol{w}_{tr}$ takes into account the change in vector $\boldsymbol{v}_{tr}$ only in the motion of transport, because it is computed as the acceleration of point m, fixed in the reference frame $Oxyz$. But the new member $\boldsymbol{w}_2$ takes into account the change in vector $\boldsymbol{v}_{tr}$ that occurs in the relative motion of particle M, since as a result of that motion M moves from position m to a new position m_1, where the value of $\boldsymbol{v}_{tr}$ is different (see also Fig. 221*b*).

We examined the case of rotational motion of transport. But in the most general case, too, when the motion of transport, as motion of a free rigid body (see § 88), is the resultant of translational and rotational motions, Eqs. (79) and (81) remain of the same form, with the only difference being that in Eq. (81) $\boldsymbol{w}_{tr}$ is calculated according to Eq. (72) instead of Eq. (80).

Now substituting the quantities (79) and (81) into Eq. (75), we obtain:

$$\boldsymbol{w}_a=\boldsymbol{w}_{rel}+\boldsymbol{w}_{tr}+\boldsymbol{w}_1+\boldsymbol{w}_2. \tag{82}$$

Let us introduce the notation

$$\boldsymbol{w}_{Cor}=\boldsymbol{w}_1+\boldsymbol{w}_2=2(\boldsymbol{\omega}\times\boldsymbol{v}_{rel}). \tag{83}$$

The quantity $\boldsymbol{w}_{Cor}$, which characterises the rate of change of the vector of relative velocity $\boldsymbol{v}_{rel}$ in the motion of transport and the rate of change of the vector of the transport velocity $\boldsymbol{v}_{tr}$ in the relative motion, is called the *supplementary*, or *Coriolis*, *acceleration* of the particle. Then, from Eq. (82) we obtain

$$\boldsymbol{w}_a=\boldsymbol{w}_{rel}+\boldsymbol{w}_{tr}+\boldsymbol{w}_{Cor}. \tag{84}$$

Eq. (84) expresses the **Coriolis theorem** *):

---

*) Gaspard Gustave de Coriolis (1792-1843), a French scientist famous for his works in theoretical and applied mechanics.

The absolute acceleration of a particle is equal to the geometrical sum of three accelerations: the relative acceleration, which characterises the time rate of change of the relative velocity in the relative motion, the transport acceleration, which characterises the time rate of change of the transport velocity in the transport motion, and the Coriolis acceleration, which characterises the time rate of change of the relative velocity in the transport motion and of the transport velocity in the relative motion.

If the motion of transport is translational, $\omega = 0$ and $w_{Cor} = 0$, and Eq. (84) becomes Eq. (78).

Calculation of Relative, Transport, and Coriolis Accelerations. We examined the question of computing the relative and transport accelerations of a particle in proving the theorem; these quantities are determined according to the known equations of kinematics.

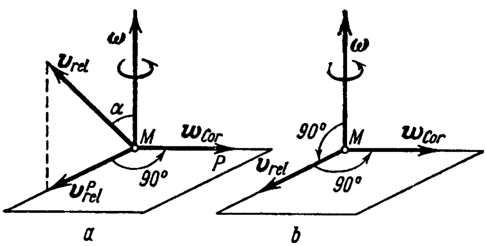

Fig. 216

For, since in calculating w_{rel} the motion of the moving axis can be disregarded, w_{rel} is computed by the conventional methods of particle kinematics (§§ 64, 67). In calculating w_{tr}, the relative motion of the particle can be disregarded, consequently w_{tr} is computed as the acceleration of a point belonging to a certain rigid body fixed relative to the reference frame *Oxyz* and moving together with it, i.e., by the methods of rigid-body kinematics (§§ 76, 84, 87, 88). The Coriolis acceleration is calculated from Eq. (83):

$$w_{Cor} = 2(\omega \times v_{rel}), \tag{85}$$

where ω is the angular velocity of the motion of transport.

Thus, *the Coriolis acceleration of a particle is equal to the double vector product of the angular velocity of the motion of transport and the relative velocity of the particle.* If the angle between the vectors v_{rel} and ω is α, then in magnitude

$$w_{Cor} = 2\omega v_{rel} \sin \alpha. \tag{86}$$

The vector w_{Cor} is of the same sense as the vector $\omega \times v_{rel}$, i.e., normal to the plane through vectors ω and v_{rel} in the direction from which a counterclockwise rotation would be seen to carry vector ω into vector v_{rel} through the smaller angle (Fig. 216*a*).

It can also be seen from Fig. 216*a* that the direction of vector w_{Cor} can be obtained by projecting vector v_{rel} on plane P, which is normal to ω, and turning the projection v_{rel}^P through 90° in the direction of the rotation of transport.

If the relative path is a plane curve moving in its plane, then angle $\alpha = 90°$ (Fig. 216*b*) and in magnitude

$$w_{Cor} = |2\omega \cdot v_{rel}|. \tag{86'}$$

It can be seen from Fig. 216*b* that in this case the direction of w_{Cor} can be obtained by turning the vector of the relative velocity v_{rel} through 90° in the direction of the rotation of transport (i.e., clockwise or counterclockwise depending on the sense of the rotation).

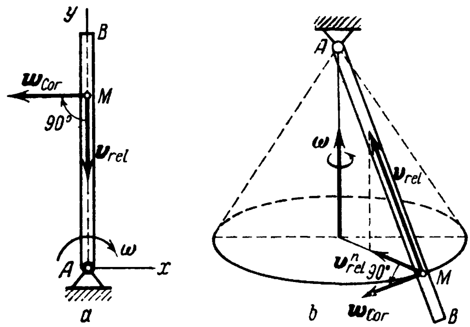

Fig. 217

To illustrate the above, Fig. 217 shows the direction of the Coriolis acceleration of a ball M, moving in a tube AB for the cases when the tube itself turns in the plane of the diagram (Fig. 217*a*) and when it describes a cone (Fig. 217*b*).

From Eq. (86) we see that the Coriolis acceleration is zero when:

(1) $\omega = 0$, i.e., if the motion of transport is translational [Eq. (78)] or if the angular velocity of the rotation of transport becomes zero at a given instant;

(2) $v_{rel} = 0$, i.e., if there is no relative motion or if the relative velocity becomes zero at a given instant;

(3) angle $\alpha = 0$ or $\alpha = 180°$, i.e., if the relative motion is parallel to the axis of the rotation of transport or if vector v_{rel} is parallel to that axis at a given instant.

§ 92. Solution of Problems

(A) *Translational Motion of Transport.* When the motion of transport is translational the nature of the problems and the methods of their solution are analogous to the problems on the composition of velocities (§ 90).

Problem 85. A wedge moving horizontally with an acceleration w_1 pushes up a rod moving in vertical slides (Fig. 218). Determine the acceleration of the rod if the angle of the wedge is α.

Solution. The absolute acceleration $\boldsymbol{w}_A$ of point A is directed vertically up. It can be regarded as consisting of a relative acceleration $\boldsymbol{w}_{rel}$ directed along the side of the wedge and a transport acceleration $\boldsymbol{w}_{tr}$, which is equal to the acceleration of the wedge $\boldsymbol{w}_1$. As

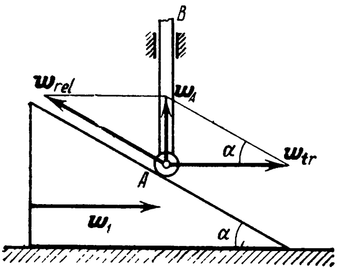

Fig. 218

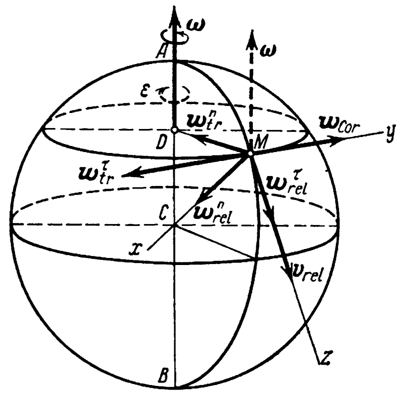

Fig. 219

the motion of transport of the wedge is translational, by drawing a parallelogram on the basis of Eq. (78) and taking into account that $w_{tr} = w_1$, we obtain:

$$w_A = w_1 \tan \alpha,$$

which is the acceleration of the rod.

(B) *Rotational Motion of Transport.* Let us investigate in general form the determination of $\boldsymbol{w}_a$ when the motion of transport is a rotation about a fixed axis.

Let a particle M move along a relative path AMB on the surface of a body (say a sphere) rotating with retardation about an axis BA (Fig. 219). In order to find the absolute acceleration of the particle at a given time t_1, we must know for that instant: (1) the position of the particle on the curve AB, (2) the relative velocity v_{rel} of the particle, (3) the angular velocity ω and the angular acceleration ε of the body (i.e., ω and ε of the motion of transport). If these quantities are not given, they should be determined from the conditions of the problem.

The next step is to depict the moving particle in its position at time t_1 and show vectors $\boldsymbol{v}_{rel}$ and $\boldsymbol{\omega}$ in the diagram. The further computations are then as follows.

(1) Determination of $\boldsymbol{w}_{\text{rel}}$. Mentally stop the rotation of the body and calculate the acceleration of the particle in its motion along AB according to the formulas of particle kinematics. If AB is given, then (§ 67)

$$w_{\text{rel}}^{\tau}=\frac{dv_{\text{rel}}}{dt}; \quad w_{\text{rel}}^{n}=\frac{v_{\text{rel}}^{2}}{\rho_{\text{rel}}},$$

where ρ_{rel} is the radius of curvature of AB at point M. If the relative motion is described by the coordinate method, then v_{rel} and w_{rel} are calculated according to the formulas of § 64.

(2) Determination of $\boldsymbol{w}_{\text{tr}}$. Calculating the acceleration of that point of the body with which the particle M coincides at the given moment, we obtain the transport acceleration from the formulas of rigid-body kinematics (§ 76):

$$w_{\text{tr}}^{\tau}=h\varepsilon, \quad w_{\text{tr}}^{n}=h\omega^{2},$$

where $h = MD$ is the distance of M from the axis of rotation at time t_1.

(3) Determination of $\boldsymbol{w}_{\text{Cor}}$. The calculation is according to the procedure explained in § 91.

(4) Determination of $\boldsymbol{w}_{\text{a}}$. Draw all the computed vectors on the diagram (taking into account their directions). From Coriolis theorem we have

$$\boldsymbol{w}_{\text{a}}=\boldsymbol{w}_{\text{rel}}^{\tau}+\boldsymbol{w}_{\text{rel}}^{n}+\boldsymbol{w}_{\text{tr}}^{\tau}+\boldsymbol{w}_{\text{tr}}^{n}+\boldsymbol{w}_{\text{Cor}}.$$

If it is difficult to obtain the sum of the vectors in the right side of the equation geometrically, draw an arbitrary set of rectangular axes $Mxyz$ (see Fig. 219) and compute the projections of all the component vectors on these axes. Then, from the theorem of the projection of a vector sum, we have

$$w_{\text{a}x}=\sum w_{ix}, \quad w_{\text{a}y}=\sum w_{iy}, \quad w_{\text{a}z}=\sum w_{iz},$$

and finally

$$w_{\text{a}}=\sqrt{w_{\text{a}x}^{2}+w_{\text{a}y}^{2}+w_{\text{a}z}^{2}}.$$

Note: In computing $\boldsymbol{w}_{\text{a}}$ be careful not to assume that

$$w_{\text{a}}=\sqrt{w_{\text{rel}}^{2}+w_{\text{tr}}^{2}+w_{\text{Cor}}^{2}},$$

as in the general case vectors $\boldsymbol{w}_{\text{rel}}$, $\boldsymbol{w}_{\text{tr}}$, and $\boldsymbol{w}_{\text{Cor}}$ are not mutually perpendicular.

Problem 86. The rocker OA in Fig. 220 turns with a constant angular velocity ω about axis O. Block B slides along the slot with a constant relative velocity $\boldsymbol{u}$. Determine the dependence of the absolute acceleration of the block on its distance x from O.

Solution. Stopping the rocker at time t_1, we find that the relative motion of the block along it is uniform and rectilinear; consequently $w_{\text{rel}} = 0$.

For the block, the motion of the rocker is that of transport; consequently, the transport acceleration $\boldsymbol{w}_{tr}$ of the block is equal to the acceleration of the point of the rocker with which the block coincides at the given time. Since that point of the rocker is moving in a circle of radius $OB = x$ and $\omega = \text{const.}$, vector $\boldsymbol{w}_{tr} = \boldsymbol{w}_{tr}^{n}$ is directed along BO. In magnitude $w_{tr} = w_{tr}^{n} = \omega^2 x$.

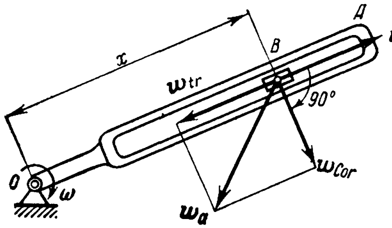

Fig. 220

As the motion is in a plane, the Coriolis acceleration $w_{Cor} = 2\ wu$. By turning the vector of the relative velocity $\boldsymbol{u}$ about point B through a right angle in the direction of the rotation of transport (clockwise), we obtain the direction of $\boldsymbol{w}_{Cor}$.

From Coriolis theorem,

$$\boldsymbol{w}_{a} = \boldsymbol{w}_{rel} + \boldsymbol{w}_{tr} + \boldsymbol{w}_{Cor}.$$

In the present case $\boldsymbol{w}_{rel} = 0$ and $\boldsymbol{w}_{Cor}$ is perpendicular to $\boldsymbol{w}_{tr}$. Consequently,

$$w_{a} = \sqrt{w_{tr}^{2} + w_{Cor}^{2}} = \omega \sqrt{\omega^2 x^2 + 4u^2}.$$

Let us use this problem as an example showing how the components of $\boldsymbol{w}_{Cor}$ in Eq. (83), w_1 and w_2, appear.

Direct a moving axis Ox along the rocker slot and denote the sliding block as a point B (Fig. 221*a*); axis Ox rotates around the centre O and is in transport motion with respect to the fixed axes Ox_1y_1. As in this case the relative velocity $\boldsymbol{u}$ does not change with the displacement of B along axis Ox, we have $\boldsymbol{w}_{rel} = 0$. But in the motion of transport vector $\boldsymbol{u}$ will, in time dt, rotate together with axis Ox through an angle $d\varphi = \omega\ dt$, receiving an increment $d'\boldsymbol{u}$ (the primed d denoting the addition increment). As the magnitude of $\boldsymbol{u}$ does not change in the rotation, vector $d'\boldsymbol{u}$ is perpendicular to $\boldsymbol{u}$, and $|d'\boldsymbol{u}| = u \cdot d\varphi = u\omega\, dt$. As a result point B receives an additional acceleration $\boldsymbol{w}_1$ in the same direction as vector $d'\boldsymbol{u}$, i.e, perpendicular to axis Ox, and in magnitude is

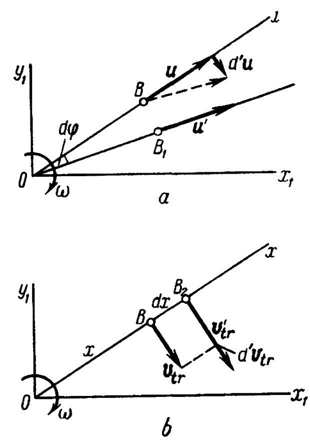

Fig. 221

$$w_1 = \frac{|d'\boldsymbol{u}|}{dt} = u\omega. \qquad \text{(a)}$$

The velocity v_{tr} of point B in the motion of transport (in the rotation of axis Ox) changes in direction, giving the point the acceleration of transport computed above and shown in Fig. 220. But vector v_{tr} also receives an increment in the relative displacement of point B to position B_2 (Fig. 221*b*), since in position B the quantity $v_{tr} = \omega x$, and in position B_2, $v'_{tr} = \omega (x + dx)$. Consequently, since $dx = u\,dt$, $|d'v_{tr}| = \omega\,dx = \omega u\,dt$. Hence, point B receives yet another acceleration w_2 in the same direction as v_{tr}, i.e., as w_1, and is equal in magnitude to

$$w_2 = \frac{|d'v_{tr}|}{dt} = \omega u. \qquad \text{(b)}$$

As vectors w_1 and w_2 are of the same sense, adding them we obtain the total additional acceleration of point B as a result of the change of vector v_{rel} in the motion of transport and of vector v_{tr} in the relative motion. This acceleration, equal in magnitude to $w_1 + w_2$, i.e., $2\omega u$, is the Coriolis acceleration of point B.

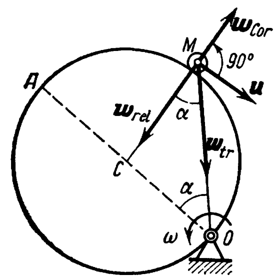

Fig. 222

Problem 87. The eccentric in Fig. 222 is a circular disc of radius R rotating with a uniform angular velocity ω about axis O through the rim of the disc. Sliding from point A along the disc with a constant relative velocity u is a pin M. Determine the absolute acceleration of the pin at any time t. The motions are directed as shown in the diagram.

Solution. At time t the pin is at a distance $s = \overset{\frown}{AM} = ut$ from A. Consequently, at that instant the angle $AOM = \alpha$ will be

$$\alpha = \frac{s}{2R} = \frac{u}{2R}t, \qquad \text{(a)}$$

as angle α is equal to half the central angle ACM.

Stopping the motion of the disc at time t, we find that the relative motion of the pin is along a circle of radius R. As $v_{rel} = u = \text{const.}$,

$$w^{\tau}_{rel} = \frac{du}{dt} = 0, \quad w^{n}_{rel} = \frac{u^2}{R}. \qquad \text{(b)}$$

The vector $w_{rel} = w^{n}_{rel}$ is directed along the radius MC.

For the pin the motion of the disc is that of transport. Hence, the transport acceleration w_{tr} of the pin is equal to the acceleration of the point of the disc with which it coincides at the given time. This point moves in a circle of radius $OM = 2R\cos\alpha$. For the disc, $\omega = \text{const.}$, hence $\varepsilon = 0$ and

$$w^{\tau}_{tr} = OM \cdot \varepsilon = 0, \quad w^{n}_{tr} = OM \cdot \omega^2 = 2R\omega^2 \cos\alpha. \qquad \text{(c)}$$

Vector $w_{tr} = w^{n}_{tr}$ is directed along MO.

As the motion is in one plane,

$$w_{Cor} = 2\omega u. \tag{d}$$

The direction of $\boldsymbol{w}_{Cor}$ is found by turning vector $\boldsymbol{v}_{rel} = \boldsymbol{u}$ round point M through 90° in the direction of the motion of transport (counterclockwise). The absolute acceleration of the pin is

$$\boldsymbol{w}_a = \boldsymbol{w}_{rel} + \boldsymbol{w}_{tr} + \boldsymbol{w}_{Cor}.$$

In this case vectors $\boldsymbol{w}_{rel}$ and $\boldsymbol{w}_{Cor}$ are collinear and can be replaced by a collinear vector $\boldsymbol{w}_1$ of magnitude $w_1 = w_{rel} - w_{Cor}$.

Adding vectors $\boldsymbol{w}_1$ and $\boldsymbol{w}_{tr}$ according to the parallelogram law, we obtain finally

$$w_a = \sqrt{w_{tr}^2 + (w_{rel} - w_{Cor})^2 + 2w_{tr}(w_{rel} - w_{Cor}) \cos \alpha},$$

where the values of α, w_{rel}, w_{tr}, and w_{Cor} are given by equations (a), (b), (c), and (d).

Problem 88. A body in the northern hemisphere is translated from north to south along a meridian with a velocity $v_{rel} = u$ m/s (Fig. 223).

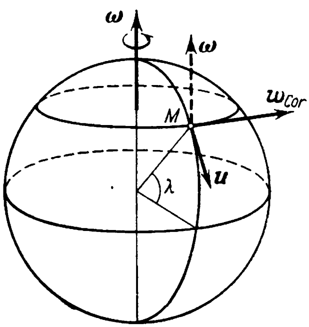

Fig. 223

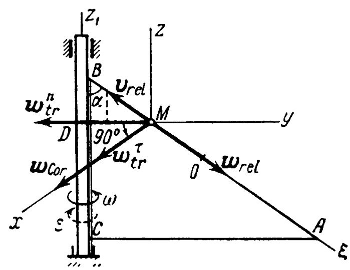

Fig. 224

Determine the magnitude and direction of the Coriolis acceleration of the body at latitude λ.

Solution. Neglecting the dimensions of the body, we treat it as a particle. The relative velocity $\boldsymbol{u}$ of the body makes an angle λ with the earth's axis. Consequently,

$$w_{Cor} = 2\omega u \sin \lambda,$$

where ω is the angular velocity of the earth's rotation.

Thus, the Coriolis acceleration is greatest at the north pole, where $\lambda = 90°$. As the body approaches the equator, the value of w_{Cor} decreases, till it reaches zero at the equator, where the vector $\boldsymbol{v}_{rel} = \boldsymbol{u}$ is parallel to the axis of rotation of the earth.

The direction of $\boldsymbol{w}_{Cor}$ is found by vector product. As $\boldsymbol{w}_{Cor} = 2(\boldsymbol{\omega} \times \boldsymbol{u})$, we find that vector $\boldsymbol{w}_{Cor}$ is perpendicular to the plane through vectors $\boldsymbol{u}$ and $\boldsymbol{\omega}$, i.e., perpendicular to the meridian plane, and is directed eastwards, from where the shortest turn from vector $\boldsymbol{\omega}$ to vector $\boldsymbol{u}$ is seen counterclockwise.

The question of how the Coriolis acceleration affects the motion of bodies at the earth's surface is studied in the course of dynamics. However, from the formula obtained it can be seen that the value of w_{Cor} is usually small, as the angular velocity of rotation of the earth is small:

$$\omega \approx \frac{2\pi}{24 \cdot 3\,600} \text{ s}^{-1}.$$

It is apparent, therefore, that for motions along the surface of the earth when the velocity u is not very great, the Coriolis acceleration can, for all practical purposes, be neglected.

Problem 89. The hypotenuse of the right triangle ABC in Fig. 224 is $AB = 2a = 20$ cm, and $\angle CBA = \alpha = 60°$. The triangle rotates about axis Cz_1 according to the law $\varphi = 10t - 2t^2$. Particle M oscillates along AB about its middle O, its equation of motion being $\xi = a \cos(\pi t/3)$ (axis $O\xi$ is directed along OA). Determine the absolute acceleration of the particle M at time $t_1 = 2$ s.

Solution. (1) Determine the position of M on its relative path AB at time t_1. From the equation of the motion we have

$$\xi_1 = a \cos\left(\frac{2\pi}{3}\right) = -\frac{a}{2},$$

i.e., at time t_1 the particle M is at the middle of segment OB. Show this position in the diagram.

(2) Determination of v_{rel}. As the relative motion is rectilinear,

$$v_{rel} = \frac{d\xi}{dt} = -\frac{\pi}{3} a \sin\left(\frac{\pi}{3} t\right).$$

At time $t_1 = 2$ s,

$$v_{rel_1} = -\frac{\pi}{6} a \sqrt{3}; \quad |v_{rel_1}| = \frac{5}{3} \pi \sqrt{3} \text{ cm/s}.$$

The minus indicates that at the given instant t_1 vector v_{rel} is pointed from M to B.

(3) Determination of ω and ε. Differentiating, we obtain:

$$\omega = \frac{d\varphi}{dt} = 10 - 4t, \quad \omega_1 = 2 \text{ s}^{-1},$$

where ω_1 is the value of ω at time $t_1 = 2$ s;

$$\varepsilon = \frac{d\omega}{dt} = -4 \text{ s}^{-2}.$$

The signs indicate that at time t_1 the rotation is counterclockwise (observed from the tip of axis Cz_1) and is retarded.

(4) Determination of $\boldsymbol{w}_{rel}$. As the relative motion is rectilinear,

$$w_{rel} = \frac{dv_{rel}}{dt} = -\frac{\pi^2}{9} a \cos\left(\frac{\pi}{3} t\right).$$

At time $t_1 = 2$ s, $w_{rel_1} = \frac{\pi^2}{18} a = \frac{5}{9} \pi^2$ cm/s².

(5) Determination of $\boldsymbol{w}_{tr}$. For the particle M the motion of the triangle is that of transport, and the transport acceleration of M is equal to the acceleration of the point of the triangle with which M coincides at the given time. This point of the triangle moves in a circle of radius $MD = h$, and at time $t_1 = 2$ s,

$$h = \frac{a}{2} \sin\alpha = 5\frac{\sqrt{3}}{2} \text{ cm}.$$

Thus, at the given instant,

$$w_{tr}^{\tau} = \varepsilon h = -10\sqrt{3} \text{ cm/s}^2, \quad w_{tr}^{n} = \omega^2 h = 10\sqrt{3} \text{ cm/s}^2.$$

Vector $\boldsymbol{w}_{tr}^{\tau}$ is normal to plane ABC in the direction opposite to that of the rotation of the triangle. Vector $\boldsymbol{w}_{tr}^{n}$ is directed along MD towards the axis of rotation Cz_1.

(6) Determination of $\boldsymbol{w}_{Cor}$. The magnitude of w_{Cor} at time $t_1 = 2$ s is

$$w_{Cor} = 2\,|\omega v_{rel}|\sin\alpha = 10\pi \text{ cm/s}^2,$$

as in this case the angle between $\boldsymbol{v}_{rel}$ and axis Cz_1 is α.

Projecting vector $\boldsymbol{v}_{rel}$ on a plane perpendicular to Cz_1 (the projection lies along MD) and turning the projection through a right angle in the direction of the rotation of transport, i.e., counterclockwise, we obtain the direction of $\boldsymbol{w}_{Cor}$ (which in the present case coincides with the direction $\boldsymbol{w}_{tr}^{\tau}$).

(7) Determination of $\boldsymbol{w}_a$. The absolute acceleration of the particle M at time t_1 is

$$\boldsymbol{w}_a = \boldsymbol{w}_{rel} + \boldsymbol{w}_{tr}^{\tau} + \boldsymbol{w}_{tr}^{n} + \boldsymbol{w}_{Cor}.$$

In order to determine the value of $\boldsymbol{w}_a$, draw a set of axes $Mxyz$ (see Fig. 224) and calculate the projections of all the vectors on them. We obtain:

$$w_{ax} = w_{Cor} + |w_{tr}^{\tau}| = 10\pi + 10\sqrt{3} \approx 48.7 \text{ cm/s}^2,$$

$$w_{ay} = w_{rel}\sin\alpha - w_{tr}^{n} = \frac{5\pi^2}{18}\sqrt{3} - 10\sqrt{3} \approx -12.6 \text{ cm/s}^2,$$

$$w_{az} = -w_{rel}\cos\alpha = -\frac{5}{18}\pi^2 \approx -2.7 \text{ cm/s}^2,$$

and, finally,

$$w_a = \sqrt{w_{ax}^2 + w_{ay}^2 + w_{az}^2} \approx 50.4 \text{ cm/s}^2.$$

Vector $\boldsymbol{w}_a$ can be constructed according to its rectangular components along the coordinate axes $Oxyz$.

Chapter 15
Resultant Motion of a Rigid Body

§ 93. Composition of Translational Motions

If a body moves with respect to a set of moving axes $Oxyz$ (see Fig. 209) which are in transport motion relative to a fixed system $Ox_1y_1z_1$, the resulting absolute motion of the body is called resultant, or combined, motion (see § 89).

The problem of kinematics in this case is to determine the relation between the relative, transport, and absolute motions. As we know, the main kinematic characteristics of the motion of a body are its translational and angular velocities and accelerations. We shall confine ourselves to establishing only the relations between the translational and angular velocities of the motions.

Consider first the case when the relative motion of a body is a translation with a velocity $\boldsymbol{v}_1$ and the motion of transport is also a translation with a velocity $\boldsymbol{v}_2$. Then in the relative motion all the points of the body will have a velocity $\boldsymbol{v}_1$, and in the motion of transport, $\boldsymbol{v}_2$. From the theorem of the composition of velocities, in the absolute motion all the points of the body will have the same velocity $\boldsymbol{v} = \boldsymbol{v}_1 + \boldsymbol{v}_2$, i.e., the absolute motion will be that of translation.

Thus, *the resultant motion of two translational motions with velocities* $\boldsymbol{v}_1$ *and* $\boldsymbol{v}_2$ *is also a translation with a velocity* $\boldsymbol{v} = \boldsymbol{v}_1 + \boldsymbol{v}_2$.

The composition of velocities in this case is reduced to a problem in particle kinematics (§ 90).

§ 94. Composition of Rotations About Two Parallel Axes

Consider the case of a relative rotational motion of a body with an angular velocity ω_1 about a shaft aa' mounted on a crank ba (Fig. 225*a*), and a rotational motion of transport of the crank ba round axis bb' with an angular velocity ω_2. If axes aa' and bb' are parallel, the body will be in plane motion perpendicular to the axes. Let us investigate the cases of rotations.

(1) Rotations of Same Sense. The figure S in Fig. 225b is a cross section of a body perpendicular to the axes of rotation. Let us denote the points of intersection of the axes of rotation with the section by the letters A and B. It will be readily noticed that, as A lies on the axis Aa', its velocity is due only to the rotation about axis Bb', whence $v_A = \omega_2 \cdot AB$. Similarly, $v_B = \omega_1 \cdot AB$. Vectors $\boldsymbol{v}_A$ and $\boldsymbol{v}_B$

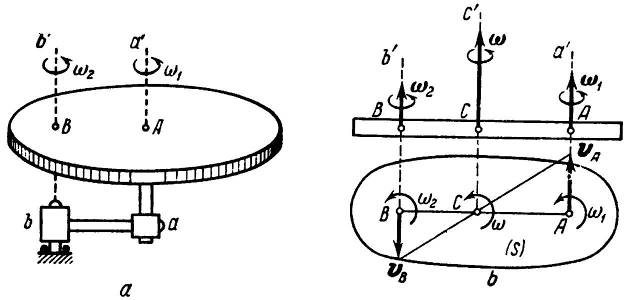

Fig. 225

are, of course parallel, both being perpendicular to AB, and of opposite sense. Therefore, point C (see § 81, Fig. 180b) is the instantaneous centre of zero velocity ($v_C = 0$), and, consequently, axis Cc', which is parallel to both Aa' and Bb', is the *instantaneous axis of rotation* of the body.

The angular velocity ω of the absolute rotation of the body about axis Cc' and the position of the axis, i.e., of point C, can be determined from Eq. (55) of § 81:

$$\omega = \frac{v_B}{BC} = \frac{v_A}{AC}.$$

By virtue of the properties of proportions, we obtain:

$$\omega = \frac{v_A + v_B}{AC + BC} = \frac{v_A + v_B}{AB}.$$

Substituting $v_A = \omega_2 \cdot AB$ and $v_B = \omega_1 \cdot AB$ into the last two equations, we finally obtain:

$$\omega = \omega_1 + \omega_2, \tag{87}$$

$$\frac{\omega_1}{BC} = \frac{\omega_2}{AC} = \frac{\omega}{AB}. \tag{88}$$

Thus, *if a body participates simultaneously in two rotations of same sense about parallel axes, the resultant motion will be an instantaneous rotation with an angular velocity* $\omega = \omega_1 + \omega_2$ *about an instanta-*

neous axis parallel to the two given axes; the position of this axis is determined by the ratio (88).

The instantaneous axis of rotation Cc' moves with time, describing a cylindrical surface.

(2) Rotations of Opposite Sense. Again draw a section S of the body under consideration (Fig. 226) and let us assume that $\omega_1 > \omega_2$. Then, reasoning in the same way as above, we find that the magnitudes of the velocities of points A and B are, respectively, $v_A = \omega_2 \cdot AB$

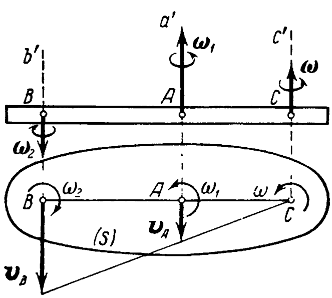

Fig. 226

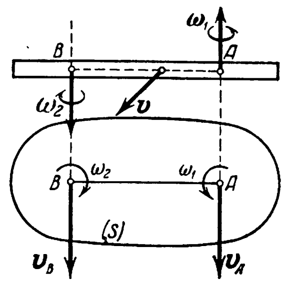

Fig. 227

and $v_B = \omega_1 \cdot AB$, the two velocities being parallel and of same sense. The instantaneous axis of rotation passes through point C (Fig. 226) and

$$\omega = \frac{v_B}{BC} = \frac{v_A}{AC} = \frac{v_B - v_A}{BC - AC},$$

or

$$\omega = \frac{v_B - v_A}{AB}.$$

Substituting the values of v_A and v_B in these equations, we obtain finally:

$$\omega = \omega_1 - \omega_2, \tag{89}$$

$$\frac{\omega_1}{BC} = \frac{\omega_2}{AC} = \frac{\omega}{AB}. \tag{90}$$

Thus, in this case, too, the resultant motion is an instantaneous rotation with absolute angular velocity $\omega = \omega_1 - \omega_2$ about the axis Cc', the position of which is specified by the ratio (90).

These results show that in rotations about parallel axes the vectors of the angular velocities are compounded like the vectors of parallel forces (§ 17).

(3) Rotation Couple. Consider the special case of rotations of opposite sense about parallel axes (Fig. 227) in which $\omega_1 = \omega_2$. Such a combination of rotations is called a *rotation couple*, and vectors ω_1 and ω_2 make an *angular velocity couple*. In this case we have

$v_A = v_B = \omega_1 \cdot AB$. Then (see § 81, Fig. 180*a*) the instantaneous centre of zero velocity lies at infinity and all points of the body have the same velocity $v = \omega_1 \cdot AB$.

Thus, the resultant motion of the body will be a *translation* (or an *instantaneous translation*) with a velocity $\boldsymbol{v}$ equal in magnitude to $\omega_1 \cdot AB$ and normal to the plane through vectors $\boldsymbol{\omega}_1$ and $\boldsymbol{\omega}_2$; the direction of vector $\boldsymbol{v}$ is determined in the same way as the direction of the moment $\boldsymbol{m}$ of a force couple in statics (§ 45). Thus, *a rotation couple is equivalent to a translation (or an instantaneous translation) with a velocity* $\boldsymbol{v}$ *equal to the moment of the corresponding angular velocity couple.*

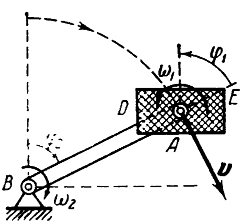

Fig. 228

An example of such motion is the translatory displacement of a bicycle pedal DE with respect to the bicycle frame (Fig. 228) resulting from the relative rotation of the pedal about axle A mounted on the crank BA and the rotational motion of transport of the crank about axle B. The angular velocities $\boldsymbol{\omega}_1$ and $\boldsymbol{\omega}_2$ of these rotations are equal in magnitude, as at any instant the angle φ_1 of rotation of the pedal relative to the crank is equal to the angle φ_2 of rotation of the crank. The velocity of the pedal's translational motion will be $v = \omega_2 \cdot AB$.

§ 95*. Toothed Spur Gearing

The results obtained in the preceding section can be used for the kinematic calculation of spur gearing transmissions. Let us consider the main types of such transmissions.

(1) A *common gear* train is one in which all the gears, which are meshed successively, rotate on fixed axes. One of the gears (e.g., gear *1* in Fig. 229) is the driving gear, or driver, and the others are the driven gears or followers. In the case of external (Fig. 229*a*) or internal (Fig. 229*b*) gearing with two wheels we have $|\omega_1| r_1 = |\omega_2| r_2$, as the velocity of the point of engagement A is the same for both gears. Since the number of teeth z of the gears is proportional to their radii, and taking into account that the rotation of the gears is in the same direction in the case of internal engagement and in opposite directions in the case of external engagement, we have:*)

$$\left(\frac{\omega_1}{\omega_2}\right)_{\text{ext}} = -\frac{r_2}{r_1} = -\frac{z_2}{z_1}, \quad \left(\frac{\omega_1}{\omega_2}\right)_{\text{int}} = \frac{r_2}{r_1} = \frac{z_2}{z_1}.$$

*) In all the formulas of this section, ω is the algebraic (numerical) value of the angular velocity, "plus" indicating counterclockwise rotation and "minus" indicating clockwise rotation.

For a train of three external gears (Fig. 229c) we have

$$\frac{\omega_1}{\omega_2} = -\frac{r_2}{r_1}, \quad \frac{\omega_2}{\omega_3} = -\frac{r_3}{r_2},$$

whence

$$\frac{\omega_1}{\omega_3} = \frac{r_3}{r_1} = \frac{z_3}{z_1}.$$

Consequently, the ratio of the angular velocities of the end gears in a straight drive is inversely proportional to their radii (the number

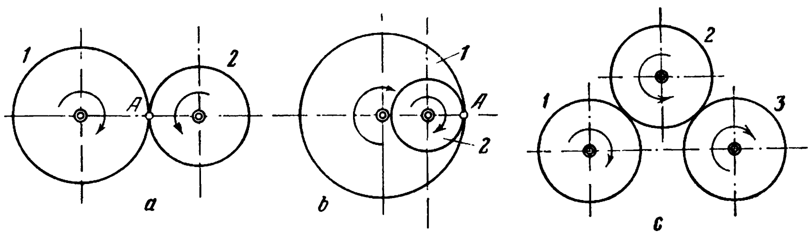

Fig. 229

of teeth) and does not depend on the radii of intermediate gears (idlers).

From the results obtained it follows that in a straight drive consisting of n gears

$$\frac{\omega_1}{\omega_n} = (-1)^k \frac{r_n}{r_1} = (-1)^k \frac{z_n}{z_1}, \tag{91}$$

where k is the number of *external* engagements (in the case of Fig. 229a there is one external engagement, in Fig. 229c there are two external engagements, in Fig. 229b there are no external engagements).

The *gear ratio* of a toothed drive is the quantity i_{1n} which gives the ratio of the angular velocity of the driver to that of the follower:

$$i_{1n} = \frac{\omega_1}{\omega_n}. \tag{92}$$

For a straight drive the value of i_{1n} is given by the right side of Eq. (91).

(2) An epicyclic, or *planetary gear train* (see Fig. 230), is one in which gear *1* is fixed while the other successively engaged gears are connected by a link AB, called an arm, which rotates about the axis of the fixed gear.

(3) A *differential gear train* is again of the type in Fig. 230, with the difference that gear *1* also rotates about its axis independently of the arm AB.

Epicyclic and differential gear trains can be calculated by mentally giving the fixed plane Ax_1y_1 a rotation with an angular velocity

$-\omega_{AB}$ equal in magnitude and opposite in sense to the angular velocity of the arm AB.

Then, by the results of § 94, the arm will be fixed in this resultant motion and any gear of radius r_k will have an angular velocity $\tilde{\omega}_k = \omega_k - \omega_{AB}$, where ω_k is the absolute angular velocity of this gear relative to the axes Ax_1y_1. The axes of all the gears will be fixed and the dependence between $\tilde{\omega}_k$ (for different k's) can be found either

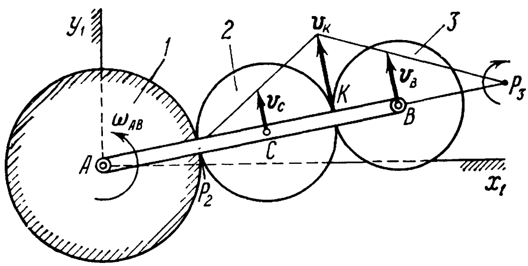

Fig. 230

by equating the velocities of the points of engagement or directly from Eq. (91).

Epicyclic and differential gear trains can also be calculated with the help of instantaneous centres of zero velocity (§ 81).

Problem 90. In the epicyclic gear train in Fig. 230, gear *1* of radius r_1 is fixed and arm AB rotates with an angular velocity ω_{AB}. Determine the angular velocity of gear *3* of radius r_3.

Solution. Let us denote the absolute angular velocities of the gears with respect to the axes Ax_1y_1 as ω_1 ($\omega_1 = 0$), ω_2, and ω_3. Rotating the whole plane Ax_1y_1 with an angular velocity $-\omega_{AB}$, we obtain:

$$\tilde{\omega}_1 = 0 - \omega_{AB}, \quad \tilde{\omega}_2 = \omega_2 - \omega_{AB},$$
$$\tilde{\omega}_3 = \omega_3 - \omega_{AB}, \quad \tilde{\omega}_{AB} = 0.$$

The number of external engagements in the resulting common gear train is $k = 2$. Then, from Eq. (91),

$$\frac{\tilde{\omega}_1}{\tilde{\omega}_3} = \frac{r_3}{r_1} \quad \text{or} \quad \frac{-\omega_{AB}}{\omega_3 - \omega_{AB}} = \frac{r_3}{r_1}.$$

From this we find the absolute angular velocity of gear *3*:

$$\omega_3 = \left(1 - \frac{r_1}{r_3}\right)\omega_{AB}.$$

If $r_3 > r_1$, the direction of rotation of gear *3* is the same as that of the arm; if $r_3 < r_1$, it is opposite. If $r_1 = r_3$, we obtain $\omega_3 = 0$, and gear *3* is in translatory motion.

The relative angular velocity of gear *3* (with respect to arm AB) is found from Eq. (87). As the absolute velocity $\omega_3 = \omega_{rel3} + \omega_{AB}$ (the angular velocity ω_{AB} of the arm is the transport velocity for gear *3*), then

$$\omega_{rel3} = \omega_3 - \omega_{AB} = -\frac{r_3}{r_1}\omega_{AB}.$$

At $r_3 = r_1$, we obtain $\omega_{rel3} = -\omega_{AB}$. The relative ω_{rel3} and transport ω_{AB} angular velocities form a couple, which in another way leads us to the same conclusion that the resultant motion of gear *3* in this case is a translation with a velocity $v = \omega_{AB} \cdot AB$.

*Alternative solution**). The velocity of point B of gear *3* is $v_B = (r_1 + 2r_2 + r_3)\,\omega_{AB}$. We shall determine the velocity of point K of the gear, which is the point of engagement of gears *2* and *3*. The velocity of gear *2* is $v_C = (r_1 + r_2)\,\omega_{AB}$. The instantaneous centre of zero velocity of the gear is at point P_2, where it coincides with gear *1*. Consequently, $v_K = 2v_C = 2(r_1 + r_2)\,\omega_{AB}$.

Then, by Eq. (56) (§ 81),

$$\omega_3 = \frac{|v_B - v_K|}{BK} = \frac{r_3 - r_1}{r_3}\omega_{AB}.$$

The same result can be obtained by constructing the centre of zero velocity P_3 of gear *3*.

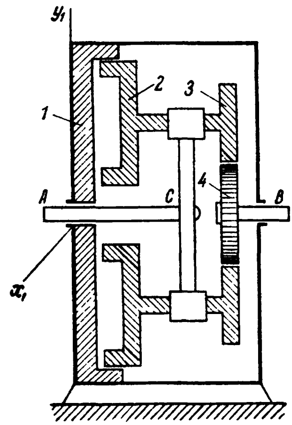

Fig. 231

Problem 91. The reduction gear in Fig. 231 consists of (a) a fixed gear *1*, (b) two pairs of twin pinions *2* and *3* mounted on a link connected with the driving shaft AC (the engagement of pinions *2* and *1* with gear *1* is internal), (c) a gear *4* mounted on the driven shaft B. The number of teeth in the gears are: $z_1 = 120$, $z_2 = 40$, $z_3 = 30$, $z_4 = 50$. The driving shaft makes $n_A = 1\,500$ rpm. Determine the rpm of the driven shaft B.

Solution. Let us denote the absolute angular velocities: of shaft AC together with the link as ω_A; of gear *4* together with shaft B as ω_B; of the pinions *2* and *3* as ω_{23} (these gears rotate as a single body). Gear *1* has an angular velocity $\omega_1 = 0$. Rotating plane x_1y_1, parallel to which the mechanism moves, with an angular velocity $-\omega_A$, we obtain that the link in this imaginary motion is fixed ($\tilde{\omega}_A = 0$) and the velocities of the gears are

$$\tilde{\omega}_1 = 0 - \omega_A, \quad \tilde{\omega}_{23} = \omega_{23} - \omega_A, \quad \tilde{\omega}_4 = \omega_B - \omega_A.$$

*) An alternative solution is given to show the possibility of employing the methods of § 81. Problems 91 and 92 could also have been solved this way, but usually this solution is more involved (especially for Problem 92).

Writing now Eg. (91) for gears *1* and *2*, and *3* and *4*, we obtain:

$$\frac{\tilde{\omega}_1}{\tilde{\omega}_{23}} = \frac{z_2}{z_1}, \quad \frac{\tilde{\omega}_{23}}{\tilde{\omega}_4} = -\frac{z_4}{z_3}.$$

Multiplying the two equations, we have:

$$\frac{\tilde{\omega}_1}{\tilde{\omega}_4} = -\frac{z_2 z_4}{z_1 z_3}, \quad \text{or} \quad \frac{-\omega_A}{\omega_B - \omega_A} = -\frac{z_2 z_4}{z_1 z_3},$$

whence, taking into account that the number of rpm is proportional to ω, we find:

$$n_B = \left(1 + \frac{z_1 z_3}{z_2 z_4}\right) n_A = 4\,200 \text{ rpm}.$$

Problem 92. Solve the last problem, assuming that gear *1* rotates at $n_1 = 1\,100$ rpm in the same direction as shaft *AC* (a differential reduction gear).

Solution. The sequence of the solution is the same as in Problem 91, with the sole difference that now $\omega_1 \neq 0$ and ω_1 and ω_A are of same sense. Consequently, $\tilde{\omega}_1 = \omega_1 - \omega_A$.

As a result, the ratio

$$\frac{\tilde{\omega}_1}{\tilde{\omega}_4} = -\frac{z_2 z_4}{z_1 z_3},$$

obtained in Problem 91, gives

$$\frac{\omega_1 - \omega_A}{\omega_B - \omega_A} = -\frac{z_2 z_4}{z_1 z_3},$$

whence, passing to rpm, we find

$$n_B = n_A + \frac{z_1 z_3}{z_2 z_4}(n_A - n_1) = 2\,220 \text{ rpm}.$$

If gear *1* rotates in the opposite direction to that of shaft *AC*, the sign of n_1 in the result obtained should be changed.

§ 96*. Composition of Rotations About Intersecting Axes

Let the relative motion of body *1* in Fig. 232*a* be a rotation with angular velocity ω_1 about an axis a_1a mounted on a crankshaft *2*, and let the motion of transport be the rotation of the crankshaft with angular velocity ω_2 about axis b_1b, which intersects with axis a_1a at *O*. Schematically such a case of composition of rotations about intersecting axes is shown in Fig. 232*b*.

It is obvious that in this case the velocity of point *O*, as lying simultaneously on both axes, is zero. Consequently, the resultant motion of the body is a motion about the fixed point O, which for every infinitesimal time interval represents an infinitesimal turn with an angular velocity $\boldsymbol{\omega}$ about an instantaneous axis through point O (§ 86).

To determine vector $\boldsymbol{\omega}$, calculate the velocity of any point M of the body whose radius vector is $\overline{OM} = \boldsymbol{r}$. In the relative motion about axis Oa, point M, by Eq. (67), has a velocity $\boldsymbol{v}_{rel} = \boldsymbol{\omega}_1 \times \boldsymbol{r}$;

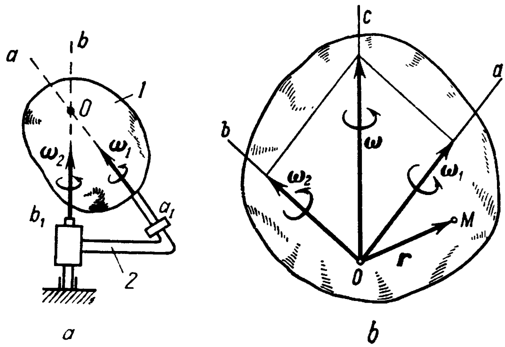

Fig. 232

in the motion of transport about axis Ob the velocity of the point is $\boldsymbol{v}_{tr} = \boldsymbol{\omega}_2 \times \boldsymbol{r}$. Hence, the absolute velocity of M is

$$\boldsymbol{v}_a = \boldsymbol{v}_{rel} + \boldsymbol{v}_{tr} = (\boldsymbol{\omega}_1 + \boldsymbol{\omega}_2) \times \boldsymbol{r}.$$

But as the resultant motion of the body is an instantaneous rotation with an angular velocity $\boldsymbol{\omega}$, we must have:

$$\boldsymbol{v}_a = \boldsymbol{\omega} \times \boldsymbol{r}.$$

Similar results can be obtained for every point of the body (i.e., for any value of $\boldsymbol{r}$). We conclude, therefore, that

$$\boldsymbol{\omega} = \boldsymbol{\omega}_1 + \boldsymbol{\omega}_2. \tag{93}$$

Thus, *the resultant of rotational motions about two axes intersecting at any point O is an instantaneous rotation about an axis Oc through O the angular velocity $\boldsymbol{\omega}$ of which is the geometrical sum of the relative angular velocity and the transport angular velocity*. The instantaneous axis of rotation Oc is collinear with vector $\boldsymbol{\omega}$, i.e., it is directed along the diagonal of a parallelogram with vectors $\boldsymbol{\omega}_1$ and $\boldsymbol{\omega}_2$ as its sides.

Axis Oc changes its position with time, describing a conic surface with its apex at O.

If a body takes part simultaneously in instantaneous rotations about several axes intersecting at O, by consecutively applying Eq. (93) we arrive at the conclusion that the resultant motion is an instantaneous rotation about an axis through O, and the angular velocity of that motion is

$$\omega = \omega_1 + \omega_2 + \ldots + \omega_n. \qquad (94)$$

Problem 93. Determine the absolute angular velocity of the roller in Fig. 233 (see Problem 80, § 87), if its radius $AC = R$, $OA = l$, and the velocity of point A is v_A.

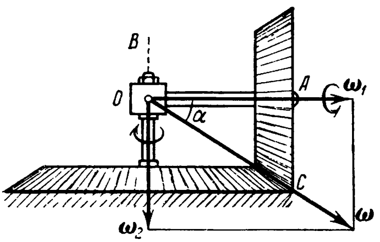

Fig. 233

Solution. The absolute motion of the roller is the result of its relative rotation about axis OA with an angular velocity ω_1 and the transport rotation of crank OA about axis OB with an angular velocity ω_2, where

$$\omega_2 = \frac{v_A}{l}.$$

The instantaneous axis of rotation and, consequently, the vector of the absolute angular velocity ω are directed along OC, as the velocity of point C is zero (see Problem 80). Constructing a parallelogram, we find that $\omega = \omega_2/\sin\alpha$.

As $\sin\alpha = \frac{R}{\sqrt{l^2+R^2}}$ we obtain finally:

$$\omega = \frac{v_A}{R}\sqrt{1+\frac{R^2}{l^2}}.$$

The same result can be obtained (taking into account that OC is the instantaneous axis of rotation) from the equation $v_A = \omega h$, where $h = l\sin\alpha$.

The motion of the roller represents a succession of infinitesimal turns with an angular velocity ω round axis OC which continuously changes its position, describing a circular cone with its apex at O.

§ 97*. Euler Kinematic Equations

Let us use the results of the previous section to determine the angular velocity ω of a body rotating about a fixed axis (§ 86) if the motion is given by Eqs. (65).

Draw the fixed axes $Ox_1y_1z_1$ and axes $Oxyz$ rigidly connected with the body and moving with it (Fig. 234) and indicate the Euler angles φ, ψ, and θ. As the angles change with time, the body performs rotations about axes Oz, Oz_1 and OK (see § 86); denote the angular velocities of these rotations, respectively, by ω_1, ω_2, and ω_3 and represent them as vectors directed along the respective axes. In magnitude,

$$\omega_1 = \frac{d\varphi}{dt} = \dot{\varphi}, \quad \omega_2 = \frac{d\psi}{dt} = \dot{\psi}, \quad \omega_3 = \frac{d\theta}{dt} = \dot{\theta}. \tag{95}$$

By Eq. (94) the body's angular velocity at the given instant is

$$\boldsymbol{\omega} = \boldsymbol{\omega}_1 + \boldsymbol{\omega}_2 + \boldsymbol{\omega}_3. \tag{96}$$

To determine vector $\boldsymbol{\omega}$, find its projections on the moving axes *Oxyz*. From Eq. (96) we have:

$$\left.\begin{aligned} \omega_x &= \omega_{1x} + \omega_{2x} + \omega_{3x}, \\ \omega_y &= \omega_{1y} + \omega_{2y} + \omega_{3y}, \\ \omega_z &= \omega_{1z} + \omega_{2z} + \omega_{3z}. \end{aligned}\right\} \tag{97}$$

The projections of vectors $\boldsymbol{\omega}_1$ and $\boldsymbol{\omega}_3$ are determined at once [see Fig. 234 and the notations (95)]:

$$\omega_{1x} = \omega_{1y} = 0, \ \omega_{1z} = \dot{\varphi}, \ \omega_{3x} = \dot{\theta}\cos\varphi,$$
$$\omega_{3y} = -\dot{\theta}\sin\varphi, \quad \omega_{3z} = 0.$$

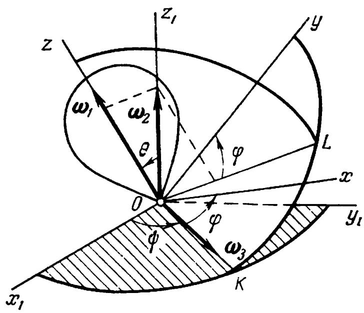

Fig. 234

To determine the projections of $\boldsymbol{\omega}_2$, draw through axes Oz_1 and Oz a plane intersecting plane *Oxy* along *OL*. As *OK* is perpendicular to the plane zOz_1, it is also perpendicular to the line *OL* ($\angle\, KOL = 90°$, and $\angle\, LOy = \varphi$). Then, projecting vector $\boldsymbol{\omega}_2$ on *OL* and in turn projecting that projection on axes *Ox* and *Oy*, we obtain [see Eqs. (5) in § 8]:

$$\omega_{2x} = \dot{\psi}\sin\theta\sin\varphi, \quad \omega_{2y} = \dot{\psi}\sin\theta\cos\varphi, \quad \omega_{2z} = \dot{\psi}\cos\theta.$$

Substituting the projections into the right-hand sides of Eqs. (97), we finally obtain:

$$\left.\begin{aligned} \omega_x &= \dot{\psi}\sin\theta\sin\varphi + \dot{\theta}\cos\varphi, \\ \omega_y &= \dot{\psi}\sin\theta\cos\varphi - \dot{\theta}\sin\varphi, \\ \omega_z &= \dot{\varphi} + \dot{\psi}\cos\theta. \end{aligned}\right\} \tag{98}$$

Eqs. (98) are called the *Euler kinematic equations* and they give the projections of the body's angular velocity on the moving axes *Oxyz* in terms of the Euler angles. It is readily apparent from Eqs. (98) that knowing Eqs. (65) we can determine ω_x, ω_y, and ω_z, i.e., vector $\boldsymbol{\omega}$, as was indicated in § 86.

§ 98*. Composition of a Translation and a Rotation. Screw Motion

Let us examine the resultant motion of a rigid body having a translation and a rotation. An example is presented in Fig. 235, where the relative motion of body *1* is a rotation with angular velocity about axis Aa mounted on the platform *2*, and the translational motion of transport is that of the platform with velocity $\boldsymbol{v}$. Also taking part simultaneously in two such motions is the wheel *3* for which the relative motion is the rotation about its axis and the motion of transport is that of the platform. Depending on the size of the angle between vectors $\boldsymbol{\omega}$ and $\boldsymbol{v}$ (for the wheel it is 90°), three cases are possible.

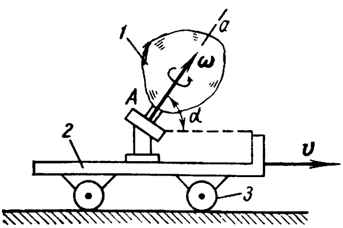

Fig. 235

(1) The Velocity of the Translation is Perpendicular to the Axis of Rotation ($\boldsymbol{v} \perp \boldsymbol{\omega}$). Let the resultant motion of the body consist of a rotation about an axis Aa with an angular velocity $\boldsymbol{\omega}$ and a translation with a velocity $\boldsymbol{v}$ perpendicular to $\boldsymbol{\omega}$ (Fig. 236). It is evident

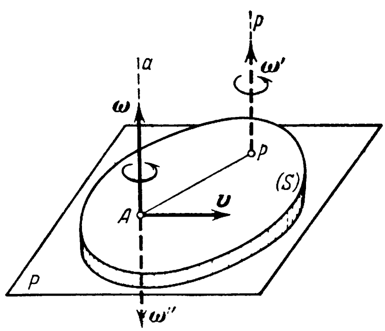

Fig. 236

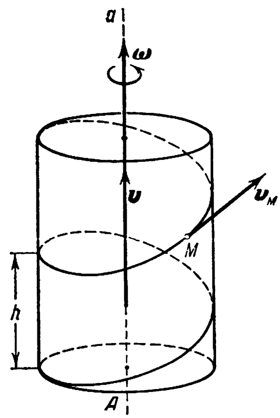

Fig. 237

that with respect to plane P, which is perpendicular to Aa, this is plane motion, which is discussed in detail in Chapter 12. If point A is taken as the pole, the motion, like any plane motion, will actually consist of a translation with a velocity $\boldsymbol{v}_A = \boldsymbol{v}$, i.e., the velocity of the pole, and a rotation round the axis Aa through the pole.

Vector $\boldsymbol{v}$ can be replaced by a couple of angular velocities $\boldsymbol{\omega}'$, $\boldsymbol{\omega}''$ (§ 94), where $\boldsymbol{\omega}' = \boldsymbol{\omega}$, and $\boldsymbol{\omega}'' = -\boldsymbol{\omega}$. The distance AP can be found from the equation $v = \omega' \cdot AP$, whence (taking into account that $\omega' = \omega$)

$$AP = \frac{v}{\omega}. \tag{99}$$

The resultant of vectors $\boldsymbol{\omega}$ and $\boldsymbol{\omega}''$ is zero, and we find that the motion of the body can be treated as an instantaneous rotation about axis Pp with an angular velocity $\boldsymbol{\omega}' = \boldsymbol{\omega}$. This result was obtained earlier in another way (§ 81). Comparing Eqs. (53) and (99), we will readily notice that point P is the instantaneous centre of zero velocity ($v_P = 0$) for section (S) of the body. Thus, we see once again that the rotation of the body about Aa and Pp is with the same angular velocity $\boldsymbol{\omega}$, i.e., the rotational component of a motion does not depend on the choice of the pole (§ 77).

(2) Screw Motion ($\boldsymbol{v} \parallel \boldsymbol{\omega}$). If the resultant motion of a body consists of a rotation about an axis Aa with an angular velocity $\boldsymbol{\omega}$ and a translation with a velocity $\boldsymbol{v}$, directed parallel to Aa (Fig. 237), it is called *screw motion*. Axis Aa is called the *axis of the screw*. When vectors $\boldsymbol{v}$ and $\boldsymbol{\omega}$ are in the same direction, then, according to the rule for the notation of $\boldsymbol{\omega}$, the screw is right-handed; if they are of opposite sense, the screw is left-handed.

The translatory displacement of any point of the body lying on the axis of the screw during one complete turn is called the *pitch h* of the screw. If v and ω are constant, the pitch of the screw is said to be *constant*. Denoting the time of one complete turn as T, we obtain in this case $vT = h$, and $\omega T = 2\pi$, whence

$$h = 2\pi \frac{v}{\omega}.$$

Any point M of a screw with a constant pitch not lying on the axis of the screw describes a screw line. The velocity of such a point M lying at a distance r from the axis of the screw is compounded of the translatory velocity $\boldsymbol{v}$ and the rotational velocity ωr perpendicular to it. Consequently,

$$v_M = \sqrt{v^2 + \omega^2 r^2}.$$

Velocity $\boldsymbol{v}_M$ is tangent to the screw line. If the cylindrical surface along which point M moves is cut along its generator and spread out, the screw lines will, evidently, turn into straight lines inclined at an angle α to the base of the cylinder such that $\tan\alpha = h/2\pi r$.

(3) The Velocity of Translation Makes an Arbitrary Angle With the Axis of Rotation. In this case the resultant motion of the body (Fig. 238*a*) is the same as discussed in Sec. 88 (the most general motion of a free rigid body).

Let us resolve vector $\boldsymbol{v}$ (Fig. 238*b*) into its components: (1) $\boldsymbol{v}'$ collinear with $\boldsymbol{\omega}$ ($v' = v\cos\alpha$) and (2) $\boldsymbol{v}''$ perpendicular to $\boldsymbol{\omega}$ ($v'' = v\sin\alpha$). Velocity $\boldsymbol{v}''$ can be replaced by an angular velocity couple such that $\boldsymbol{\omega}' = \boldsymbol{\omega}$ and $\boldsymbol{\omega}'' = -\boldsymbol{\omega}$ (as in Fig. 236), and vectors $\boldsymbol{\omega}$ and $\boldsymbol{\omega}''$ can be discarded. The distance AC is found from formula (99):

$$AC = \frac{v''}{\omega} = \frac{v\sin\alpha}{\omega}.$$

The body is thus left with a rotation with an angular velocity $\boldsymbol{\omega}'$ and a translation with a velocity $\boldsymbol{v}'$. Thus, the distribution of the

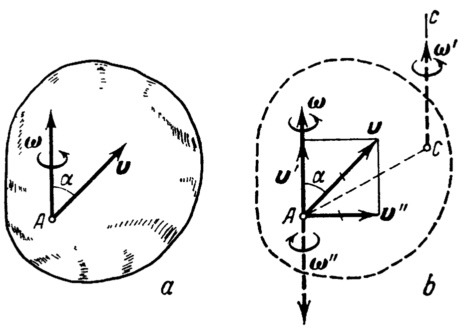

Fig. 238

velocities of all the points of the body at any given instant is the same as in a screw motion about axis *Cc* with an angular velocity $\omega' = \omega$ and a translatory velocity $v' = v\cos\alpha$.

As a result of these operations (Fig. 238*b*), we have passed from pole *A* to pole *C*, which demonstrates that even in the most general motion of a body the angular velocity does not depend on the choice of the pole ($\boldsymbol{\omega}' = \boldsymbol{\omega}$), and only the translatory velocity is affected ($\boldsymbol{v}' \neq \boldsymbol{v}$).

As in the motion of free rigid bodies $\boldsymbol{v}$, $\boldsymbol{\omega}$, α in general change continuously, so will the position of axis *Cc*, which is therefore called the *instantaneous screw axis*. Thus, the motion of a free rigid body can also be treated as the resultant of a series of instantaneous screw motions about continuously changing screw axes.

Part 3
PARTICLE DYNAMICS

Chapter 16
Introduction to Dynamics. Laws of Dynamics

§ 99. Basic Concepts and Definitions

Dynamics is the section of mechanics which treats of the laws of motion of material bodies subjected to the action of forces.

The motion of bodies from a purely geometrical point of view was discussed in kinematics. Unlike kinematics, in dynamics the motion of bodies is investigated in connection with the acting forces and the inertia of the material bodies themselves.

The concept of force as a quantity characterising the measure of mechanical interaction of material bodies was introduced in the course of statics. But in statics we treated all forces as constant, without considering the possibility of their changing with time. In real systems, alongside of constant forces (gravity can generally be regarded as an example of a constant force), a body is often subjected to the action of variable forces whose magnitudes and directions change when the body moves. Variable forces may be the applied (active) forces or the reactions of constraints.

Experience shows that variable forces may depend in some specific ways on *time*, on the *position* of a body, and on its *velocity* (examples of dependence on *time* are furnished by the tractive force of an electric locomotive whose rheostat is gradually switched on or off or the force causing the vibration of the foundation of a motor with a poorly centred shaft; the Newtonian force of gravitation or the elastic force of a spring depending on the *position* of a body; the resistance experienced by a body moving through air or water depends on the *velocity*. In dynamics we shall deal with such forces alongside of constant forces. The laws for the composition and resolution of variable forces are the same as for constant forces.

The concept of inertia of bodies arises when we compare the results of the action of an identical force on different material bodies. Experience shows that if the same force is applied to two different bodies initially at rest and free from any other actions, in the most general case the bodies will travel different distances and acquire different velocities in the same interval of time.

Inertia is the property of material bodies to resist a change in their velocity under the action of applied forces. If, for example, the velocity of one body changes slower than that of another body subjected to the same force, the former is said to have greater inertia, and vice versa.

*The quantitative measure of the inertia of a body is a physical quantity called the mass of that body.**) In mechanics mass m is treated as a quantity which is positive and constant for every body. The units of mass will be discussed in § 101.

In the most general case the motion of a body depends not only on its total mass and the applied forces; the nature of motion may also depend on the shape of the body or, more precisely, on the mutual position of its particles (i.e., on the distribution of mass).

In the initial course of dynamics, so as to neglect the influence of the shape (distribution of the mass) of a body, the concept of a *material point*, or *particle* is introduced.

A particle is a material body (a body possessing mass) the size of which can be neglected in investigating its motion.

Actually any body can be treated as a particle when the distances travelled by its points are very great as compared with the size of the body itself. For example, in studying the motion of a planet about the sun or determining the range of an artillery shell, etc., the planet or shell can be treated as particles. Furthermore, as will be shown in the dynamics of systems, a body in *translational* motion can always be considered as a particle of mass equal to the mass of the whole body. Finally, the parts into which we shall mentally divide bodies in analysing any of their dynamical characteristics can also be treated as material points.

Obviously, the investigation of the motion of a single particle should precede the investigation of systems of particles, and in particular of rigid bodies. Accordingly, the course of dynamics is conventionally subdivided into particle dynamics and the dynamics of systems of particles.

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*) Mass is also a measure of a body's gravitational properties as, according to the law of universal gravitation, two bodies attract each other with a force varying directly as the product of their masses and inversely as the square of the distance between them.

§ 100. The Laws of Dynamics

The study of dynamics is based on a number of laws generalising the results of a wide range of experiments and observations of the motions of bodies—laws which have been verified in the long course of human history. These laws were first systematised and formulated by Isaac Newton in his classical work *Principia Mathematica* published in 1687.

The **First Law** (the Inertia Law), discovered by Galileo in 1638, states: *A particle free from any external influences continues its state of rest or uniform rectilinear motion, except and so far as it is compelled to change that state by impressed forces.* The motion of a body not subjected to any force is called motion under no forces, or *inertial motion.*

The inertia law states one of the basic properties of matter: that of being always in motion. It establishes the equivalence, for material bodies, of the states of rest and of motion under no forces. It follows, then, that if $\boldsymbol{F} = 0$, a particle is at rest or moves with a velocity of constant magnitude and direction ($\boldsymbol{v} =$ const.); the acceleration of the particle is, evidently, zero ($\boldsymbol{w} = 0$); if the motion of a particle is not uniform and rectilinear, there must be some force acting on it.

A frame of reference for which the inertia law is valid is called an *inertial frame* (or, conventionally, a fixed frame). Experience shows that, for our solar system, an inertial frame of reference has its origin in the centre of the sun and its axes pointed towards the so-called "fixed" stars. In solving most engineering problems a sufficient degree of accuracy is obtained by assuming any frame of reference connected with the earth to be an inertial system. The validity of this assumption will be proved in Chapter 20.

The **Second Law** (the Fundamental Law of Dynamics) establishes the mode in which the velocity of a particle changes under the action of a force: *The product of the mass of a particle and the acceleration imparted to it by a force is proportional to the acting force; the acceleration takes place in the direction of the force.*

Mathematically this law is expressed by the vector equation

$$m\boldsymbol{w} = \boldsymbol{F}. \tag{1}$$

The dependence between the magnitudes of the acceleration and the force is

$$mw = F. \tag{2}$$

The second law of dynamics, like the first, is valid only for an inertial frame. It can be immediately seen from the law that the measure of the inertia of a particle is its mass, since two different particles subjected to the action of the same force receive the same

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acceleration only if their masses are equal; if their masses are different, the particle with the larger mass (i.e., the more inert one) will receive a smaller acceleration, and vice versa.

A set of forces acting on a particle can, as we know, be replaced by a single resultant $\boldsymbol{R}$ equal to the geometrical sum of those forces. In this case the equation expressing the fundamental law of dynamics acquires the form

$$m\boldsymbol{w} = \boldsymbol{R} \text{ or } m\boldsymbol{w} = \sum \boldsymbol{F}_k. \tag{3}$$

This result can also be obtained by applying, instead of the parallelogram principle, the *law of independent action of forces*, according to which each of a number of forces acting on a particle imparts to it the same acceleration as it would have imparted if acting alone.

Weight and Mass. All bodies close to the surface of the earth are subject to the force of gravity $\boldsymbol{P}$, equal in magnitude to a body's weight. It has been established experimentally that under the action of force $\boldsymbol{P}$ all bodies falling to the earth (from a small height and in vacuo) possess the same acceleration g; this is known as the *acceleration of gravity* or *of free fall**). Applying Eq. (2), for free fall we have

$$P = mg \quad \text{or} \quad m = \frac{P}{g}. \tag{4}$$

Eq. (4) gives the body's mass in terms of its weight, and vice versa; it establishes that *a body's weight is the product of its mass and acceleration of gravity, and its mass is the quotient of its weight divided by the acceleration of gravity*. Weight, like the acceleration of gravity g, changes with latitude and altitude; mass is a constant for every given body (or particle).

The **Third Law** (the Law of Action and Reaction) establishes the character of mechanical interaction between material bodies. For two particles it states: *Two particles exert on each other forces equal in magnitude and acting in opposite directions along the straight line connecting the two particles.*

It should be noted that the forces of interaction between free particles (or bodies) do not form a balanced system, as they act on different objects. For example, if a piece of iron and a magnet are placed near each other on a smooth surface, they will move towards each other under the influence of their mutual attraction and not remain at rest. Since the magnitude of the force acting on each body is the same, it follows from the second law of dynamics that the accelerations of the two bodies will be inversely proportional to their masses.

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*) The law of free fall of bodies was discovered by Galileo. The value of g varies for different localities, depending on geographic latitude and elevation above sea level. At sea level on the latitude of Moscow $g = 9.8156$ m/s^2.

The third law of dynamics, which establishes the character of interaction of material particles, plays an important part in the dynamics of systems.

§ 101. Systems of Units

Three basic units are sufficient for the measurement of all mechanical quantities. Two of these are, conventionally, the units of length and time, already introduced in the course of kinematics (§ 58). The third most suitable unit is either mass or force. As, given Eq. (2), these units cannot both be arbitrary, two basically different systems of units are possible.

First Type of Unit Systems. The primary units in these systems are the unit of length, the unit of time and the unit of mass, with force measured by a derived unit.

To this type of system belongs the *International System of Units* (SI), in which the basic mechanical units are the *metre* (m), the *kilogram of mass* (kg), and the *second* (s). The unit of force is thus a derived unit, called the *newton* (N), and is the force that imparts to a mass of 1 kg an acceleration of 1 m/s² $\left(1\text{N} = 1\frac{\text{kg-m}}{\text{s}^2}\right)$.

Another similar system is the CGS, widely used in physics, in which the basic units are the *centimetre*, *gram of mass* and *second*, and force is a derived unit called the *dyne*.

Second Type of Unit Systems. The basic units in these systems are the unit of length, the unit of time, and the unit of force, mass being a derived unit.

To these systems belongs the mkg(f)s system, used extensively in engineering, in which the basic units are the *metre* (m), the *kilogram of force* (kgf) and the *second* (s). In this system the unit of mass is $1\,\frac{\text{kg-s}^2}{\text{m}}$, i.e., a mass to which a force of 1 kgf imparts an acceleration of 1 m/s².

The equivalence between the units of force in the SI and mkg(f)s systems is: 1 kgf $\approx$ 9.8 N, and 1 N $\approx$ 0.102 kgf.

The difference between the two types of systems is that in the one the mass unit is taken as a primary dynamical unit, while in the other force is a primary unit

§ 102. The Problems of Dynamics for a Free and a Constrained Particle

The problems of dynamics for a *free* particle are: (1) knowing the equation of motion of a particle, determine the force acting on it (*the first problem of dynamics*); (2) knowing the forces acting on a particle, determine its equation of motion (*the second*, or *principal*, *problem of dynamics*).

Both problems are solved with the help of Eq. (1) or (3), which express the fundamental law of dynamics, since they give the relation between acceleration $\boldsymbol{w}$, i.e., the quantity characterising the motion of a particle, and the forces acting on it.

In engineering it is often necessary to investigate the *constrained* motion of a particle, i.e., cases when constraints attached to a particle compel it to move along a given fixed surface or curve.

In such cases we shall use, as in statics, the axiom of constraints which states that *any constrained particle can be treated as a free body detached from its constraints provided the latter is represented by their reactions* $\boldsymbol{N}$. Then the *fundamental law of dynamics for the constrained motion of a particle* takes the form

$$m\boldsymbol{w} = \sum \boldsymbol{F}_k^a + \boldsymbol{N}, \tag{5}$$

where $\boldsymbol{F}_k^a$ denotes the applied forces acting on the particle.

For constrained motion, the first problem of dynamics will usually be: determine the reactions of the constraints acting on a particle if the motion and applied forces are known. The second (principal) problem of dynamics for such motion will pose two questions, namely, knowing the applied forces, to determine: (a) the equation of motion of the particle and (b) the reaction of its constraints.

§ 103. Solution of the First Problem of Dynamics (Determination of the Forces if the Motion is Known)

If the acceleration of a moving particle is given, the applied force or the reaction of the constraint can be immediately found from Eq. (1) or (5). To calculate the reaction it is also necessary to know the applied forces. When the acceleration is not specified but the equation of motion is known, it is necessary first to calculate the acceleration from the formulas of kinematics (see §§ 64, 67) and then to find the force (or reaction).

Problem 94. A balloon of weight P descends with an acceleration w. What weight (ballast) Q must be thrown overboard in order to give the balloon an equal upward acceleration?

Solution. The forces acting on the falling balloon are its weight $\boldsymbol{P}$ and buoyancy force $\boldsymbol{F}$ (Fig. 239*a*). Projecting Eq. (5) on the vertical, we obtain:

$$\frac{P}{g} w = P - F.$$

After the ballast has been thrown down (Fig. 239*b*), the weight of the balloon becomes $P-Q$, the buoyancy force remaining the same. Hence, taking into account that now the balloon is rising, we have:

$$\frac{P-Q}{g} w = F-(P-Q).$$

Eliminating the unknown force F from the equations, we obtain:

$$Q = \frac{2P}{1+\frac{g}{w}}.$$

Problem 95. An elevator of weight P (Fig. 240) starts ascending with an acceleration w. Determine the tension in the cable.

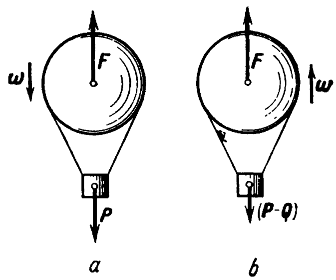

Fig. 239

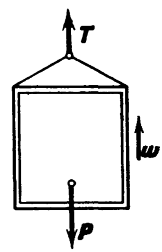

Fig. 240

Solution. Considering the lift as a free body, replace the action of the constraint (the cable) by its reaction $\boldsymbol{T}$. From Eq. (5) we obtain:

$$\frac{P}{g} w = T-P,$$

whence

$$T = P\left(1+\frac{w}{g}\right).$$

If the lift starts descending with the same acceleration, the tension in the cable will be

$$T_1 = P\left(1-\frac{w}{g}\right).$$

Problem 96. The radius of curvature of a bridge at point A is R (Fig. 241). Determine the pressure exerted on the bridge at A by an automobile of mass m moving with a velocity v.

Solution. The normal acceleration of the car at point A is $w_n = v^2/R$. Acting on it are the force of gravity $\boldsymbol{P}$ and the reaction of the constraint $\boldsymbol{N}$. Then, from Eq. (5) projected on the normal,

$$m\frac{v^2}{R} = = mg - N,$$

whence,

$$N = m\left(1 - \frac{v^2}{R}\right).$$

The pressure on the bridge is equal to N in magnitude but is directed downward.

Problem 97. A crank OA of length l (Fig. 242) rotates with a uniform angular velocity ω and translates the slotted bar K of weight P

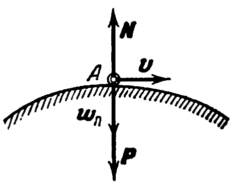

Fig. 241

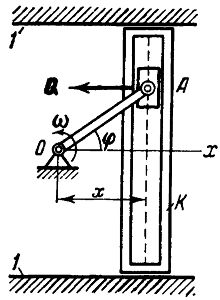

Fig. 242

along slides 1, $1'$. Neglecting friction, determine the pressure exerted by the slide block A on the slotted bar.

Solution. The position of the bar is specified by its coordinate

$$x = l\cos\omega t.$$

Eq. (5) for the motion of the bar in terms of its projection on axis x gives $mw_x = Q_x$. But $w_x = \frac{d^2x}{dt^2} = -l\omega^2\cos\omega t = -\omega^2 x$; whence, as $Q_x = -Q$,

$$-\frac{P}{g}\omega^2 x = -Q, \qquad Q = \frac{P}{g}\omega^2 x.$$

Thus, the pressure of the slide block on the slotted bar is proportional to its distance x from O.

The examples show that the first problem of dynamics is fairly simple. If the acceleration of the moving particle is not given, it can be determined by purely kinematic computations. Therefore, and also because of its practical importance, the second problem is considered the principal problem of dynamics.

Chapter 17

Differential Equations of Motion for a Particle and Their Integration

§ 104. Rectilinear Motion of a Particle

We know from kinematics that in rectilinear motion the velocity and acceleration of a particle are continuously directed along the same straight line. As the direction of acceleration is coincident with the direction of force, it follows that a free particle will move in a straight line whenever the force acting on it is of constant direction and the velocity at the initial moment is either zero or is collinear with the force.

Consider a particle moving rectilinearly under the action of an applied force $\boldsymbol{R} = \sum \boldsymbol{F}_k$. The position of the particle on its path is

Fig. 243

specified by its coordinate x (Fig. 243). In this case the principal problem of dynamics is: knowing $\boldsymbol{R}$, find the equation of motion of the particle $x = f(t)$. Eq. (3) gives the relation between x and R. Projecting both sides of the equation on axis Ox, we obtain:

$$mw_x = R_x \equiv \sum F_{kx},$$

or, as

$$w_x = \frac{d^2x}{dt^2},$$

$$m\frac{d^2x}{dt^2} = \sum F_{kx}. \qquad (6)$$

Eq. (6) is called the ***differential equation of rectilinear motion of a particle****). It is often more convenient to replace Eq. (6) with two

*) This is a differential equation, as the required quantity x is under the derivative symbol. Eq. (6) can be used to solve the first problem of dynamics: knowing the equation of motion $x = f(t)$ for a particle, determine the applied force F_x.

differential equations containing first derivatives:

$$m\frac{dv_x}{dt}=\sum F_{kx}, \tag{7}$$

$$\frac{dx}{dt}=v_x. \tag{7'}$$

Whenever the solution of a problem requires that the velocity be found as a function of the coordinate x instead of time t (or when the forces themselves depend on x), Eq. (7) is converted to the variable x. As $\frac{dv_x}{dt}=\frac{dv_x}{dx}\cdot\frac{dx}{dt}=\frac{dv_x}{dx}v_x$, Eq. (7) takes the form

$$mv_x\frac{dv_x}{dx}=\sum F_{kx}. \tag{8}$$

The principal problem of dynamics is, essentially, to develop the equation of motion $x=f(t)$ for a particle from the above equations, the forces being known. For this it is necessary to integrate the corresponding differential equation. In order to make clear the nature of the mathematical problem, it should be recalled that the forces in the right side of Eq. (6) can depend on time t, on the position of the particle x, and on the velocity $v_x=\frac{dx}{dt}$ (see § 99). Consequently, in the general case Eq. (6) is, mathematically, a differential equation of the second order of the form

$$\frac{d^2x}{dt^2}=\Phi\left(t,\ x,\ \frac{dx}{dt}\right). \tag{9}$$

The equation can be solved for every specific problem after determining the form of its right-hand member, which depends on the applied forces. When Eq. (9) is integrated for a given problem, the general solution will include two constants of integration C_1 and C_2, and the *general solution* will be

$$x=f(t,\ C_1,\ C_2). \tag{10}$$

To solve a concrete problem, it is necessary to determine the values of the constants C_1 and C_2. For this we introduce the so-called *initial conditions*.

Investigation of any motion begins with some specified instant called the *initial time* $t=0$, usually the moment when the motion under the action of the given forces starts. The position occupied by a particle at the initial time is called its *initial displacement*, and its velocity at that time is its *initial velocity* (a particle can have an initial velocity either because at time $t=0$ it was moving under no force or because up to time $t=0$ it was subjected to the action of some other forces). To solve the principal problem of dynamics, we must know, besides the applied forces, the *initial conditions*, i.e., the position and velocity of the particle at the initial time.

In the case of rectilinear motion, the initial conditions are specified in the form

$$\text{at } t = 0, \ x = x_0 \text{ and } v_x = v_0. \tag{11}$$

From the initial conditions we can determine the constants C_1 and C_2 and find the *partial solution* of Eq. (9), which gives the equation of motion of the particle:

$$x = f(t, \ x_0, \ v_0). \tag{12}$$

The following simple example will explain the above.

Let there be a force Q of constant magnitude and direction acting on a particle. Then Eq. (7) acquires the form

$$m \frac{dv_x}{dt} = Q_x.$$

As $Q_x = \text{const.}$, multiplying both members of the equation by dt and integrating, we obtain:*)

$$v_x = \frac{Q_x}{m} t + C_1. \tag{13}$$

Substituting the value of v_x into Eq. (7′), we have:

$$\frac{dx}{dt} = \frac{Q_x}{m} t + C_1.$$

Multiplying by dt and integrating once again, we obtain:

$$x = \frac{1}{2} \frac{Q_x}{m} t^2 + C_1 t + C_2. \tag{14}$$

This is the general solution of Eq. (9) for the specific problem in the form given by Eq. (10).

Now let us determine the integration constants C_1 and C_2, assuming for the specific problem that the initial conditions are given by (11). Solutions (13) and (14) must satisfy any moment of time, including $t = 0$. Therefore, substituting zero for t in Eqs. (13) and (14), we should obtain v_0 and x_0 instead of v_x and x, i.e., we should have

$$v_0 = C_1, \quad x_0 = C_2.$$

These equations give the values of the constants C_1 and C_2 which satisfy the initial conditions of a given problem. Substituting these values into Eq. (14), we finally obtain the relevant equation of motion in the form expressed by Eq. (12):

$$x = x_0 + v_0 t + \frac{1}{2} \frac{Q_x}{m} t^2. \tag{15}$$

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*) We assume that the constant of integration has been transposed from the left to the right and included in C_1.

We see from Eq. (15) that a particle subjected to a constant force performs uniformly variable motion. This could have been foreseen; for, if Q = const., w = const., too. An example of this type of motion is the motion of a particle under the force of gravity, in which case in Eq. (15) $\frac{Q_x}{m} = g$ and axis Ox is directed vertically down.

§ 105. Solution of Problems

Solution of problems of dynamics by integrating the differential equations of motion includes the following operations:

(1) Writing the Differential Equation of Motion. For this,

(a) Choose an origin (usually coinciding with the initial position of the particle) and draw a coordinate axis along the line of motion, as a rule in the direction of motion; if, for the applied forces, a particle has a position of equilibrium, it is convenient to choose the origin to coincide with the position of static equilibrium.

(b) Depict the moving particle in an arbitrary position (but such that $x > 0$ and $v_x > 0$; the latter condition is important when the applied forces include forces depending on velocity), and draw all the forces acting on the particle.

(c) Compound the projections of all the forces on the coordinate axis and substitute the sum into the right side of the differential equation of motion. *It is important to express all the variable forces in terms of the quantities* (t, x or v) *on which they depend.*

(2) Integrating the Differential Equation of Motion. The integration is carried out according to the rules of higher mathematics, depending on the form of the obtained equation, i.e., on the form of the right-hand member of Eq. (9). When besides the constant forces there is one variable force that depends *only* on time t, or *only* on distance x, or *only* on velocity v, the equation of rectilinear motion can generally be integrated by the method of separating the variables (see Problems 98-100). If only the velocity has to be determined, it is often possible to solve the problem by integrating either Eq. (7) or Eq. (8).

(3) Determining the Constants of Integration. In order to determine the constants of integration, it is necessary from the conditions of the problem to define the initial conditions in the form (11). The values of the constants are found from the initial conditions as shown in § 104, and they can be determined directly after each integration.

If the differential equation of motion is an equation with separable variables, instead of introducing integration constants we can immediately evaluate the definite integrals on both sides of the equation over the appropriate range; an example is given in Problem 100.

(4) Determining the Required Quantities and Analysing the Obtained Results. In order to be able to analyse the solution and also to verify the dimensions, the whole solution should be carried out in the most general form (in letter notation), inserting the numerical data only in the final results.

These general rules also hold for curvilinear motion.

Let us consider three specific problems, in which the force depends on time, distance, or velocity of the motion of the particle.

1. The Force Depends on Time

Problem 98. A load of weight P starts moving from rest along a smooth horizontal plane under the action of a force $\boldsymbol{R}$ the magnitude of which increases in proportion to the time: $R = kt$. Develop the equation of motion of the load.

Solution. Place the origin O in the initial position of the load and direct the axis Ox in the direction of motion (see Fig. 243). Then the initial conditions are: at $t = 0$, $x = 0$ and $v_x = 0$. Depict the load in an arbitrary position and the forces acting on it. We have $R_x = R = kt$, and Eq. (7) takes the form

$$\frac{P}{g}\frac{dv_x}{dt} = kt.$$

Multiplying by dt, we immediately separate the variables and obtain:

$$v_x = \frac{kg}{P}\frac{t^2}{2} + C_1.$$

Substituting the initial values into this equation, we find that $C_1 = 0$. Then, substituting dx/dt for v_x, we have:

$$\frac{dx}{dt} = \frac{kg}{2P}t^2.$$

Multiplying by dt, we again separate the variables and, integrating, we find:

$$x = \frac{kg}{2P}\frac{t^3}{3} + C_2.$$

Substitution of the initial values gives $C_2 = 0$, and we obtain the equation of motion for the load in the form

$$x = \frac{kg}{6P}t^3.$$

Thus, the displacement of the load is proportional to the cube of time.

2. The Force Depends on Distance

Problem 99. Neglecting the resistance of air and friction, determine the time it would take a body to travel from end to end of a tunnel AB dug through the earth along a chord (Fig. 244). Assume the earth's radius to be $R = 6\ 370$ km.

Note: The theory of gravitation states that a body inside the earth is attracted towards the centre of the earth with a force F directly proportional to the distance r from the centre. Taking into account that at $r = R$ (i.e., at the surface of the earth) force F is equal to the weight of the body ($F = mg$), we find that inside the earth

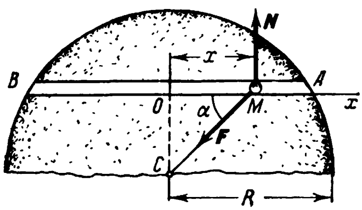

Fig. 244

$$F = \frac{mg}{R} r,$$

where $r = MC$ is the distance of point M from the centre of the earth.

Solution. Place the origin O in the middle of the chord AB (where a body in the tunnel would be in equilibrium) and direct the axis Ox along OA. If we assume the chord to be of length $2a$, the initial conditions will be: at $t = 0$, $x = a$ and $v_x = 0$.

The forces acting on the body in an arbitrary position are F and N. Consequently,

$$\sum F_{kx} = -F \cos \alpha = -\frac{mg}{R} r \cos \alpha = -\frac{mg}{R} x,$$

as it is evident from the diagram that $r \cdot \cos \alpha = x$.

We see that the acting force depends on the coordinate x of point M. In order to separate the variables in the differential equation of motion, write it in the form (8). Then, eliminating m and introducing the quantity

$$\frac{g}{R} = k^2,$$

we obtain:

$$v_x \frac{dv_x}{dx} = -k^2 x.$$

Multiplying by dx, we separate the variables and, integrating, obtain:

$$\frac{v_x^2}{2} = -k^2 \frac{x^2}{2} + C_1.$$

From the initial conditions, at $x = a$, $v_x = 0$; hence $C_1 = \frac{1}{2} k^2 a^2$. Substituting this expression for C_1, we have:

$$v_x = \pm k \sqrt{a^2 - x^2}.$$

As in the investigated position the velocity is directed from M to O, we see that $v_x < 0$ and the sign before the radical should be minus. Then, substituting dx/dt for v_x, we have

$$\frac{dx}{dt} = -k\sqrt{a^2 - x^2}.$$

Separating the variables, we write the equation in the form

$$k\,dt = -\frac{dx}{\sqrt{a^2 - x^2}},$$

and, integrating, we obtain:

$$kt = \arccos\frac{x}{a} + C_2.$$

Substituting the initial data (at $t = 0$, $x = a$) in this equation, we find that $C_2 = 0$. The equation of motion for the body in the tunnel is

$$x = a\cos kt.$$

Thus, the body is in harmonic motion with an amplitude a.

Now let us determine the time t_1 when the body will reach the end B of the tunnel. At B the coordinate $x = -a$. Substituting this value in the equation of motion, we obtain $\cos kt_1 = -1$, whence $kt_1 = \pi$ and $t_1 = \frac{\pi}{k}$. But we have assumed $k = \sqrt{\frac{g}{R}}$. Calculating, we find that the time of the motion through the tunnel, given the conditions of the problem, does not depend on the length of the tunnel and is always equal to

$$t_1 = \pi\sqrt{\frac{R}{g}} \approx 42 \text{ min } 11 \text{ s}.$$

This extremely interesting result has given rise to a number of projects—so far utopian—of such a tunnel.

Let us also find the maximum velocity of the body. From the expression for v_x we see that $v = v_{max}$ at $x = 0$, i.e., at the origin O. The magnitude of the velocity is

$$v_{max} = ka = a\sqrt{\frac{g}{R}}.$$

If, for example, $2a = 0.1R = 637$ km, then $v_{max} \approx 395$ m/s $=$ $= 1422$ km/h.

The vibration of a particle under the action of a force proportional to distance will be studied in greater detail in Chapter 21, where another method of integrating the differential equations of motion will be discussed.

3. The Force Depends on Velocity

Problem 100. A boat of mass $m = 40$ kg is pushed and receives an initial velocity $v_0 = 0.5$ m/s. Assuming the resistance of water at low velocities to be proportional to the first power of the velocity and changing according to the equation $R = \mu v$, where the factor $\mu = 9.1$ kg/s, determine the time in which the velocity will drop by one-half and the distance the boat will travel in that time. Determine also the distance the boat will travel till it stops.

Solution. Let us choose the origin O to coincide with the initial position of the boat, pointing axis Ox in the direction of the motion (Fig. 245). In this case the initial conditions will be: at $t = 0$, $x = 0$ and $v_x = v_0$.

Depict the boat in an arbitrary position with the acting forces $\boldsymbol{P}$, $\boldsymbol{N}$, and $\boldsymbol{R}$.

Note. There are no other forces acting on the boat. The force which pushed the boat acted *before* the instant $t = 0$. The result of its

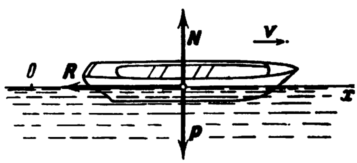

Fig. 245

action is taken into account by stating the initial velocity v_0 imparted to the boat by the force (see § 104). To be quite sure about the forces actually acting on a body during its motion, always remember that a force is the result of the interaction of bodies. In the present case, the force $\boldsymbol{P}$ is the result of the action of the earth on the boat, while forces $\boldsymbol{R}$ and $\boldsymbol{N}$ are the results of the action of the water on the boat. No other material bodies act on the boat in its motion and, consequently, there are no other forces applied to it. Attention is drawn to this because it is often a cause of errors in problem solutions.

Calculating the projections of the acting forces, we find:

$$\sum F_{kx} = -R = -\mu v.$$

To determine the duration of the motion, we write differential equation (7). Noting that $v_x = v$, we have:

$$m\frac{dv}{dt} = -\mu v.$$

Integrate this equation, evaluating both sides, after separating the variables, for definite integrals. The lower limit of each of the integrals is the value of the integration variable at the initial time, and the upper is its value at an arbitrary time.

Then, taking into account that the problem specifies that at $t = 0$ we have $v = v_0$, we obtain:

$$\int_{v_0}^{v} \frac{dv}{v} = -\frac{\mu}{m} \int_{0}^{t} dt \quad \text{or} \quad \ln v - \ln v_0 = -\frac{\mu}{m} t,$$

whence finally,

$$t = \frac{m}{\mu} \ln \frac{v_0}{v}. \tag{a}$$

The required time t_1 is determined by assuming $v = 0.5\ v_0$. We see that in this case the time does not depend on the value of v_0. As $\ln\ 2 = 0.69$,

$$t_1 = \frac{m}{\mu} \ln 2 \approx 3 \text{ s}.$$

To determine the distance, it is best to write the differential equation of motion in the form (8), as it immediately establishes the relation between x and v*). We thus obtain:

$$mv \frac{dv}{dx} = -\mu v,$$

whence, eliminating v and separating the variables and taking into account that $v = v_0$ when $x = 0$, we obtain:

$$\int_{v_0}^{v} dv = -\frac{\mu}{m} \int_{0}^{x} dx \quad \text{or} \quad v - v_0 = -\frac{\mu}{m} x.$$

Consequently,

$$x = \frac{m}{\mu} (v_0 - v). \tag{b}$$

Assuming $v = 0.5\ v_0$, we find the required displacement:

$$x_1 = \frac{mv_0}{2\mu} \approx 1.1 \text{ m}.$$

To find the distance travelled by the boat till it stops, in equation (b) we assume $v = 0$. Then $x_2 = \frac{mv_0}{\mu} = 2.2$ m.

Determining the duration of the motion until complete rest, we find from equation (a) that at $v = 0$, $t_2 = \infty$. This means that, under the given law of resistance, $R = \mu v$, the boat will approach its final

*) The distance can also be found by rewriting Eq. (a) as an expression of the velocity v in terms of time t in the form $v = v_0 e^{-\frac{\mu}{m} t}$, and then substituting dx/dt for v and integrating the new equation, but this solution is more involved.

position (specified by the coordinate x_2) asymptotically. Actually, the duration of the motion till complete rest is finite, as, with the decrease of velocity, the resistance equation changes and the dependence of v on t changes accordingly (see, for example, Problem 110 in § 118).

Another important example of motion under the action of a force depending on velocity is discussed in the next article.

§ 106*. Body Falling in a Resisting Medium (in Air)

A body moving in some medium is subjected to a resistance, which depends on the shape and size of the body, its velocity, and the properties of the medium.

Experience shows that at velocities which are not very small and not approaching the velocity of sound, the resistance is proportional to the square of the velocity v and can be expressed by the formula*)

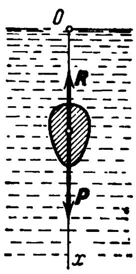

Fig. 246

$$R = \frac{1}{2} c_x \rho S v^2, \tag{16}$$

where ρ is the density of the medium $\left(\text{for air at } 15°\text{C and 760 mm Hg pressure, } \rho = \frac{1}{8}\,\frac{\text{kgf-s}^2}{\text{m}^4}\right)$, S m^2 is the area of the projection of the body on a plane perpendicular to the direction of motion (the middle section), and c_x is a dimensionless resistance factor depending on the shape of the body. For example, for a parachute $c_x = 1.4$, for a sphere $c_x = 0.5$, for highly streamlined spindle-shaped bodies $c_x < 0.03$.

Let us consider a body falling in the atmosphere from a low height in comparison with the earth's radius (low enough to assume the weight P of the body and the density of the air ρ to be constant).

Directing the coordinate axis Ox vertically down (Fig. 246), let us investigate how the velocity of fall depends on the displacement x, assuming $v_0 = 0$.

The forces acting on the falling body are $\boldsymbol{P}$ and $\boldsymbol{R}$; consequently,

$$\sum F_{kx} = P - R = P - \frac{1}{2} c_x \rho S v^2.$$

In order to obtain directly the dependence of v on x, we write the differential equation of motion in the form (8). Then, taking into account that $v_x = v$, we have

$$\frac{P}{g} v \frac{dv}{dx} = P - \frac{1}{2} c_x \rho S v^2.$$

*) For fall in air, formula (16) holds for velocities up to 300 m/s.

Introducing the expression

$$\frac{2P}{c_x \rho S} = a^2, \tag{17}$$

the previous equation takes the form

$$v\frac{dv}{dx} = g\left(1 - \frac{v^2}{a^2}\right)$$

or, after separating the variables,

$$-\frac{v\,dv}{a^2 - v^2} = -\frac{g}{a^2}\,dx.$$

Integrating, we have

$$\ln(a^2 - v^2) = -2\frac{g}{a^2}x + C_1.$$

According to the initial conditions, as $x = 0$, the velocity $v = 0$, consequently $C_1 = \ln a^2$. Substituting this value for C_1, we obtain:

$$\ln\frac{a^2 - v^2}{a^2} = -2\frac{g}{a^2}x,$$

or

$$\frac{a^2 - v^2}{a^2} = e^{-2\frac{g}{a^2}x},$$

and finally

$$v = a\sqrt{\left(1 - e^{-2\frac{g}{a^2}x}\right)}. \tag{18}$$

Formula (18) gives the dependence of velocity on displacement for a body falling in the atmosphere.

As x increases, the quantity $e^{-2\frac{g}{a^2}x}$ decreases and tends to zero as $x \to \infty$. It follows, then, that the velocity of fall v increases as x increases, tending to a constant value a. This quantity is called the *limiting velocity of fall* and is denoted v_{lim}. From Eq. (17) we find that

$$v_{\text{lim}} = \sqrt{\frac{2P}{c_x \rho S}}. \tag{19}$$

Thus, at $v_0 = 0$, *a body falling in air cannot exceed a limiting velocity* v_{lim}. The greater the weight of a body and the smaller the values of c_x, ρ, and S, the greater the limiting velocity of fall.

Let us determine the speed with which the velocity of a falling body approaches its limiting value. For this see Table 1, which

Table 1

$\frac{g}{v_{\lim}^2}x$	$\frac{v}{v_{\lim}}$
0	0
0.5	0.80
1.0	0.93
1.2	0.95
1.5	0.97
2.0	0.99

gives the dependence of $\frac{v}{v_{\lim}}$ upon $\frac{g}{v_{\lim}^2}x$, as calculated from Eq. (18). It follows from the table that

$$\left.\begin{aligned} &\text{at} \quad \frac{g}{v_{\lim}^2}x = 1.2, \quad v = 0.95\, v_{\lim}, \\ &\text{at} \quad \frac{g}{v_{\lim}^2}x = 2.0, \quad v = 0.99\, v_{\lim}. \end{aligned}\right\} \tag{20}$$

Consequently, the velocity of fall approaches its limiting value fairly rapidly, if only the quantities c_x and S are not very small (see Problem 101).

The existence of a limiting velocity of fall can be established by the following simple reasoning: The velocity of a falling body in air increases; consequently, force R increases. If we consider that force R cannot, obviously, be greater than the weight of the body P (see Fig. 246), then $R_{\lim} = P$. Substituting the expression for $R_{\lim}$ from Eq. (16), we obtain $\frac{1}{2}c_x\rho S v_{\lim}^2 = P$, whence we obtain the value of $v_{\lim}$ as in Eq. (19). This reasoning, however, provides no clue to the rate with which the velocity of fall v tends towards $v_{\lim}$. This important characteristic can be obtained only from Eq. (18).

Problem 101. Determine the limiting velocity of fall of a parachute jumper weighing $P = 75$ kgf (weight of parachute included) (a) in free fall, assuming $S = 0.4$ m^2, $c_x = 1.0$; (b) with open canopy, assuming $S = 36$ m^2, $c_x = 1.4$.

Determine for both cases the distance H_1 after which the jumper will have attained a velocity $v_1 = 0.95v_{\text{lim}}$ (i.e., 5% less than the limiting velocity), and the distance H_2 at which his velocity of fall $v_2 = 0.99v_{\text{lim}}$.

Solution. Determine the limiting velocity of fall from Eq. (19), assuming for air $\rho = \frac{1}{8}\frac{\text{kgf-s}^2}{m^4}$. We find H_1 and H_2 from Eq. (20). As $v_1 = 0.95v_{\text{lim}}$ at $\frac{g}{v_{\text{lim}}^2}x = 1.2$, the required distance $H_1 = 1.2\frac{v_{\text{lim}}^2}{g}$. Similarly, we find that $H_2 = 2\frac{v_{\text{lim}}^2}{g}$.

Performing the necessary computations, we find:

(a) in free fall $v_{\text{lim}} \approx 55$ m/s, $H_1 \approx 370$ m, $H_2 \approx 610$ m;

(b) with open canopy $v_{\text{lim}} \approx 5$ m/s, $H_1 \approx 3$ m, $H_2 \approx 5$ m.

We see that when the resistance is great, the limiting velocity is rapidly achieved.

§ 107. Curvilinear Motion of a Particle

Consider a free particle moving under the action of forces $F_1, \ldots, F_n$. Let us draw a fixed set of axes $Oxyz$ (Fig. 247). Projecting both members of the equation $mw = \sum F_k$ on these axes, and taking into account that $w_x = \frac{d^2x}{dt^2}$, etc., we obtain the *differential equations of curvilinear motion of a body* in terms of the projections on rectangular cartesian axes:

$$m\frac{d^2x}{dt^2} = \sum F_{kx}, \quad m\frac{d^2y}{dt^2} = \sum F_{ky},$$
$$m\frac{d^2z}{dt^2} = \sum F_{kz}. \tag{21}$$

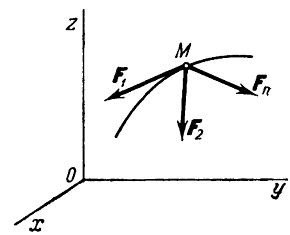

Fig. 247

As the forces acting on the particle may depend on time, the displacement or the velocity of the particle, then by analogy with Eq. (9) in § 104, the right-hand members of Eq. (21) may contain the time t, the coordinates x, y, z of the particle, and the projections of its velocity $\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}$. Furthermore, the right side of each equation may include all these variables.

Eqs. (21) can be used to solve both the first and the second (principal) problems of dynamics. To solve the principal problem of dynamics we must know, besides the acting forces, the initial conditions, i.e., the position and velocity of the particle at the initial time. The initial conditions for a set of coordinate axes $Oxyz$ are specified in the form

$$\text{at } t=0, \quad \left. \begin{array}{lll} x=x_0, & y=y_0, & z=z_0, \\ v_x=v_{x0}, & v_y=v_{y0}, & v_z=v_{z0}. \end{array} \right\} \tag{22}$$

Knowing the acting forces, by integrating Eq. (21) we find the coordinates x, y, z of the moving particle as functions of time t, i.e., the equation of motion for the particle. The solutions will contain six constants of integration C_1, C_2, . . ., C_6, the values of which must be found from the initial conditions (22). An example of integrating Eqs. (21) is given in § 108.

Differential equations of motion can also be written in terms of projections on the axes of other coordinate systems as, for instance, in § 118.

§ 108. Motion of a Particle Thrown at an Angle to the Horizon in a Uniform Gravitational Field

Let us investigate the motion of a projectile thrown with an initial velocity v_0 at an angle α to the horizon, considering it as a material particle of mass m, neglecting the resistance of the atmosphere, assuming that the horizontal range and height of path are small compared with the radius of the earth, and considering the gravitational field to be uniform ($P =$ $=$ const.).

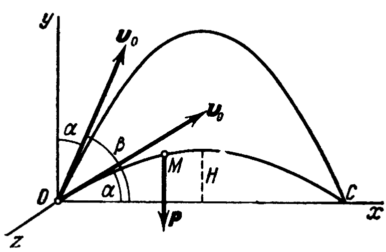

Fig. 248

Place the origin of the coordinate axes O at the initial position of the particle, direct axis y vertically up, axis x in the plane through Oy and vector v_0, and axis z perpendicular to the first two (Fig. 248). The angle between vector v_0 and axis x will be α.

Depict the moving particle M anywhere on its path. Acting on the particle is only the force of gravity P (see note to Problem 100, p. 286), the projections of which on the coordinate axes are

$$P_x = 0, \quad P_y = -P = -mg, \quad P_z = 0.$$

Substituting these values into Eqs. (21), noting that $\frac{d^2x}{dt^2} = \frac{dv_x}{dt}$, etc., and eliminating m, we obtain:

$$\frac{dv_x}{dt}=0, \quad \frac{dv_y}{dt} = -g, \quad \frac{dv_z}{dt}=0.$$

Multiplying these equations by dt and integrating, we find:

$$v_x = C_1, \quad v_y = -gt + C_2, \quad v_z = C_3.$$

The initial conditions of our problem have the form:

$$x = 0, \quad y = 0, \quad z = 0,$$

at $t = 0$,

$$v_x = v_0 \cos\alpha, \quad v_y = v_0 \sin\alpha, \quad v_z = 0.$$

Satisfying the initial conditions, we have:

$$C_1 = v_0 \cos\alpha, \quad C_2 = v_0 \sin\alpha, \quad C_3 = 0.$$

Substituting these values of C_1, C_2, and C_3 in the solutions above and replacing v_x, v_y, v_z by $\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}$, we arrive at the equations

$$\frac{dx}{dt} = v_0 \cos\alpha, \quad \frac{dy}{dt} = v_0 \sin\alpha - gt, \quad \frac{dz}{dt} = 0.$$

Integrating, we obtain

$$x = v_0 t \cos\alpha + C_4, \quad y = v_0 t \sin\alpha - \frac{gt^2}{2} + C_5, \quad z = C_6.$$

Substituting the initial conditions, we have $C_4 = C_5 = C_6 = 0$. And finally we obtain the equations of motion of particle M in the form

$$x = v_0 t \cos\alpha, \quad y = v_0 t \sin\alpha - \frac{gt^2}{2}, \quad z = 0. \tag{23}$$

From the last equation it follows that the motion takes place in the plane Oxy.

Knowing the equations of motion of a particle, it is possible to determine all the characteristics of the given motion by the methods of kinematics.

(1) **Path.** Eliminating the time t between the first two of Eqs. (23), we obtain the equation of the path of the particle:

$$y = x \tan\alpha - \frac{gx^2}{2v_0^2 \cos^2\alpha}. \tag{24}$$

This is an equation of a parabola the axis of which is parallel to axis y. Thus, *a heavy particle thrown at an angle to the horizon in a vacuum follows a parabolic path* (Galileo).

(2) **Horizontal Range.** The horizontal range is the distance $OC = X$ along axis x. Assuming in Eq. (24) $y = 0$, we obtain the points of intersection of the path with the x axis. From the equation

$$x\left(\tan\alpha - \frac{gx}{2v_0^2 \cos^2\alpha}\right) = 0$$

we obtain

$$x_1 = 0, \quad x_2 = \frac{2v_0^2 \cos^2\alpha \tan\alpha}{g}.$$

The first solution gives point O, the second point C. Consequently $X = x_2$, and finally

$$X = \frac{v_0^2}{g} \sin 2\alpha. \tag{25}$$

From Eq. (25) we see that the horizontal range X is the same for angle β, where $2\beta = 180° - 2\alpha$, i.e., if $\beta = 90° - \alpha$. Consequently, a particle thrown with a given initial velocity v_0 can reach the same point C by two paths: flat (low) ($\alpha < 45°$) or curved (high) ($\beta = 90° - \alpha > 45°$).

With a given initial velocity v_0, the maximum horizontal range in a vacuum is obtained when $\sin 2\alpha = 1$, i.e., when angle $\alpha = 45°$.

(3) Height of Path. If in Eq. (24) we assume $x = \frac{1}{2}X = \frac{v_0^2}{g} \sin\alpha \cos\alpha$, we obtain the height H of the path:

$$H = \frac{v_0^2}{2g} \sin^2 \alpha. \tag{26}$$

(4) Time of Flight. It follows from Eq. (23) that the total time of flight is defined by the equation $X = v_0 T \cos\alpha$. Substituting the expression for X, we obtain:

$$T = \frac{2v_0}{g} \sin \alpha. \tag{27}$$

At the maximum range angle $\alpha^* = 45°$, all the quantities become, respectively,

$$X^* = \frac{v_0^2}{g}, \quad H^* = \frac{v_0^2}{4g} = \frac{1}{4} X^*, \quad T^* = \frac{v_0}{g}\sqrt{2}. \tag{28}$$

These results can be used to estimate the flight characteristics of missiles (rockets) with a horizontal range of 200 to 600 km, as at such ranges (and at $\alpha \approx 45°$) a projectile travels most of its path in the stratosphere, where atmospheric resistance can be neglected. At closer ranges the resistance of the air has a considerable effect, while at ranges exceeding 600 km the gravitational force can no longer be considered constant.

Example. It is known*) that at an altitude of 20 km a vertically launched German V-2 rocket had a velocity $v_0 \approx 1\,700$ m/s and an angle $\alpha \approx 45°$ (the rocket was inclined by means of special instruments and control vanes). The further flight of the rocket was actually motion of a projectile in a vacuum. Therefore, from Eqs. (28), its characteristics must have been

$$X^* \approx 300 \text{ km}, \quad H^* \approx 75 \text{ km}, \quad T^* \approx 245 \text{ s}.$$

These results are very close to the actual performance of V-2 rockets.

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*) See *Ballistics of the Future* by J. M. Kooy and J. W. H. Uytenbogaart. The Technical Publishing Company H. Stam. Haarlem—Holland.

Chapter 18
General Theorems of Particle Dynamics

In solving many problems of dynamics it will be found that the so-called *general theorems*, representing corollaries of the fundamental law of dynamics, are more conveniently applied than the method of integration of differential equations of motion.

The importance of the general theorems is that they establish visual relationships between the principal dynamic characteristics of motion of material bodies, thereby presenting broad possibilities for analysing the mechanical motions widely employed in practical engineering. Furthermore, the general theorems make it possible to study for practical purposes the specific aspects of a given phenomenon without investigating the phenomenon as a whole. Finally, the use of the general theorems makes it unnecessary to carry out for every problem the operations of integration performed once and for all in proving the theorems, which simplifies the solution. Let us see how these theorems apply to one material particle.

§ 109. Momentum and Kinetic Energy of a Particle

The basic dynamic characteristics of particle motion are *momentum* (or linear momentum) and *kinetic energy.*

The momentum of a particle is defined as a vector quantity $m\mathbf{v}$ *equal to the product of the mass of the particle and its velocity.* The vector $m\mathbf{v}$ is directed in the same direction as the velocity, i.e., tangent to the path of the particle.

The kinetic energy of a particle is defined as a quantity equal to half the product of the mass of the particle and the square of its velocity $\left(\frac{1}{2}mv^2\right)$.

The units of measurement of these quantities are:

(a) In the SI system

$$\text{kg-m/s} \quad (\text{for } mv), \quad \text{and} \quad \text{kg-m}^2/\text{s}^2 \ \left(\text{for } \frac{mv^2}{2}\right);$$

(b) In the mkg(f)s system

$$\frac{\text{kgf-s}^2}{\text{m}}\,\frac{\text{m}}{\text{s}} = \text{kgf-s} \quad (\text{for } mv), \quad \text{and} \quad \frac{\text{kgf-s}^2}{\text{m}}\,\frac{\text{m}^2}{\text{s}^2} = \text{kgf-m} \ \left(\text{for } \frac{mv^2}{2}\right).$$

The reason for the introduction of two dynamic characteristics is that a single characteristic does not cover all the aspects of particle motion.

For example, knowing the momentum of an automobile (i.e., the quantity $Q = mv$, but not the separate values of m and v) and the force acting on it when it slows down, we can determine the time it will take for the car to stop; the information, however, is not enough to determine the path travelled during the braking time. Conversely, knowing the initial kinetic energy of the car and the braking force, we can determine the braking distance, but not the time of the braking*).

§ 110. Impulse of a Force

The concept of *impulse* (or linear impulse) *of a force* is used to characterise the effect on a body of a force acting during a certain interval of time. First let us introduce the concept of elementary impulse, i.e., impulse in an infinitesimal time interval dt. *Elementary impulse is defined as a vector quantity* $d\boldsymbol{S}$ *equal to the product of the vector of the force* $\boldsymbol{F}$ *and the time element* dt:

$$d\boldsymbol{S} = \boldsymbol{F}\,dt. \tag{29}$$

The elementary impulse is directed along the action line of the force.

The impulse $\boldsymbol{S}$ of any force during a finite time interval t_1 is computed as the integral sum of the respective elementary impulses:

$$\boldsymbol{S} = \int_0^{t_1} \boldsymbol{F}\,dt. \tag{30}$$

Thus, *the impulse of a force in any time interval* t_1 *is equal to the integral of the elementary impulse over the interval from zero to* t_1.

In the special case when the force $\boldsymbol{F}$ is of constant magnitude and direction ($\boldsymbol{F}$ = const.), we have $\boldsymbol{S} = \boldsymbol{F}t_1$. In the general case the magnitude of an impulse can be computed from its projections.

*) See Problem 103 (§ 115).

We can find the projections of an impulse on a set of coordinate axes if we remember that an integral is the limit of a sum, and the projection of a vector sum on an axis is equal to the sum of the projections of the component vectors on the same axis. Hence,

$$S_x = \int_0^{t_1} F_x\,dt, \quad S_y = \int_0^{t_1} F_y\,dt, \quad S_z = \int_0^{t_1} F_z\,dt. \tag{31}$$

With these projections we can construct the vector $\boldsymbol{S}$ and find its magnitude and the angles it makes with the coordinate axes. The dimension of linear impulse are: kg-m/s in SI units, and kgf-s in mkg(f)s units.

To solve the principal problem of dynamics, it is important to establish the forces whose impulses can be computed *without* knowing the equation of motion of the particle moving under the action of those forces. From Eq. (31) it is apparent that to these forces belong only *constant forces* and *forces depending on time*.

In order to calculate the impulses of forces depending on the coordinates or the velocity of a particle, we have to know the equations of its motion, i.e., $x = f_1(t)$, $y = f_2(t)$, $z = f_3(t)$. By expressing x, y, z or v_x, v_y, v_z through t, we can compute the integrals (31). Without knowing the equation of motion of a particle, i.e., without solving the principal problem of dynamics, the impulse of such forces cannot be calculated.

§ 111. Theorem of the Change in the Momentum of a Particle

As the mass of a particle is constant and its acceleration $\boldsymbol{w} = \frac{d\boldsymbol{v}}{dt}$, Eq. (3), which expresses the fundamental law of dynamics, can be expressed in the form

$$\frac{d(m\boldsymbol{v})}{dt} = \sum \boldsymbol{F}_k. \tag{32}$$

Eq. (32) at the same time states the **theorem of the linear momentum of a particle in differential form:** *The derivative of the linear momentum of a particle with respect to time equals the geometric sum of the forces acting on that particle.* Let us solve the equations.

Let a particle of mass m moving under the action of a force $\boldsymbol{R} = \sum \boldsymbol{F}_k$ (Fig. 249) have a velocity $\boldsymbol{v}_0$ at time $t = 0$, and at time t_1 let its velocity be $\boldsymbol{v}_1$. Now multiply both sides of Eq. (32) by dt and take definite integrals. On the right side, where we integrate with respect to time, the limits are zero and t_1; on the left side, where we integrate the velocity, the limits are the respective values of $\boldsymbol{v}_0$

and $\boldsymbol{v}_1$. As the integral of $d\,(m\boldsymbol{v})$ is $m\boldsymbol{v}$, we have:

$$m\boldsymbol{v}_1 - m\boldsymbol{v}_0 = \sum \int_0^{t_1} \boldsymbol{F}_k\,dt.$$

By Eq. (30), the integrals on the right side are the impulses of the acting force. Hence, we finally have:

$$m\boldsymbol{v}_1 - m\boldsymbol{v}_0 = \sum \boldsymbol{S}_k. \tag{33}$$

Eq. (33) states the **theorem of the change in the linear momentum of a particle in final form**: *The change in the momentum of a particle*

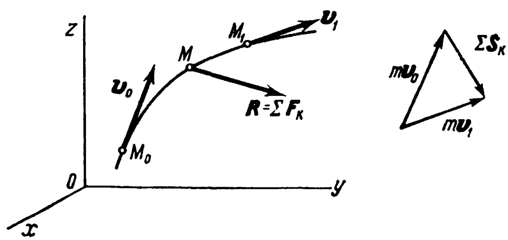

Fig. 249

during any time interval is equal to the geometrical sum of the impulses of all the forces acting on the particle during that interval of time (Fig. 249).

In problem solutions, projection equations are often used instead of the vector equation (33). Projecting both sides of Eq. (33) on a set of coordinate axes, we have:

$$\left.\begin{aligned} mv_{1x} - mv_{0x} &= \sum S_{kx}, \\ mv_{1y} - mv_{0y} &= \sum S_{ky}, \\ mv_{1z} - mv_{0z} &= \sum S_{kz}. \end{aligned}\right\} \tag{34}$$

In the case of rectilinear motion along the x axis, the theorem is stated by the first of these equations.

§ 112. Work Done by a Force. Power

The concept of *work* is introduced as a measure of the action of a force on a body in a given displacement, specifically that action which is represented by the change in the magnitude of the velocity of a moving particle.

First let us introduce the concept of elementary work done by a force in an infinitesimal displacement ds. The elementary work done by a force F (Fig. 250) is defined as a scalar quantity

$$dA = F_\tau ds, \qquad (35)$$

where F_τ is the projection of the force on the tangent to the path in the direction of the displacement, and ds is an infinitesimal displacement of the particle along that tangent.

This definition corresponds to the concept of work as a characteristic of that action of a force which tends to change the magnitude of velocity. For if force F is resolved into components F_τ and F_n, only the component F_τ, which imparts the particle its tangential acceleration, will change the magnitude of the velocity. As for component F_n, it either changes the direction of the velocity vector v (gives the particle its normal acceleration) or, in the case of constrained motion, changes the pressure on the constraint. Component F_n does not affect the magnitude of the velocity, or, as they say, force F_n "does no work".

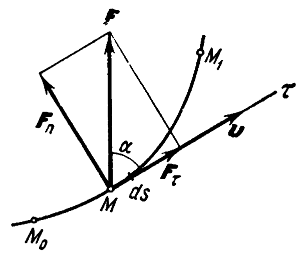

Fig. 250

Noting that $F_\tau = F \cdot \cos\alpha$, we further obtain from Eq. (35):

$$dA = F\, ds \cos\alpha. \qquad (36)$$

Thus, *the elementary work done by a force is equal to the product of the projection of that force on the direction of displacement of the particle and the infinitesimal displacement ds* (Eq. 35) or, *the elementary work done by a force is the product of the magnitude of that force, the infinitesimal displacement ds, and the cosine of the angle between the direction of the force and the direction of the displacement* (Eq. 36).

If angle α is acute, the work is of positive sense. In particular, at $\alpha = 0$, the elementary work $dA = F\, ds$.

If angle α is obtuse, the work is of negative sense. In particular, at $\alpha = 180°$, the elementary work $dA = -F\, ds$.

If angle $\alpha = 90°$, i.e., if a force is directed perpendicular to the displacement, the elementary work done by the force is zero.

The sign of the work has the following meaning: the work is positive when the tangential component of the force is pointed in the direction of the displacement, i.e., when the force accelerates the motion; the work is negative when the tangential component is pointed opposite the displacement, i.e., when the force retards the motion.

As we know from kinematics, the vector of the elementary displacement of a particle $d\boldsymbol{r} = \boldsymbol{v}\, dt$, and $ds = |\boldsymbol{v}|\, dt$, whence $ds = |d\boldsymbol{r}|$. Using the concept of the scalar product of two vectors employed in vector algebra*), Eq. (36) can be represented in the form

$$dA = \boldsymbol{F}\cdot d\boldsymbol{r}. \tag{36'}$$

Consequently, *the elementary work done by a force equals the scalar product of the force vector and the vector of the elementary displacement of its point of application.*

Let us now find the analytical expression for elementary work. For this we resolve force $\boldsymbol{F}$ into components $\boldsymbol{F}_x, \boldsymbol{F}_y, \boldsymbol{F}_z$ parallel to the coordinate axes (Fig. 251; the force $\boldsymbol{F}$ is not shown in the diagram). The infinitesimal displacement $MM' = ds$ is compounded of the displacements dx, dy, dz parallel to the coordinate axes, where x, y, z are the coordinates of point M. The work done by force $\boldsymbol{F}$ in the displacement ds can be calculated as the sum of the work done by its components $\boldsymbol{F}_x$, $\boldsymbol{F}_y$, $\boldsymbol{F}_z$ in the displacements dx, dy, dz. But the work in the displacement dx is done only by component $\boldsymbol{F}_x$ and is equal to $F_x\,dx$. The work in the displacements dy and dz is calculated similarly. Thus, we finally obtain

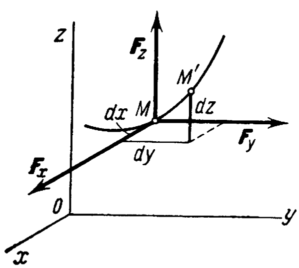

Fig. 251

$$dA = F_x dx + F_y dy + F_z dz. \tag{37}$$

Eq. (37) gives the *analytical expression of the elementary work* done by a force.

Eq. (37) can be obtained directly from Eq. (36′) if the scalar product is expressed in terms of the projections of the vectors. Then, taking into account that the projections of the radius vector $\boldsymbol{r}$ of point M on the axes $Oxyz$ are equal to its cartesian coordinates x, y, z, we obtain at once (see the footnote) $dA = F_x dx + F_y dy + F_z dz$.

The work done by a force in any finite displacement M_0M_1 (see Fig. 250) is calculated as the integral sum of the corresponding elementary works and is equal to

$$A_{(M_0M_1)} = \int_{(M_0)}^{(M_1)} F_\tau\, ds, \tag{38}$$

or

$$A_{(M_0M_1)} \int_{(M_0)}^{(M_1)} (F_x dx + F_y dy + F_z dz). \tag{38'}$$

Thus, *the work done by a force in any displacement M_0M_1 is equal to the integral of the elementary work taken along this displacement.*

*) The scalar product of two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is a scalar quantity equal to the product of the magnitudes of those vectors and the cosine of the angle α between them, i.e., $\boldsymbol{a}\cdot\boldsymbol{b} = ab\cdot\cos\alpha$. The scalar product in terms of the projections of the multiplied vectors has the form $\boldsymbol{a}\cdot\boldsymbol{b} = a_xb_x + a_yb_y + a_zb_z$.

The limits of the integral correspond to the values of the variables of integration at points M_0 and M_1 (or, more exactly, the integral is taken along the curve M_0M_1, i.e., it is curvilinear).

If the quantity F_τ is constant (F_τ = const.), then from Eq. (38), denoting the displacement $\overset{\frown}{M_0M_1}$ by the symbol s_1, we obtain:

$$A_{(M_0M_1)} = F_\tau s_1. \tag{38''}$$

In particular, such a case is possible when the acting force is constant in magnitude and direction (F = const.) and the point of

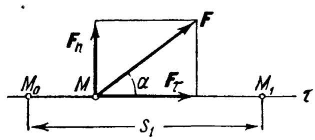

Fig. 252

application is in rectilinear motion (Fig. 252). In this case $F_\tau = F\cos\alpha$ = const., and the work done by the force $A_{(M_0M_1)} = Fs_1\cos\alpha$.

The unit of work in the SI system is the *joule* (1J = 1N-m), and in the mkg(f)s system, the kgf-m.

In order to solve the principal problem of dynamics, it is important to establish the forces whose work can be calculated immediately without knowing the equation of motion of the particle on which they are acting (compare with § 110). From Eq. (38′) it can be seen that to these forces belong *only constant forces and forces which depend on the position (coordinates) of a moving particle.*

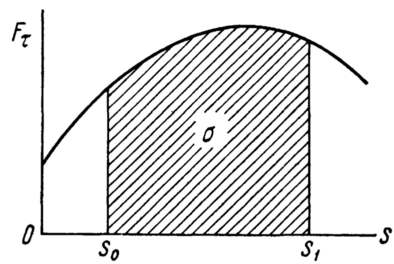

Fig. 253

In order to calculate the work done by forces depending on time or velocity of a particle, we must know the equation of its motion, i.e., the coordinates x, y, z as functions of time. Then all the variables can be expressed through the time t and the integral (38′) may be calculated. Without knowing the equation of motion of the particle, i.e., without solving the principal problem of dynamics, the work done by such forces cannot be determined.

Graphical Method of Calculating Work. If a force depends on the displacement s and a graph is given showing the dependence of F_τ on s (Fig. 253), the work done by a force F can be calculated graphically. Let a particle at M_0 be at a distance s_0 from the origin, and at M_1 let its displacement be s_1. Then, from Eq. (38), taking into account the geometric interpretation of integration, we have:

$$A_{(M_0M_1)} = \int_{s_0}^{s_1} F_\tau \, ds = \sigma,$$

where σ is the product of the shaded area in Fig. 253 and a scale factor.

Power. The term *power* is defined as the work done by a force in a unit of time (the time rate of doing work). If work is done at a constant rate, the power

$$W = \frac{A}{t_1},$$

where t_1 is the time in which the work A is done. In the general case,

$$W = \frac{dA}{dt} = \frac{F_\tau \, ds}{dt} = F_\tau v.$$

Thus, power is the product of the tangential component of a force by the velocity.

The unit of power in the SI system is the *watt* (1 W = 1 J/s), and in the mkg(f)s system, the kgf-m/s. In engineering the unit of power commonly used is horsepower (hp), which is equal to 75 kgf-m/s, or 736 W.

The work done by a machine can be expressed as the product of its power and the time of work. This has given rise to the commonly used technical unit of work, the kilowatt-hour (1 kW-h = $3.6 \cdot 10^6$ J ≈ 367100 kgf-m).

It can be seen from the equation $W = F_\tau v$ that if a motor has a given power W, the tractive force F_τ is inversely proportional to the velocity v. That is why, for instance, on an upgrade or poor road an automobile goes into lower gear, thereby reducing the speed and developing a greater tractive force with the same power.

§ 113. Examples of Calculation of Work

The examples considered below give results which can be used immediately in solving problems.

(1) Work Done by Gravity. Let a particle M subjected to the force of gravity $\boldsymbol{P}$ move from a point M_0 (x_0, y_0, z_0) to a point M_1 (x_1, y_1, z_1). Choose a coordinate system so that the axis Oz points vertically up (Fig. 254). Then $P_x = 0$, $P_y = 0$, $P_z = -P$. Substituting these expressions into Eq. (38′) and taking into account that the integration variable is z, we obtain:

$$A_{(M_0M_1)}=\int_{(M_0)}^{(M_1)}(-P)\,dz=-P\int_{z_0}^{z_1}dz=P\,(z_0-z_1).$$

If point M_0 is higher than M_1, then $z_0 - z_1 = h$, where h is the vertical displacement of the particle; if M_0 is below M_1, then $z_0 - z_1 = -(z_1 - z_0) = -h$. Finally we have:

$$A_{(M_0M_1)}=\pm Ph. \qquad (39)$$

Thus, *the work done by gravity is equal to the product of the magnitude of the force and the vertical displacement of the point to which it is applied, taken with the appropriate sign*. The work is positive if the initial point is higher than the final one and negative if it is lower.

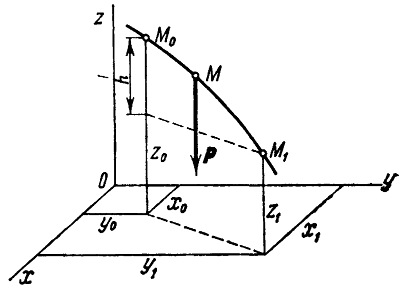

Fig. 254

It follows from this that the work done by gravity does not depend on the path along which the point of its application moves. Forces possessing this property are called *conservative forces* (see § 151).

(2) Work Done by an Elastic Force. Consider a weight M lying in a horizontal plane and attached to the free end of a spring (Fig. 255*a*).

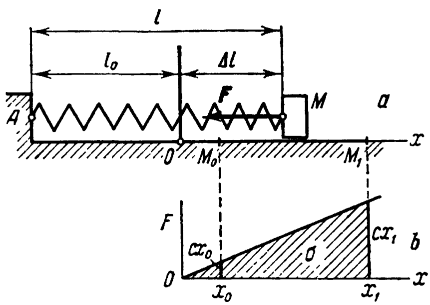

Fig. 255

Let point O on the plane represent the position of the end of the spring when it is not in tension ($AO = l_0$ is the length of the unextended spring) and let it be the origin of our coordinate system. Now if we draw the weight from its position of equilibrium O, stretching the spring to length l, acting on the weight will be the elastic force of the spring $\boldsymbol{F}$ directed towards O. According to Hooke's law, the magnitude of this force is proportional to the extension of the spring

$\Delta l = l - l_0$. As in our case $\Delta l = x$, in magnitude

$$F = c \mid \Delta l \mid = c \mid x \mid.$$

The factor c is called the *stiffness* of the spring, or the *spring constant*, and its dimension is $[c]$ = kgf/cm. Numerically, the stiffness c is equal to the force required to extend the spring by 1 cm.

Let us find the work done by the elastic force in the displacement of the weight from position M_0 (x_0) to position M_1 (x_1). As in this case $F_x = -F = -cx$, $F_y = F_z = 0$, substituting these expressions into Eq. (38′) we obtain:

$$A_{(M_0M_1)} = \int_{(M_0)}^{(M_1)} (-cx)\,dx = -c\int_{x_0}^{x_1} x\,dx = \frac{c}{2}(x_0^2 - x_1^2).$$

[The same result could be obtained from the graph of F depending on x (Fig. 255b) by calculating the area σ of the shaded trapezium in the diagram and taking into account the sign of the work.] In the obtained formula x_0 is the initial extension of the spring Δl_{in}, and x_1 is the final extension Δl_{fin}. Hence,

$$A_{(M_0M_1)} = \frac{c}{2}[(\Delta l_{\text{in}})^2 - (\Delta l_{\text{fin}})^2], \tag{40}$$

i.e., *the work done by an elastic force is equal to half the product of the stiffness and the difference between the squares of the initial and final extensions (or compressions) of a spring.*

The work is positive if $\mid \Delta l_{\text{in}} \mid > \mid \Delta l_{\text{fin}} \mid$, i.e., when the end of the spring moves towards the position of equilibrium, and negative when $\mid \Delta l_{\text{in}} \mid < \mid \Delta l_{\text{fin}} \mid$, i.e., when the end of the spring moves away from the position of equilibrium.

It can be proved that Eq. (40) holds for the case when the displacement of point M is not rectilinear. It follows, therefore, that the work done by the force F depends only on the quantities Δl_{in} and Δl_{fin} and does not depend on the actual path travelled by M. Consequently, an elastic force is also a *conservative force*.

(3) Work Done by Friction. Consider a particle moving on a rough surface (Fig. 256) or a rough curve. The magnitude of the frictional force acting on the particle is fN, where f is the coefficient of friction and N is the normal reaction of the surface. Frictional force is directed opposite to the displacement of the particle, whence $F_{\text{fr}\,\tau} = -fN$, and from Eq. (38),

$$A_{(M_0M_1)} = -\int_{(M_0)}^{(M_1)} F_{\text{fr}}\,ds = -\int_{(M_0)}^{(M_1)} fN\,ds.$$

If the friction force is constant, then $A_{(M_0M_1)} = -F_{\text{fr}}s$, where s is the length of the arc M_0M_1 along which the particle moves.

Thus, *the work done by kinetic friction is always negative*. It depends on the length of the arc M_0M_1, and consequently *friction is a non-conservative force*.

(4)* Work Done by a Gravitational Force. Treating the earth (a planet) as a homogeneous sphere (or a sphere of homogeneous concentric layers), a particle M of mass m above or on the surface is subject to a gravitational (attractive) force $\boldsymbol{F}$ directed towards the centre O (Fig. 257) and varies as the inverse square of distance $\boldsymbol{r}$ from point M to O:

$$F = k\frac{m}{r^2}.$$

The coefficient k can be determined from the condition that at the surface of the earth ($r = R$, where R is the radius of the earth) the

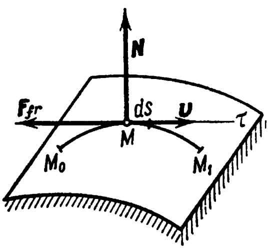

Fig. 256

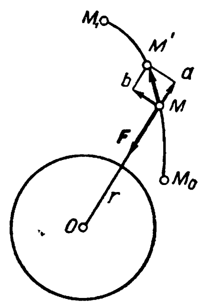

Fig. 257

force of attraction equals mg, where g is the acceleration of gravity at the surface of the earth, in which case

$$mg = k\frac{m}{R^2}, \quad \text{or} \quad k = gR^2.$$

Let us first calculate the elementary work done by force $\boldsymbol{F}$. It will be observed from Fig. 257 that the elementary displacement $\overline{MM'}$ of point M can be resolved into a displacement $\overline{Ma}$, equal in magnitude to the increment dr of the distance $OM = r$ and directed along OM, and a displacement $\overline{Mb}$ perpendicular to OM and, consequently, to force $\boldsymbol{F}$ (see also § 71, Fig. 155). As the work done by force $\boldsymbol{F}$ in the latter displacement is zero and the displacement $\overline{Ma}$ is in the opposite direction of the force,

$$dA = -F\,dr = -k\frac{m}{r^2}\,dr. \tag{41}$$

Assume now that the particle moves from point M_0, for which $r = r_0$, to point M_1, where $r = r_1$. Then,

$$A_{(M_0M_1)} = \int_{(M_0)}^{(M_1)} dA = -km \int_{r_0}^{r_1} \frac{dr}{r^2} = km \int_{r_0}^{r_1} d\left(\frac{1}{r}\right),$$

and, finally,

$$A_{(M_0M_1)} = km\left(\frac{1}{r_1} - \frac{1}{r_0}\right) = mgR^2\left(\frac{1}{r_1} - \frac{1}{r_0}\right). \tag{41'}$$

The work is positive when $r_0 > r_1$, i.e., when the particle's final position is closer to the surface than the initial, and negative when $r_0 < r_1$. As Eq. (41′) shows, the work done by a gravitational force does not depend on the particle's path. The gravitational force is, therefore, *a conservative force.*

§ 114. Theorem of the Change in the Kinetic Energy of a Particle

Consider a particle of mass m displaced by acting forces from a position M_0 where its velocity is v_0 to a position M_1 where its velocity is v_1.

To obtain the required relation, consider the equation $m\boldsymbol{w} = \sum \boldsymbol{F}_k$, which expresses the fundamental law of dynamics. Projecting this equation on the tangent $M\tau$ to the path of the particle in the direction of motion, we obtain:

$$mw_\tau = \sum F_{k\tau}.$$

The tangential acceleration in the left side of the equation can be written in the form

$$w_\tau = \frac{dv}{dt} = \frac{dv}{ds}\frac{ds}{dt} = \frac{dv}{ds}v,$$

whence, we have:

$$mv\frac{dv}{ds} = \sum F_{k\tau}.$$

Multiplying both sides of the equation by ds, bring m under the differential sign. Then, noting that $F_{k\tau}\,ds = dA_k$, where dA_k is the elementary work done by the force $\boldsymbol{F}_k$, we obtain an expression of the **theorem of the change in kinetic energy in differential form:**

$$d\left(\frac{mv^2}{2}\right) = \sum dA_k.$$

Integrating both parts in the limits of corresponding values of the variables at points M_0 and M_1, we finally obtain:

$$\frac{mv_1^2}{2} - \frac{mv_0^2}{2} = \sum A_{(M_0M_1)}. \tag{42}$$

Eq. (42) states the **theorem of the change in the kinetic energy of a particle in the final form:** *The change in the kinetic energy of a particle in any displacement is equal to the algebraic sum of the work done by all the forces acting on the particle in the same displacement.*

The Case of Constrained Motion. If the motion of a particle is constrained, then, from Eq. (5), the right side of Eq. (42) will include the work done by the given (active) forces F_k^a and the work done by the reaction force of the constraint. Let us limit ourselves to the case of a particle moving on a fixed smooth surface or curve. In this case the reaction N (see Fig. 256) is normal to the path of the particle, and $N_\tau = 0$. Then by Eq. (38), *the work done by the reaction force of a fixed smooth surface (or curve) in any displacement of a particle is zero*, and from Eq. (42) we obtain:

$$\frac{mv_1^2}{2} - \frac{mv_0^2}{2} = \sum A^a_{(M_0M_1)}. \qquad (42')$$

Thus, *in a displacement of a particle on a fixed smooth surface (or curve) the change in the kinetic energy of the particle is equal to the sum of the work done in this displacement by the active forces applied to that particle.*

If the surface (curve) is not smooth, the work done by frictional force will be added to the work done by the active forces (see § 113). If the surface (curve) is itself in motion, the absolute displacement of the particle M may not be perpendicular to N and the work done by the reaction N will not be zero (for instance, the work done by the reaction of the floor of an elevator).

§ 115. Solution of Problems

The first thing in approaching any problem is to see whether one of the above theorems can be directly applied to solve it. The following considerations should be borne in mind.

The **theorem of the change in the momentum** of a particle is conveniently used to solve problems in which:

(a) the acting forces are constant or depend only on time;

(b) the given and required quantities include the acting forces, the duration of the motion, and the initial and final velocities of the particle (i.e., F, t, v_0, and v_1).

The **theorem of the change in the kinetic energy** of a particle is conveniently used to solve problems in which:

(a) the acting forces are constant or depend only on the distance;

(b) the given and required quantities include the acting forces, the displacement of the particle and the velocities at the beginning and the end of the displacement (i.e., F, s, v_0, and v_1).

Both theorems can be combined to solve problems in which both the time of motion and the displacement of a particle are given (or required).

If the acting forces include a force depending on the velocity, the principal problem of dynamics cannot be solved with the help of any of the general theorems (it is impossible immediately to calculate the work done by, or the impulse of, a force). In this case the method of integration of differential equations should be used (Chapter 17).

The following steps for the solution may be suggested:

(1) From the statement of the problem determine the theorem which can be used for its solution.

(2) Depict the moving particle in an arbitrary position together with all the active forces and the reactions of the constraints (in the case of constrained motion).

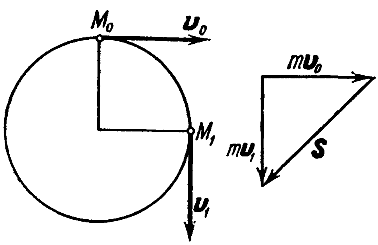

Fig. 258

(3) Calculate with the corresponding formulas the impulses or work done by all the forces during the motion.

(4) Applying Eq. (34) or (42), write the appropriate equations and find the required quantities. In the computations special attention should be paid to expressing all the quantities in the same system of units.

The theorems proved above are also convenient to determine the impulses or the work done by the forces acting on a particle from the change in its momentum or kinetic energy (the first problem of dynamics).

Problem 102. A load of weight $P = 0.1$ kgf moves in a circle with a constant velocity $v = 2$ m/s. Determine the impulse and the work done by the force acting on the load during the time the load takes to travel one quarter of the circle.

Solution. From the theorem of the change in momentum,

$$\boldsymbol{S} = m\boldsymbol{v}_1 - m\boldsymbol{v}_0.$$

Constructing geometrically the difference between these momenta (Fig. 258), we find from the right-angled triangle:

$$S = m\sqrt{v_1^2 + v_0^2}.$$

But from the conditions of the problem $v_1 = v_0 = v$, consequently

$$S = \frac{P}{g} v\sqrt{2} = 0.029 \text{ kgf-s}.$$

From Eq. (42) we determine the work:

$$A_{(M_0M_1)}=\frac{m}{2}(v_1^2-v_0^2)=0.$$

Problem 103. A load of mass m lying on a horizontal plane is suddenly pushed and imparted an initial velocity v_0. The motion of the load is then retarded by a constant force F. Determine the time it takes the load to stop and how far it will have travelled.

Solution. From the conditions of the problem we see that the first of our theorems can be used to calculate the time of the motion and the second to determine the distance.

Depict the load in an arbitrary position M (Fig. 259), M_0 and M_1 being its initial and final positions. Acting on the load are its weight P, the reaction of the plane N, and the opposing force F. Pointing axis Ox in the direction of the motion, we have from Eq. (34):

$$mv_{1x}-mv_{0x}=\sum S_x.$$

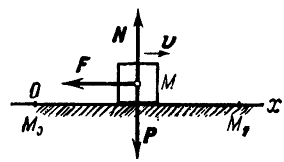

Fig. 259

In this case $v_{1x}=0$ (v_1 is the velocity at the instant when the load stops) and $v_{0x}=$ $=v_0$. Force F is the only one projected on axis x. As it is constant, $S_x=F_xt_1=-Ft_1$, where t_1 is the braking time. Substituting these expressions into our equation, we obtain $-mv_0=-Ft_1$, whence the required time is

$$t_1=\frac{mv_0}{F}. \tag{a}$$

To determine the braking distance we use the theorem of the change in the kinetic energy:

$$\frac{mv_1^2}{2}-\frac{mv_0^2}{2}=\sum A_{(M_0M_1)}.$$

Here again $v_1=0$ and only force F does any work: $A(F)=-Fs$, where s is the braking distance. The work done by forces P and N is zero as they are perpendicular to the displacement. Hence we obtain $-\frac{mv_0^2}{2}=-Fs_1$, and the braking distance is

$$s_1=\frac{mv_0^2}{2F}. \tag{b}$$

We see from equations (a) and (b) that the braking time for a given force F is proportional to the initial velocity v_0, and the braking distance is proportional to the square of the initial velocity.

If the braking force is a frictional force and the coefficient of friction f is known, then $F=fP=fmg$, and equations (a) and (b) give

$$t_1=\frac{v_0}{fg},\quad s_1=\frac{v_0^2}{2fg}.$$

Note that if we were given the initial momentum $Q_0 = mv_0$ (for example, $Q_0 = 2$ kg-m/s) instead of the mass m and initial velocity v_0, knowing F we could find the braking time from equation (a), but the data would be insufficient to determine the braking distance s_1.

Conversely, if we knew the initial kinetic energy of the load $T_0 = \frac{mv_0^2}{2}$ and the force F, we could determine the braking distance s_1 from formula (b), but we would not be able to determine the braking time t_1. This was mentioned in § 109.

Problem 104. The resultant $\boldsymbol{R}$ of all the forces acting on the piston in Fig. 260 changes during a certain time-interval according to the equation $R = 0.4P\ (1 - kt)$, where P is the weight of the piston,

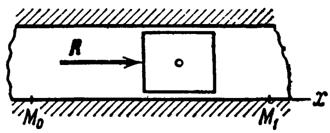

Fig. 260

t is the time in seconds, and k a factor equal to $1.6\ s^{-1}$. Determine the velocity of the piston at time $t_1 = 0.5$ s, if at time $t_0 = 0$ it was $v_0 = 0.2$ m/s.

Solution. As the acting force depends on the time and the given and required quantities include t_1, v_0, and v_1, we make use of Eq. (34):

$$mv_{1x} - mv_{0x} = S_x. \tag{a}$$

In this case

$$S_x = \int_0^{t_1} R_x\,dt = 0.4P \int_0^{t_1} (1 - kt)\,dt = 0.4Pt_1\left(1 - \frac{k}{2}t_1\right).$$

Furthermore, $v_{0x} = v_0$, $v_{1x} = v_1$, $P = mg$. Substituting these expressions into Eq. (a) and taking into account that $k = 1.6\ s^{-1}$ and $t_1 = 0.5$ s, we obtain

$$v_1 = v_0 + 0.4\,gt_1\left(1 - \frac{k}{2}t_1\right) \approx 1.4 \text{ m/s}.$$

Problem 105. A weight attached to a string of length l (Fig. 261a) is displaced from the vertical at an angle φ_0 and released from rest. Determine the velocity of the weight at the instant when the thread makes an angle φ with the vertical.

Solution. As the conditions of the problem include the displacement of the weight, defined by the angle through which the thread passes, and the velocities v_0 and v_1, we make use of the theorem of the change in kinetic energy:

$$\frac{mv_1^2}{2}-\frac{mv_0^2}{2}=\sum A_{(M_0M_1)}.$$

Acting on the weight is the force of gravity $\boldsymbol{P}$ and the reaction of the thread $\boldsymbol{N}$. The work done by force $\boldsymbol{N}$ is zero, as $N_\tau = 0$. For force $\boldsymbol{P}$ we have from Eq. (39): $A(\boldsymbol{P}) = Ph = mgh$. As $v_0 = 0$, we obtain $\frac{mv_1^2}{2} = mgh$, whence

$$v_1=\sqrt{2gh}.$$

This is the famous formula of Galileo. Evidently, the result is the same for the velocity v of a freely falling weight (Fig. 261*b*).

In our problem $h = l\cos\varphi - l\cos\varphi_0$, and finally

$$v_1=\sqrt{2gl(\cos\varphi-\cos\varphi_0)}.$$

Problem 106. The length l_0 of an uncompressed valve spring is 6 cm. When the valve is completely open, it is lifted to a height

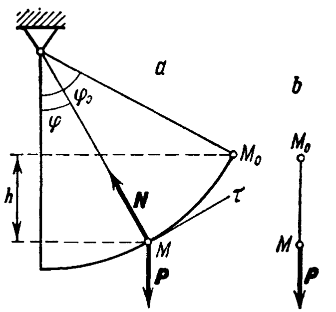

Fig 261

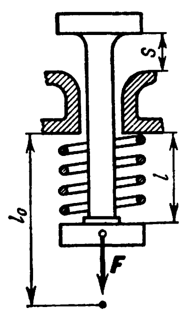

Fig. 262

$s = 0.6$ cm and the length of the compressed spring is $l = 4$ cm (Fig. 262). The stiffness of the spring is $c = 0.1$ kgf/cm and the weight of the valve is $P = 0.4$ kgf. Neglecting the gravitational and resisting forces, determine the velocity of the valve at the moment of its closure.

Solution. The elastic force $\boldsymbol{F}$ acting on the valve depends on the displacement s of the valve, which is given. Therefore, we use Eq. (42):

$$\frac{mv_1^2}{2}-\frac{mv_0^2}{2}=A_{(M_0M_1)}.$$

According to the conditions of the problem, the only force doing work is the elastic force of the spring. Hence, from Eq. (40), we have:

$$A_{(M_0M_1)}=\frac{c}{2}\left[(\Delta l_{\text{in}})^2-(\Delta l_{\text{fin}})^2\right].$$

In our case

$$\Delta l_{in} = l_0 - l = 2 \text{ cm}, \quad \Delta l_{fin} = l_0 - l - s = 1.4 \text{ cm}.$$

Furthermore, $v_0 = 0$ and $m = \frac{P}{g}$. Substituting these expressions into Eq. (42), we obtain:

$$v_1 = \sqrt{\frac{cg}{P}[(\Delta l_{in})^2 - (\Delta l_{fin})^2]} \approx 0.22 \text{ m/s}.$$

In the computations take care of the dimensions (as Δl is in centimetres, $g = 980$ cm/s²).

Problem 107. An elastic beam supporting a weight at the centre of its span (Fig. 263) deflects by an amount δ_{st} (the static deflection of the beam). Neglecting the weight of the beam, determine its

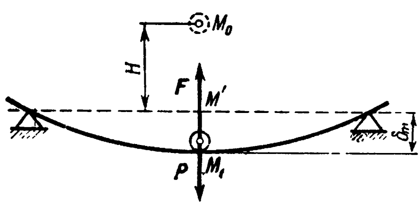

Fig. 263

maximum deflection δ_{max} if the weight is dropped on it from a height H.

Solution. As in the previous problem, we apply Eq. (42). The initial velocity v_0 and the final velocity v_1 (at the instant of the maximum deflection of the beam) of the weight are each zero, and Eq. (42) takes the form

$$\sum A_k = 0. \tag{a}$$

The forces doing work are the force of gravity $\boldsymbol{P}$ in the displacement M_0M_1 and the elastic force $\boldsymbol{F}$ of the beam in the displacement $M'M_1$. Also, $A(\boldsymbol{P}) = P(H + \delta_{max})$, $A(\boldsymbol{F}) = -\frac{c}{2}\delta_{max}^2$, as for the beam $\Delta l_{in} = 0$, $\Delta l_{fin} = \delta_{max}$. Substituting these expressions into equation (a), we obtain:

$$P(H + \delta_{max}) - \frac{c}{2}\delta_{max}^2 = 0.$$

But when the weight on the beam is in equilibrium, the force of gravity is balanced by the elastic force. Consequently, $P = c\delta_{st}$, and the last equation can be written in the form

$$\delta_{max}^2 - 2\delta_{st}\delta_{max} - 2\delta_{st}H = 0.$$

Solving this quadratic equation and taking into account that according to the conditions of the problem $\delta_{max} > 0$, we find:

$$\delta_{max} = \delta_{st} + \sqrt{\delta_{st}^2 + 2H\delta_{st}}.$$

It is interesting to note that at $H = 0$ we have $\delta_{max} = 2\delta_{st}$. Thus, if the weight is placed at the centre of a horizontal beam, the maximum deflection in the downward motion of the weight will be double the static deflection. The beam will vibrate together with the weight about the position of equilibrium. Resisting forces will damp the vibrations till the system is balanced at the position when the deflection of the beam is equal to δ_{st}.

Problem 108. Determine the initial velocity, v_0, that should be imparted to a body projected vertically up from the surface of the earth for it to reach a given altitude H. Assume the gravitational force to be changing inversely to the square of the distance from the centre of the earth and neglect the resistance of the air.

Solution. This problem is solved by applying the theorem of the change in kinetic energy, the body being considered as a particle:

$$\frac{mv_1^2}{2} - \frac{mv_0^2}{2} = A_{(M_0M_1)}, \qquad \text{(a)}$$

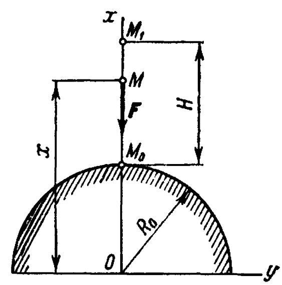

Fig. 264

where m is the mass of the particle.

Let us compute the work done by force F not using Eq. (41′). Place the origin of a coordinate system in the centre of the earth (the centre of attraction) and direct axis x in the direction of the motion (Fig. 264). Depict the moving particle M in an arbitrary position and the force F applied to it. From the conditions of the problem,

$$F = \frac{km}{x^2},$$

where k is a scalar factor.

To determine k, note that when the particle is on the surface of the earth ($x = R_0$, where R_0 is the radius of the earth), the gravitational force is mg_0, where g_0 is the acceleration of gravity at the surface of the earth. Hence, $mg_0 = km/R_0^2$, or $k = R_0^2 g_0$.

Noting that

$$F_x = -F = -\frac{mk}{x^2}, \quad F_y = F_z = 0,$$

whence, according to Eq. (38′), we obtain:

$$A_{(M_0M_1)} = \int\limits_{(M_0)}^{(M_1)} \left(-\frac{km}{x^2}\right) dx = -km \int\limits_{R_0}^{R_0+H} \frac{dx}{x^2}.$$

Integrating, we find:

$$A_{(M_0M_1)} = km\left(\frac{1}{R_0+H} - \frac{1}{R_0}\right) = -\frac{kmH}{R_0(R_0+H)}.$$

This result may be immediately obtained from Eq. (41′) if we take into account that in our case $r_0 = R_0$ and $r_1 = R_0 + H$.

As in its highest position M_1 the velocity $v_1 = 0$, by substituting these expressions of the work and of k in equation (a) we finally obtain:

$$v_0 = \sqrt{\frac{2g_0R_0H}{R_0+H}}.$$

Let us consider some special cases.

(a) If H is very small as compared with R_0, H/R_0 is very small. Dividing the numerator and denominator by R_0, we obtain,

$$v_0 = \sqrt{\frac{2g_0H}{1+\frac{H}{R_0}}} \approx \sqrt{2g_0H}.$$

Thus, at small values of H we obtain Galileo's formula.

(b) Let us find the initial velocity which will carry a projectile into infinity. Dividing the numerator and denominator by H, we obtain:

$$v_0 = \sqrt{\frac{2g_0R_0}{1+\frac{R_0}{H}}}.$$

At $H = \infty$ and assuming the mean radius of the earth to be $R_0 \approx 6\,370$ km, we obtain:

$$v_0 = \sqrt{2g_0R_0} \approx 11.2 \text{ km/s}.$$

Thus, a body projected from the surface of the earth with a velocity of 11.2 km/s will escape the earth's gravitational field forever.

It can be calculated that at initial velocities lying approximately within the limits 8 km/s $\leqslant v_0 \leqslant$ 11 km/s a body projected at a tangent to the earth's surface will not fall back and will become an artificial satellite. At initial velocities below 8 km/s or when a body is not projected horizontally, it will describe an elliptical trajectory and fall back to the earth. All these results refer to motion in a vacuum (see Chapter 22).

§ 116. Theorem of the Change in the Angular Momentum of a Particle (the Principle of Moments)

Of the two principal dynamic characteristics introduced in § 109, only mv is a vector quantity. Often, in analysing the motion of a particle, it is necessary to consider the change not in the vector mv itself but in its moment. The moment of the vector mv with respect to any centre O or axis z is denoted by the symbol $m_0(mv)$ or $m_z(mv)$ and is called the *moment of momentum* or *angular momentum* with respect to that centre or axis. The moment of vector mv is calculated in the same way as the moment of a force. Vector mv is considered to be applied to the moving particle. In magnitude $|m_0(mv)| = mvh$, where h is the perpendicular distance from O to the position line of the vector mv (see Fig. 265).

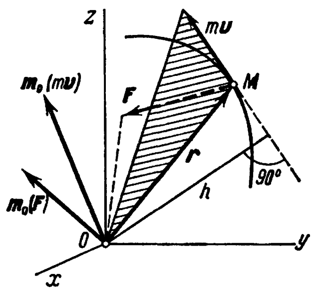

Fig. 265

(1) Principle of Moments About an Axis*). Consider a particle of mass m moving under the action of a force F. Let us establish the dependence between the moments of the vectors mv and F with respect to any fixed axis z. From the formulas of § 43,

$$m_z(F) = xF_y - yF_x. \tag{43}$$

Similarly, form $m_z(mv)$, and taking m out of the parentheses, we have:

$$m_z(mv) = m(xv_y - yv_x). \tag{43'}$$

Differentiating this equation with respect to time, we obtain:

$$\frac{d}{dt}[m_z(mv)] = m\left(\frac{dx}{dt}v_y - \frac{dy}{dt}v_x\right) + \left(xm\frac{dv_y}{dt} - ym\frac{dv_x}{dt}\right).$$

The first member in the right-hand side of the equation is zero, as $\frac{dx}{dt} = v_x$ and $\frac{dy}{dt} = v_y$. From Eq. (43), the second member is equal to $m_z(F)$, since, from the fundamental law of dynamics,

$$m\frac{dv_y}{dt} = F_y, \quad m\frac{dv_x}{dt} = F_x.$$

Finally, we have:

$$\frac{d}{dt}[m_z(mv)] = m_z(F). \tag{44}$$

*) This result can be obtained as a special case of item (2).

This equation states the **principle of moments about an axis:** *The derivative of the angular momentum of a particle about any axis with respect to time is equal to the moment of the acting force about the same axis*. An analogous theorem can be proved for moments about any centre O. Its mathematical expression is given below in Eq. (45′).

From Eq. (44) it follows that if $m_z(\boldsymbol{F}) = 0$, then $m_z(m\boldsymbol{v}) = \text{const.}$, i.e. if the moment of the acting force about an axis is zero, the angular momentum of this particle about this axis is constant in magnitude and direction.

(2)* Principle of Moments About a Centre. Let us find for a particle moving under the action of a force $\boldsymbol{F}$ (Fig. 265) the relation between the moments of vectors $m\boldsymbol{v}$ and $\boldsymbol{F}$ with respect to any fixed centre O. It was shown at the end of § 42 that $\boldsymbol{m}_0(\boldsymbol{F}) = \boldsymbol{r} \times \boldsymbol{F}$. Similarly,

$$\boldsymbol{m}_0(m\boldsymbol{v}) = \boldsymbol{r} \times m\boldsymbol{v}.$$

Vector $\boldsymbol{m}_0(\boldsymbol{F})$ is normal to the plane through O and vector $\boldsymbol{F}$, while vector $\boldsymbol{m}_0(m\boldsymbol{v})$ is normal to the plane through O and vector $m\boldsymbol{v}$. Differentiating the expression $\boldsymbol{m}_0(m\boldsymbol{v})$ with respect to time, we obtain:

$$\frac{d}{dt}(\boldsymbol{r} \times m\boldsymbol{v}) = \left(\frac{d\boldsymbol{r}}{dt} \times m\boldsymbol{v}\right) + \left(\boldsymbol{r} \times m\frac{d\boldsymbol{v}}{dt}\right) = (\boldsymbol{v} \times m\boldsymbol{v}) + (\boldsymbol{r} \times m\boldsymbol{w}).$$

But $\boldsymbol{v} \times m\boldsymbol{v} = 0$, as the vector product of two parallel vectors, and $m\boldsymbol{w} = \boldsymbol{F}$. Hence,

$$\frac{d}{dt}(\boldsymbol{r} \times m\boldsymbol{v}) = \boldsymbol{r} \times \boldsymbol{F}, \tag{45}$$

or

$$\frac{d}{dt}[\boldsymbol{m}_0(m\boldsymbol{v})] = \boldsymbol{m}_0(\boldsymbol{F}). \tag{45'}$$

This is the **principle of moments about a centre:** *The derivative of the angular momentum of a particle about any fixed centre with respect to time is equal to the moment of the force acting on the particle about the same centre*. An analogous theorem is true for the moments of vector $m\boldsymbol{v}$ and force $\boldsymbol{F}$ with respect to any axis z, which is evident if we project both sides of Eq. (45′) on that axis. This was proved directly in item (1). The mathematical statement of the theorem of moments about an axis is given in Eq. (44) above.

A comparison of Eqs. (45′) and (32) shows that the moments of vectors $m\boldsymbol{v}$ and $\boldsymbol{F}$ are linked by the same relationship as the vectors $m\boldsymbol{v}$ and $\boldsymbol{F}$ themselves.

From Eq. (45′) it follows that if $\boldsymbol{m}_0(\boldsymbol{F}) = 0$, then $\boldsymbol{m}_0(m\boldsymbol{v}) = \text{const.}$, i.e., if the moment of the acting force relative to a centre is zero, the angular momentum of this particle about the same centre is constant in magnitude and direction. This result is of great importance in the case of motion under the action of a *central force* (see § 117).

Problem 109. A bead M is attached to a string MBA whose part BA passes through a vertical pipe (Fig. 266). At the moment when the bead is at a distance h_0 from the axis z of the pipe, it receives an initial velocity v_0 perpendicular to the plane MBA. At the same time the string is slowly pulled into the pipe. Determine the velocity v_1 of the bead when its distance from axis z is h_1.

Solution. Acting on the bead are the force of gravity $\boldsymbol{P}$ and the reaction of the string $\boldsymbol{T}$. The moments of these forces about axis z, which is coincident with the pipe, are zero. Then, from Eq. (44), we have

$$\frac{d}{dt}[m_z(m\boldsymbol{v})]=0,$$

whence $m_z(m\boldsymbol{v}) = mvh =$ const. As the mass m is constant, it follows that during the motion of the bead $v_0h_0 = v_1h_1$, whence

$$v_1=\frac{h_0}{h_1}v_0.$$

The closer the bead approaches the rod, the greater is its velocity.

§ 117*. Motion Under the Action of a Central Force. Law of Areas

A central force is one whose line of action always passes through a given centre O. An example of a central force is the force with which the planets are attracted by the sun or a satellite by the earth.

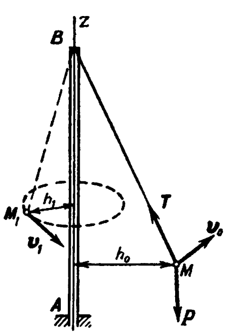

Fig. 266

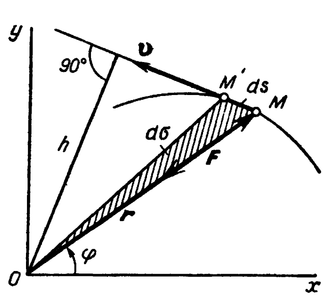

Fig. 267

Using Eq. (45′), let us investigate how a particle M will move under the action of a central force $\boldsymbol{F}$ (Fig. 267). As in this case $\boldsymbol{m}_0(\boldsymbol{F}) = 0$, we have $\boldsymbol{m}_0(m\boldsymbol{v}) = \boldsymbol{r}\times m\boldsymbol{v} =$ const., or, as the mass m is constant, $\boldsymbol{m}_0(\boldsymbol{v}) = \boldsymbol{r}\times\boldsymbol{v} =$ const., i.e., the vector $\boldsymbol{m}_0(\boldsymbol{v})$ is constant both in magnitude and direction. Recalling that vector $\boldsymbol{m}_0(\boldsymbol{v}) = \boldsymbol{r} \times \boldsymbol{v}$ is perpendicular to the plane through vectors $\boldsymbol{r}$ and $\boldsymbol{v}$, if vector $\boldsymbol{r} \times \boldsymbol{v}$ is constant in direction, the radius vector $\boldsymbol{r} = \overline{OM}$ of particle M and its velocity vector $\boldsymbol{v}$ remain in the same plane. Hence, the path of particle M is a *plane curve*. Also, $|\boldsymbol{m}_0(\boldsymbol{v})| = vh = \text{const.}$

Thus, a particle subjected to the action of a central force moves in a plane curve; its velocity $\boldsymbol{v}$ changes in such a way that the moment of vector $\boldsymbol{v}$ about the centre O is constant ($vh = \text{const.}$).

This result has a graphic interpretation. As $vh = \frac{ds}{dt}\cdot h$ and $ds\cdot h = 2d\sigma$, where $d\sigma$ is the area of the little triangle OMM', it follows that $vh = 2\frac{d\sigma}{dt}$. The quantity $d\sigma/dt$ represents the rate of increase of the area swept out by the radius vector OM in the motion of the particle M and is called the *areal velocity*. In the present case this velocity is constant

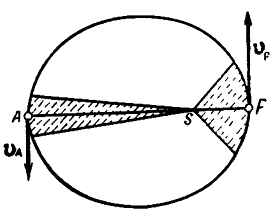

Fig. 268

$$\frac{d\sigma}{dt} = \frac{1}{2}|\boldsymbol{m}_0(\boldsymbol{v})| = \text{const.} \qquad (46)$$

Thus, *in motion under the action of a central force the radius vector of a particle sweeps out equal areas in equal intervals of time* (the Law of Areas). This applies to planetary motion and is one of Kepler's laws.

Example. The orbit of a planet moving under the force of the sun's gravity is an ellipse, with the sun located in one of the focuses S (Fig. 268). As gravity is a central force, the law of areas applies. Therefore, at the point of closest approach to the sun, P (the *perihelion*), the velocity v_P of the planet is greatest, and at its farthest point A (the *aphelion*) the velocity v_A is smallest. This follows from Eq. (46), which for points A and P yields $v_A \cdot AS = v_P \cdot PS$. The same result can be obtained when it is remembered that the areas of the shaded sectors in Fig. 268 swept out in equal times must be equal; consequently, in the same time the planet travels a longer path in the neighbourhood of point P than in the neighbourhood of point A.

The same conclusions hold for satellite motion.

Chapter 19
Constrained Motion of a Particle

§ 118. Equations of Motion of a Particle Along a Given Fixed Curve

We have seen that some problems of constrained motion can be solved with the help of the general theorems of dynamics. Let us consider another method of solving such problems, which can be used for any acting forces and which makes it possible to find both the equation of motion of a particle and the reactions of its constraints.

Let particle M in Fig. 269 be moving along a given smooth curve under the action of the applied forces $\boldsymbol{F}_1^a, \boldsymbol{F}_2^a, \ldots, \boldsymbol{F}_n^a$, let O be the origin of a frame of reference lying on the curve, and let the motion of the particle be described by the arc-coordinate $s = \widehat{OM}$ (see § 59). If we replace the constraint by its reaction $\boldsymbol{N}$, the fundamental law of dynamics will take the form (see § 102):

$$m\boldsymbol{w} = \sum \boldsymbol{F}_k^a + \boldsymbol{N}. \tag{47}$$

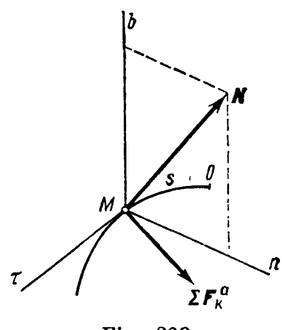

Fig. 269

Let us draw through point M the *tangent* $M\tau$ (in the positive direction of s), the *principal normal* Mn (inward to the curve), and an axis Mb perpendicular to them, which is called the *binormal*, and project both sides of Eq. (47) on these axes. As the curve is smooth, the reaction $\boldsymbol{N}$ is perpendicular to it, i.e., lies in the plane Mbn, and consequently $\boldsymbol{N}_\tau = 0$. Thus we have:

$$mw_\tau = \sum F_{k\tau}^a, \quad mw_n = \sum F_{kn}^a + N_n, \quad mw_b = \sum F_{kb}^a + N_b.$$

But $w_\tau = \frac{dv}{dt} = \frac{d^2s}{dt^2}$, $w_n = \frac{v^2}{\rho}$, and $w_b = 0$, as the acceleration vector $\boldsymbol{w}$ lies in the osculating plane $M\tau n$. This gives us the following *differential equations of motion of a particle along a given curve*:

$$m\frac{dv}{dt}=\sum F^{a}_{k\tau}, \quad \text{or} \quad m\frac{d^2s}{dt^2}=\sum F^{a}_{k\tau}; \tag{48}$$

$$\frac{mv^2}{\rho}=\sum F^{a}_{kn}+N_n, \quad 0=\sum F^{a}_{kb}+N_b. \tag{49}$$

These equations can be used to solve the two problems of constrained motion mentioned in § 102.

Eq. (48), which does not contain the unknown reaction N, is convenient for developing the equation of motion of a particle along a curve, i.e., the relation $s = f(t)$. Eqs. (49) are used to determine the reaction of the constraint (see § 119).

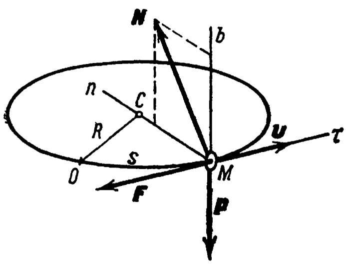

Fig. 270

These equations can also be used if the curve is not smooth, in which case the frictional force should be added to the forces F^{a}_{k}.

Eqs. (48) and (49) also hold for the motion of a free particle, in which case we assume $N = 0$.

Problem 110. A heavy ring M sliding on a horizontal wire circle receives an initial velocity v_0 tangent to the circle (Fig. 270). Acting on the ring is a resisting force $F = km\sqrt{v}$, where m is the mass of the ring, v is its velocity, and k is a constant factor. Determine when the ring will come to a halt.

Solution. Placing the origin O of the frame of reference at the initial position of the ring, we draw the ring in an arbitrary position and the axes $M\tau$, Mn, and Mb. Acting on the ring are the force of gravity P, the reaction N, and the frictional force F. Writing Eq. (48) and taking into account that $P_\tau = N_\tau = 0$ and $F_\tau = -F = -km\sqrt{v}$, we obtain:

$$m\frac{dv}{dt}=-km\sqrt{v}.$$

Whence, separating the variables and taking into account that $v = v_0$ at $t = 0$, we have:

$$\int_{v_0}^{v}\frac{dv}{\sqrt{v}}=-k\int_{0}^{t}dt, \quad \text{or} \quad 2\left(\sqrt{v}-\sqrt{v_0}\right)=-kt.$$

At the time $t = t_1$, when the ring comes to a halt, $v = 0$. Hence, assuming in the obtained equation $v = 0$, we have:

$$t_1=\frac{2\sqrt{v_0}}{k}.$$

For the given law of resistance the duration of the motion until complete rest is finite (see Problem 100 in § 105).

Problem 111. In the preceding problem determine the distance s_1 the ring will travel along the circle before it stops if acting on it instead of the resisting force dependent on velocity is the force of friction $F = fN$. Given: radius of the circle $R = 0.3$ m, initial velocity $v_0 = 2$ m/s, coefficient of friction of the ring on the circle $f = 0.3$.

Solution. Placing the origin O of the frame of reference and drawing the axes $M\tau$, Mn, and Mb as in the previous problem (see Fig. 270), the forces acting on the ring are $\boldsymbol{P}$, $\boldsymbol{N}$, and $\boldsymbol{F}$, where $\boldsymbol{F}$ is now the force of friction. Writing Eqs. (48) and (49), we obtain:

$$m\frac{dv}{dt} = -F, \quad \frac{mv^2}{R} = N_n, \quad N_b - P = 0.$$

In magnitude $F = fN = f\sqrt{N_b^2 + N_n^2}$ (it would be wrong to compute the friction as the arithmetical sum of forces fN_b and fN_n). Noting that $N_b = P = mg$, we find:

$$F = fm\sqrt{g^2 + \frac{v^4}{R^2}}.$$

We see that the friction depends, through the reaction N, on the velocity of the ring. That is why this problem cannot be solved by applying the theorem of the change in kinetic energy.

In order to find immediately the dependence of s on v, note that $\frac{dv}{dt} = \frac{dv}{ds}\frac{ds}{dt} = \frac{dv}{ds}v$. Then, eliminating m, the equation of motion of the ring takes the form

$$v\frac{dv}{ds} = -\frac{f}{R}\sqrt{g^2R^2 + v^4}.$$

Separating the variables and evaluating both sides of the equation for the respective definite integrals, we have:

$$\int_{v_0}^{v} \frac{d(v^2)}{\sqrt{g^2R^2 + v^4}} = -\frac{2f}{R}\int_0^s ds,$$

whence

$$-\frac{2f}{R}s = \ln\left(v^2 + \sqrt{g^2R^2 + v^4}\right) - \ln\left(v_0^2 + \sqrt{g^2R^2 + v_0^4}\right),$$

and finally

$$s = \frac{R}{2f}\ln\frac{v_0^2 + \sqrt{g^2R^2 + v_0^4}}{v^2 + \sqrt{g^2R^2 + v^4}}.$$

When the ring stops, $v = 0$. Thus, assuming $g \approx 10$ m/s², the required path is

$$s_1 = \frac{R}{2f}\ln\frac{v_0^2 + \sqrt{v_0^4 + g^2R^2}}{gR} \approx \frac{1}{2}\ln 3 \approx 0.55 \text{ m}.$$

§ 119. Determination of the Reactions of Constraints

The reaction of the constraint when a particle moves along a given curve is found with the help of Eq. (49). The moving particle should be drawn in the position for which the reaction is to be determined. If the velocity v in Eq. (49) is not immediately known, it can in some cases be found from the theorem of the change in kinetic energy (§ 114).

From Eq. (49) it can be seen that in curvilinear motion the dynamic reaction $\boldsymbol{N}$, unlike static reaction, depends not only on the

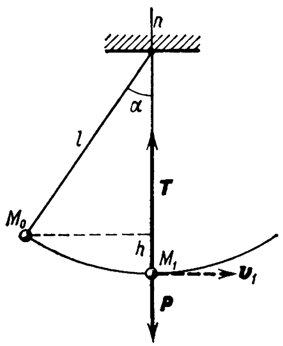

Fig. 271

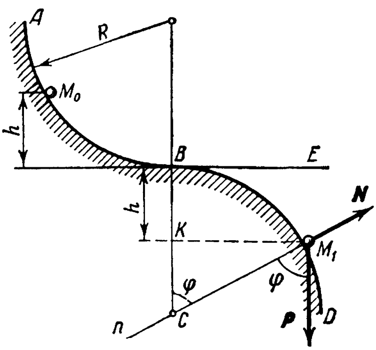

Fig. 272

applied active forces and the type of constraint but also on the velocity of the motion.

Problem 112. A load of weight P attached to a string of length l is displaced through an angle α from the vertical to a position M_0 and released from rest (Fig. 271). Determine the tension in the thread when the load is in its lowest position M_1.

Solution. Depict the load in the position for which the tension in the string has to be found, i.e., in position M_1. Acting on the load is its weight $\boldsymbol{P}$ and the reaction of the thread $\boldsymbol{T}$. Draw the inward normal M_1n and write Eq. (49), taking into account that in the present case $\rho = l$. We have:

$$\frac{mv_1^2}{l} = T - P, \quad \text{or} \quad T = P + \frac{mv_1^2}{l},$$

where v_1 is the velocity of the load at position M_1. To determine v_1, we make use of Eq. (42'):

$$\frac{mv_1^2}{2} - \frac{mv_0^2}{2} = A^{a}_{(M_0M_1)}. \tag{a}$$

On the section M_0M_1 only force P does any work. Therefore, $A^{a} = Ph = Pl\ (1 - \cos\alpha)$.

As $v_0 = 0$, substituting the expression for the work into equation (a), we obtain $mv_1^2 = 2Pl\ (1 - \cos\alpha)$, and finally,

$$T = P\ (3 - 2\cos\alpha).$$

In the special case, when the initial angle of deflection is 90°, the tension in the string when it is in vertical position will be $3P$, i.e., treble the weight of the load.

The result shows that dynamic reactions can differ considerably from static reactions.

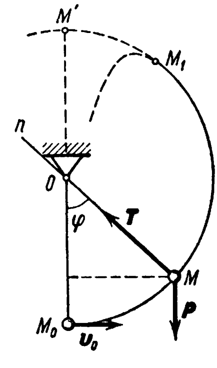

Fig. 273

Problem 113. A grooved track makes two circular arcs AB and BD of radius R in a vertical plane; the tangent BE through their point of conjugation is horizontal (Fig. 272). Neglecting friction, determine the height h from BE at which a heavy ball should be placed on the track so that it would shoot out of the track at point M_1 lying at the same distance h below BE.

Solution. The ball will leave the track at a point M_1 such that its pressure on the track (or the reaction N of the track) is zero. Consequently, our problem is reduced to the determination of N. Draw the ball at M_1. Acting on it are the force of gravity $\boldsymbol{P}$ and the reaction of the track $\boldsymbol{N}$. Writing Eq. (49) for the projection on the *inward* normal M_1n, we have:

$$\frac{mv_1^2}{R} = P\cos\varphi - N.$$

Since at the point of departure $N = 0$ and taking into account that $R\cos\varphi = KC = R - h$, we obtain the equation for determining h:

$$mv_1^2 = P\,(R - h). \tag{a}$$

The value of mv_1^2 can be found from the theorem of the change in kinetic energy. As $v_0 = 0$, Eq. (42') gives

$$\frac{mv_1^2}{2} = A^{a}_{(M_0M_1)}.$$

The only force that does work is $\boldsymbol{P}$, and $A\ (\boldsymbol{P}) = P2h$. Consequently $mv_1^2 = 4Ph$. Substituting this expression of mv_1^2 into equation (a), we obtain $4h = R - h$, whence

$$h = 0.2\ R.$$

Problem 114. A load M is attached to a string of length l (Fig. 273). What is the least initial velocity v_0 perpendicular to the string that should be imparted to the load for it to describe a complete circle?

Solution. The load will describe a complete circle if nowhere along its path (except, possibly, point M') will the tension in the string become zero, i.e., if the string remains taut. If, on the other hand, at any point M_1 where $v_1 \neq 0$ the tension becomes zero, the string will no longer constrain the load, which will continue to move as a free body (along a parabola).

To solve the problem we must determine the tension T in the string at any point M defined by angle φ and then require that $T > 0$ for any angle $\varphi \neq 180°$.

Acting on the load at point M are its weight $\boldsymbol{P}$ and the tension of the thread $\boldsymbol{T}$. Writing Eq. (49) for the projections on the inward normal Mn, we obtain:

$$\frac{mv^2}{l} = T - P\cos\varphi, \tag{a}$$

where v is the velocity of the load at point M. To determine v we apply the theorem of the change in kinetic energy:

$$\frac{mv^2}{2} - \frac{mv_0^2}{2} = A^{\text{a}}_{(M_0M)}.$$

In our case $A^{\text{a}}_{(M_0M)} = -Ph = -Pl\,(1 - \cos\varphi)$, and consequently,

$$mv^2 = mv_0^2 - 2Pl\,(1 - \cos\varphi).$$

Substituting this expression of mv^2 into equation (a) and solving it for T, we obtain

$$T = P\left(\frac{v_0^2}{gl} - 2 + 3\cos\varphi\right).$$

The least value of T is at $\varphi = 180°$;

$$T_{\min} = P\left(\frac{v_0^2}{gl} - 5\right).$$

The condition for T never to become zero (except, possibly, at point M') is that $T_{\min} \geqslant 0$. Hence

$$\frac{v_0^2}{gl} \geqslant 5 \quad \text{or} \quad v_0 \geqslant \sqrt{5gl}.$$

Thus, the least initial velocity at which the load will describe a complete circle is given by the equation

$$v_{0\,\min} = \sqrt{5gl}.$$

Let us assume that the load is attached not to a thread but to a rigid light (weightless) rod of length l. In this case (since, unlike a thread, a rod can work both in tension and in compression) the load will describe a complete circle if the velocity does not become zero anywhere (except, possibly, at point M'). Applying Eq. (42') for the displacement M_0M' and assuming $v = 0$ at point M', we obtain $-\frac{mv_0^2}{2} = -mg \cdot 2l$. Hence

$$v_{0\,\min} = \sqrt{4gl}.$$

Chapter 20
Relative Motion of a Particle

§ 120. Equations of Relative Motion and Rest of a Particle

The laws of dynamics and the equations and theorems based on them, which were obtained in the previous chapters, are valid only for so-called absolute motion of a particle, i.e., motion with respect to an inertial (fixed) reference system.

The present chapter deals with relative motion of a particle, i.e., motion with respect to non-inertial, arbitrarily moving reference systems.

Consider a particle M moving under the action of applied forces $\boldsymbol{F}_1, \boldsymbol{F}_2, \ldots, \boldsymbol{F}_n$ resulting from its interaction with other material bodies. Let us investigate the motion of this particle with respect to a set of coordinate axes $Oxyz$ (Fig. 274), which are in turn moving in some known way with respect to a set of fixed axes $O_1x_1y_1z_1$.

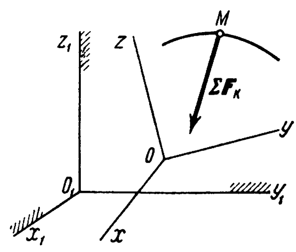

Fig. 274

Let us find the relation between the relative acceleration of the particle $\boldsymbol{w}_{\text{rel}}$ and the forces acting on it. For absolute motion the fundamental law of dynamics has the form

$$m\boldsymbol{w}_{\text{a}} = \sum \boldsymbol{F}_k. \tag{50}$$

But we know from kinematics that $\boldsymbol{w}_{\text{a}} = \boldsymbol{w}_{\text{rel}} + \boldsymbol{w}_{\text{tr}} + \boldsymbol{w}_{\text{Cor}}$, where $\boldsymbol{w}_{\text{rel}}$, $\boldsymbol{w}_{\text{tr}}$, and $\boldsymbol{w}_{\text{Cor}}$ are the relative, transport, and Coriolis accelerations of the particle. Substituting this expression of $\boldsymbol{w}_{\text{a}}$ into Eq. (50) and assuming for the future that $\boldsymbol{w}_{\text{rel}} = \boldsymbol{w}$, as this is the acceleration of the relative motion under consideration, we obtain:

$$m\boldsymbol{w} = \sum \boldsymbol{F}_k + (-m\boldsymbol{w}_{\text{tr}}) + (-m\boldsymbol{w}_{\text{Cor}}).$$

Let us introduce the following notation:

$$\boldsymbol{F}^{\text{i}}_{\text{tr}} = -m\boldsymbol{w}_{\text{tr}}, \quad \boldsymbol{F}^{\text{i}}_{\text{Cor}} = -m\boldsymbol{w}_{\text{Cor}}.$$

In dimension the quantities $\boldsymbol{F}^{\mathrm{i}}_{\mathrm{tr}}$ and $\boldsymbol{F}^{\mathrm{i}}_{\mathrm{Cor}}$ are forces, which we shall call respectively the *transport* and *Coriolis inertial forces.* Then the foregoing equation will take the form

$$m\boldsymbol{w}=\sum \boldsymbol{F}_k+\boldsymbol{F}^{\mathrm{i}}_{\mathrm{tr}}+\boldsymbol{F}^{\mathrm{i}}_{\mathrm{Cor}}. \tag{51}$$

Eq. (51) states the *Fundamental Law of Dynamics for the relative motion of a particle.* A comparison of Eqs. (50) and (51) leads us to the conclusion: *All equations and theorems of mechanics for relative motion of a particle can be written exactly like the equations of absolute motion, provided that the transport and Coriolis inertial forces are added to the forces of interaction with other bodies acting on the particle.* The addition of forces $\boldsymbol{F}^{\mathrm{i}}_{\mathrm{tr}}$ and $\boldsymbol{F}^{\mathrm{i}}_{\mathrm{Cor}}$ takes into account the effect of the displacement of the moving axes on the relative motion of the particle.

Let us consider several special cases.

(1) If the moving axes are in translatory motion, $\boldsymbol{F}^{\mathrm{i}}_{\mathrm{Cor}}=0$, as $\omega_{\mathrm{tr}}=0$ (ω is the angular velocity of rotation of the moving axes *Oxyz*), and the equation of relative motion acquires the form

$$m\boldsymbol{w}=\sum \boldsymbol{F}_k+\boldsymbol{F}^{\mathrm{i}}_{\mathrm{tr}}.$$

(2) If the moving axes are in uniform rectilinear translational motion, $\boldsymbol{F}^{\mathrm{i}}_{\mathrm{tr}}=\boldsymbol{F}^{\mathrm{i}}_{\mathrm{Cor}}=0$, and the equation of relative motion is in the same form as the equation of motion relative to fixed axes. Therefore, such a reference system is also *inertial.*

It follows from this that it is impossible to determine by mechanical experiment whether a given reference system is at rest or in uniform rectilinear translatory motion. This is the basic *principle of relativity of classical mechanics* discovered by Galileo.

(3) If a particle is at rest with respect to a moving set of axes, $\boldsymbol{w}=0$ and $\boldsymbol{v}_{\mathrm{rel}}=\boldsymbol{v}=0$, and consequently $\boldsymbol{F}^{\mathrm{i}}_{\mathrm{Cor}}=0$, since the Coriolis acceleration $w_{\mathrm{Cor}}=2\omega_{\mathrm{tr}}\,v_{\mathrm{rel}}\sin\alpha$, and Eq. (51) takes the form

$$\sum \boldsymbol{F}_k+\boldsymbol{F}^{\mathrm{i}}_{\mathrm{tr}}=0. \tag{52}$$

Eq. (52) is the *equation of relative equilibrium (rest) of a particle.* It follows from it that *equations of relative equilibrium can be written like the equations of equilibrium in fixed axes, provided that the transport inertial force is added to the forces of interaction with other bodies acting on the particle.*

(4) In developing the equations of relative motion for cases when $\boldsymbol{F}^{\mathrm{i}}_{\mathrm{Cor}}\neq 0$, it should be remembered that

$$\boldsymbol{F}^{\mathrm{i}}_{\mathrm{Cor}}=-m\boldsymbol{w}_{\mathrm{Cor}}=-2m\,(\boldsymbol{\omega}_{\mathrm{tr}}\times\boldsymbol{v}_{\mathrm{rel}}).$$

Consequently, $\boldsymbol{F}^{\mathrm{i}}_{\mathrm{Cor}}$ is *perpendicular* to $\boldsymbol{v}_{\mathrm{rel}}=\boldsymbol{v}$, and hence to the tangent to the relative path of the particle.

Therefore:

(a) the projection of the Coriolis inertial force on the tangent $M\tau$ to the relative path of a particle is always zero ($F^{i}_{\text{Cor}\,\tau}=0$) and Eq. (48) for relative motion (assuming $v = v_{\text{rel}}$) takes the form

$$m\frac{dv}{dt}=\sum F_{k\tau}+F^{i}_{\text{tr}\,\tau};\tag{53}$$

(b) the work done by the Coriolis inertial force in any relative displacement is zero [see § 112, Eq. (38)], and in relative motion the *theorem of the change in kinetic energy* takes the form (v_1 and v_0 denote relative velocities):

$$\frac{mv_1^2}{2}-\frac{mv_0^2}{2}=\sum A_k+A(\boldsymbol{F}^{i}_{\text{tr}}).\tag{54}$$

In the general case both the transport and Coriolis inertial forces will enter into all the other equations of relative motion.

Problem 115. Neglecting the mass of all the rotating parts of the centrifugal-type governor in Fig. 275 as compared with the mass

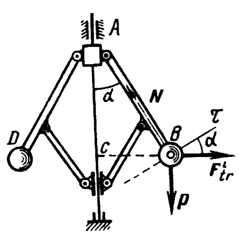

Fig. 275

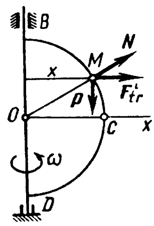

Fig. 276

of the balls B and D, determine the angle α defining the position of relative equilibrium of rod AB of length l if the governor rotates with a constant angular velocity ω.

Solution. In order to determine the position of relative equilibrium (with respect to a set of axes rotating together with the governor) add, according to Eq. (52), the transport inertia force $\boldsymbol{F}^{i}_{\text{tr}}$ to the force of gravity $\boldsymbol{P}$ and the reaction $\boldsymbol{N}$ acting on ball B. As $\omega = \text{const.}$, $w_{\text{tr}} = w^{n}_{\text{tr}} = BC\cdot\omega^2 = l\omega^2\sin\alpha$, whence $F^{i}_{\text{tr}} = ml\,\omega^2\sin\alpha$. $\boldsymbol{F}^{i}_{\text{tr}}$ is directed opposite to $\boldsymbol{w}^{n}_{\text{tr}}$, i.e., along CB. Writing the equilibrium equation for the projection on axis $B\tau$, which is perpendicular to AB, we have

$$-P\sin\alpha+F^{i}_{\text{tr}}\cos\alpha=0.$$

Hence, substituting for F^{i}_{tr} its expression and eliminating $\sin\alpha$ (not considering the solution for $\alpha = 0$), we obtain:

$$-g + l\omega^2\cos\alpha = 0,$$

whence,

$$\cos\alpha = \frac{g}{l\omega^2}.$$

As $\cos\alpha \leqslant 1$, equilibrium at $\alpha \neq 0$ is possible only when $\omega^2 > \frac{g}{l}$.

Problem 116. The semicircle BCD of radius R in Fig. 276 rotates about a vertical axis with a constant angular velocity ω. A ring M starts slipping along it without friction from a point B slightly off the axis of rotation. Determine the relative velocity v_1 of the ring at point C if its initial velocity $v_0 = 0$.

Solution. The velocity v_1 can be determined from the theorem of the change in kinetic energy. In order to write Eq. (54), which expresses the theorem, compute the work done by forces P and F^{i}_{tr}, where $F^{i}_{tr} = m\omega^2 x$ (the work done by the reaction N is zero). Assuming approximately $x_B = 0$, we obtain:

$$A_{(BC)}(F^{i}_{tr}) = \int_{(B)}^{(C)} F^{i}_{tr\,x}\,dx = m\omega^2 \int_0^R x\,dx = \frac{1}{2} m\omega^2 R^2.$$

Furthermore, $A_{(BC)}(P) = PR$. Substituting these expressions into Eq. (54) and taking into account that $v_0 = 0$, we have:

$$\frac{mv_1^2}{2} = mR\left(g + \frac{1}{2}\omega^2 R\right),$$

whence,

$$v_1 = \sqrt{2gR\left(1 + \frac{\omega^2 R}{2g}\right)}.$$

The problem can also be solved by writing Eq. (53) for the projection on the tangent $M\tau$ and then transforming its left side as in Problem 111.

An example of the integration of equations of relative motion is given in § 122.

§ 121. Effect of the Rotation of the Earth on the Equilibrium and Motion of Bodies

In solving most engineering problems we consider any reference system connected with the earth as fixed (inertial). We thereby neglect the diurnal rotation of the earth and its orbital motion about the sun. But in the case of the latter motion the corresponding transport inertial force in Eq. (51) is in effect balanced by the gravitational attraction of the sun (more on this see in § 128). It follows then that in considering a reference system connected with the earth as inertial, we neglect its diurnal rotation together with the earth relative to the stars. The velocity of this rotation is one revolution in 23 hours 56 minutes 4 seconds, i.e., its angular velocity is

$$\omega = \frac{2\pi}{86\,164} \approx 0.0000729 \text{ s}^{-1}.$$

Let us investigate the effect of this rather slow rotation on the equilibrium and motion of bodies.

(1) Relative Rest on the Surface of the Earth. Gravity. Consider a particle lying on a smooth "horizontal" plane which is motionless relative to the earth (Fig. 277). From Eq. (52), the condition of its equilibrium with respect to the earth is $\boldsymbol{F}_{gr} + \boldsymbol{N} + \boldsymbol{F}^{i}_{tr} = 0$, where $\boldsymbol{F}_{gr}$ is the force of the earth's gravitational attraction, $\boldsymbol{N}$ is the reaction of the surface, and $\boldsymbol{F}^{i}_{tr}$ is the transport inertia force. As ω = const., force $\boldsymbol{F}^{i}_{tr}$ has only a normal component, which is perpendicular to the earth's axis of rotation. Let us compound forces $\boldsymbol{F}_{gr}$ and $\boldsymbol{F}^{i}_{tr}$ and introduce the notation

$$\boldsymbol{F}_{gr} + \boldsymbol{F}^{i}_{tr} = \boldsymbol{P}.$$

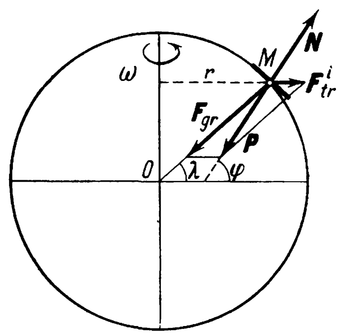

Fig. 277

Acting now on the particle M are two forces $\boldsymbol{P}$ and $\boldsymbol{N}$, which balance one another. Force $\boldsymbol{P}$ is what we call the *force of gravity*. Its direction gives the vertical at any given point of the earth's surface, and the plane normal to $\boldsymbol{P}$ is the horizontal plane. In magnitude $F^{i}_{tr} = mr\omega^2$ (where r is the distance of M from the earth's axis) and is very small in comparison with F_{gr}, since the value of ω^2 is very small. Therefore, the direction of $\boldsymbol{P}$ will differ only slightly from the direction of $\boldsymbol{F}_{gr}$ *).

When we weigh a body, we determine the force $\boldsymbol{P}$, which is the force with which the body presses on the pan of a balance. Thus, by introducing force $\boldsymbol{P}$ into our equations, we introduce the force $\boldsymbol{F}^{i}_{tr}$ as well, i.e., we actually take into account the rotation of the earth.

(2)* Relative Motion Near the Surface of the Earth. To take into account the rotation of a system of axes connected with the earth, we must add to the applied forces acting on a particle the forces $\boldsymbol{F}^{i}_{tr}$ and $\boldsymbol{F}^{i}_{Cor}$. But force $\boldsymbol{F}^{i}_{tr}$ is included in force $\boldsymbol{P}$ and is taken into account

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*) The value of F^{i}_{tr} is largest at the equator, where $r = R$ and where it is approximately 0.34% of the absolute gravitational attraction. The greatest difference between the angles λ (the geocentric latitude) and φ (the astronomical latitude) in Fig. 277 is at $\lambda = 45°$ and is approximately equal to 11′.

by the introduction of the latter into the equations of motion. Consequently, *when we consider a set of axes connected with the earth to be fixed, we actually neglect only the Coriolis inertia force*

$$F^{i}_{\text{Cor}} = 2m\omega v \sin \alpha,$$

where ω is the angular velocity of the earth and α is the angle between the relative velocity $\boldsymbol{v}$ of the particle and the earth's axis.

As ω is very small, then, if the velocity v is not very large, the force $\boldsymbol{F}^{i}_{\text{Cor}}$ can be neglected in comparison with the force of gravity. For example, even at $v = 700$ m/s (the velocity of a gun shell) and $\alpha = 90°$, F^{i}_{Cor} is only one per cent of P. Therefore, in most engineering problems involving the motion of bodies coordinate systems connected with the earth can safely be treated as inertial (fixed).

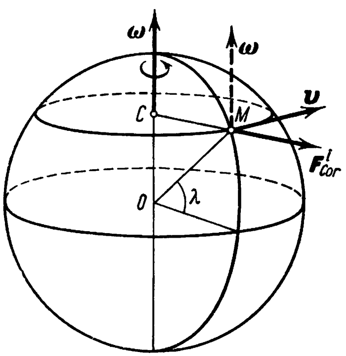

Fig. 278

The rotation of the earth becomes a factor of practical importance either at very large velocities (the flight of long-range missiles) or for motions of very long duration (river, atmospheric and ocean currents).

Let us investigate the qualitative effect the earth's rotation has on the motion of bodies.

(a) Motion on the Surface of the Earth. When a body moves down a meridian from north to south in the northern hemisphere, the Coriolis acceleration $\boldsymbol{w}_{\text{Cor}}$ is directed eastwards (§ 92, Problem 88) and the force $\boldsymbol{F}^{i}_{\text{Cor}}$ westwards. If the motion is reversed, force $\boldsymbol{F}^{i}_{\text{Cor}}$ will, evidently, be eastwardly. In both cases, we find, the force will tend to displace a particle to the right with respect to the direction of motion.

If a particle moves eastwards along a parallel, the acceleration $\boldsymbol{w}_{\text{Cor}}$ will be directed along the radius MC of that parallel (Fig. 278), force $\boldsymbol{F}^{i}_{\text{Cor}}$ pointing in the opposite direction. The vertical component of this force (along OM) will somewhat change the weight of the body, while the horizontal component will point southward, thus also deflecting the particle to the right of the direction of motion. The result is analogous if we take the motion westwardly along a parallel.

We conclude, then, that *in the northern hemisphere a body moving along the earth's surface in any direction is deflected by the rotation of the earth to the right of the direction of its motion.* In the southern hemisphere the deflection is to the left.

It is this that explains why in the Northern Hemisphere rivers undermine their right banks (Baer's Law). This, too, is what deflects steady winds (the Trade Winds) and ocean currents.

(b) Vertical Fall. In order to determine the direction of the Coriolis inertia force F^i_{Cor} acting on a freely falling particle, one must know the direction of its relative velocity v. As F^i_{Cor} is very small as compared with the force of gravity, the vector v can be considered in the first approximation to be directed vertically down, i.e., along MO in Fig. 278. In this case, apparently, the vector w_{Cor} will point westwards, and the force F^i_{Cor} eastwards (i.e., in the direction of vector v in Fig. 278). Thus, in the first approximation, *a freely falling particle (body) is deflected by the earth's rotation to the east of the vertical.* A body thrown vertically up will, evidently, be deflected towards the west. These deflections are very small and are noticeable only if the height is large enough, as is demonstrated by the calculations in § 122.

§ 122*. Deflection of a Falling Particle From the Vertical by the Earth's Rotation

Consider a particle falling from a small height H (as compared with the radius of the earth) to the surface of the earth. We shall assume that the force of gravity P is constant and neglect the

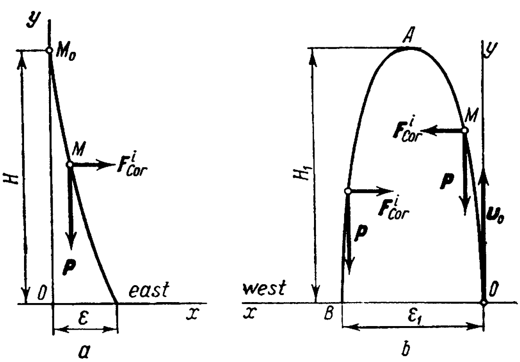

Fig. 279

resistance of the air. Direct axis Oy vertically up and axis Ox to the east (Fig. 279*a*)*). To take into account the rotation of the earth, we must add to the forces acting on the particle, besides force P that already includes force F^i_{tr}, the force F^i_{Cor}, which in the first approxi-

*) The scale in the direction of axis x in Fig. 279 is considerably enlarged.

mation is directed, as has been established, to the east. Then the differential equations of relative motion will take the form

$$m\frac{d^2x}{dt^2}=F^{\mathrm{i}}_{\mathrm{Cor}},\quad m\frac{d^2y}{dt^2}=-P=-mg, \tag{55}$$

and the initial conditions will be

$$\text{at } t=0,\ x=0,\ y=H,\ v_x=0,\ v_y=0.$$

Solving the second of Eqs. (55) and determining the constants of integration from the initial conditions, we find:

$$v_y=\frac{dy}{dt}=-gt,\quad y=H-\frac{gt^2}{2}.$$

In calculating the magnitude of $F^{\mathrm{i}}_{\mathrm{Cor}}$ we neglect, as we did previously in determining the direction of $F^{\mathrm{i}}_{\mathrm{Cor}}$, the x component of the velocity as compared with the y component (as $F^{\mathrm{i}}_{\mathrm{Cor}}$ is much less than P), and for our approximate solution we assume $v=|\,v_y\,|=gt$. The velocity v is in this case directed vertically down (along MO in Fig. 278) and makes an angle $\alpha=90°-\lambda$ with the earth's axis of rotation, where λ is the latitude. Hence, $F^{\mathrm{i}}_{\mathrm{Cor}}=2m\omega gt\cos\lambda$, and the first of Eqs. (55) takes the form

$$\frac{d^2x}{dt^2}=2\,(\omega g\cos\lambda)\,t.$$

As the quantity inside the parentheses is constant, integrating the equation, we obtain:

$$\frac{dx}{dt}=(\omega g\cos\lambda)\,t^2+C_1,$$

$$x=\frac{1}{3}(\omega g\cos\lambda)\,t^3+C_1t+C_2.$$

Substituting the initial data, we find that $C_1=C_2=0$. Thus, the equations approximately describing the relative motion of the particle are

$$x=\frac{1}{3}(\omega g\cos\lambda)\,t^3,\quad y=H-\frac{gt^2}{2}.$$

The motion is not rectilinear, and the falling particle is actually deflected to the east. Eliminating time t from the last two equations, we obtain the path of the particle in the first approximation (a semicubical parabola):

$$x^2=\frac{8}{9}\frac{\omega^2}{g}\cos^2\lambda\,(H-y)^3.$$

Assuming $y = 0$, we obtain the eastward deflection ε of the particle when it reaches the earth*).

$$\varepsilon = \frac{2}{3}\omega\cos\lambda\sqrt{\frac{2H^3}{g}}. \tag{56}$$

We see that the deflection ε is proportional to the earth's angular velocity ω and, consequently, is a very small quantity. On the latitude of Moscow, for example ($\lambda = 55°47'$, $g = 9.816$ m/s^2), $\varepsilon = 1.2$ cm for a fall from a height of 100 m.

Experiments carried out in many parts of the globe by different researchers confirm the validity of Eq. (56).

Consider now the motion of a particle thrown from a point O vertically up with an initial velocity $\boldsymbol{v}_0$. In the first approximation the force $\boldsymbol{F}^{\mathrm{i}}_{\mathrm{Cor}}$ in the ascent is directed westward. Then, if we direct axis x to the west (Fig. 279*b*), the differential equation of motion will be of the form (55) and the initial conditions will be: at $t = 0$, $x = y = 0$, $v_x = 0$, $v_y = v_0$.

Thus the second of the Eqs. (55) will give:

$$v_y = v_0 - gt, \quad y = v_0 t - \frac{gt^2}{2}. \tag{57}$$

Then, assuming approximately, as in the previous problem, that $v = v_y$, we obtain $F^{\mathrm{i}}_{\mathrm{Cor}} = 2m\omega\,(v_0 - gt)\cos\lambda$, and the first of Eqs. (55) takes the form

$$\frac{d^2x}{dt^2} = 2\omega\cos\lambda\,(v_0 - gt).$$

This equation describes the motion of a particle in its downward fall as well, since the change in the sense of the vector $\boldsymbol{F}^{\mathrm{i}}_{\mathrm{Cor}}$ is taken into account by the change in the sign of the factor $(v_0 - gt) = v_y$.

Solving the obtained equation for the initial conditions of the problem, we find finally:

$$x = \omega\cos\lambda\left(v_0 t^2 - \frac{1}{3}gt^3\right). \tag{58}$$

Assuming in Eq. (57) $y = 0$, we can determine the time it takes the particle to reach the ground: $t_1 = 2\frac{v_0}{g}$. Taking into account at the same time that $v_0 = \sqrt{2gH_1}$, where H_1 is the height of the ascent,

---

*) In determining the magnitude and direction of $\boldsymbol{F}^{\mathrm{i}}_{\mathrm{Cor}}$ in the first approximation we neglected the x component of the velocity, which is eastwardly directed. Due to this velocity, $\boldsymbol{F}^{\mathrm{i}}_{\mathrm{Cor}}$ has an additional component which deflects the particle to the south. As $x = \frac{1}{3}(\omega g\cos\lambda)\,t^3$, the quantity $v_x = \frac{dx}{dt}$ is proportional to ω, and the southward deflection is proportional to ω^2, i.e., it is a small quantity of second order.

we can determine from Eq. (58) the westward deflection of the particle at the moment it touches the ground:

$$OB = \varepsilon_1 = \frac{4}{3}\,\omega\cos\lambda\,\frac{v_0^3}{g^2} = \frac{8}{3}\,\omega\cos\lambda\sqrt{\frac{2H_1^3}{g}}. \tag{59}$$

Eqs. (56) and (59) show that at $H_1 = H$ the deflection $\varepsilon_1 = 4\varepsilon$. If the particle can continue moving (the point of projection O is not on the ground), the path of the particle from point B will deflect continuously to the east.

All these calculations, as has been pointed out, refer to motion in a vacuum and take the earth's rotation into account only in the first approximation.

Chapter 21
Rectilinear Vibration of a Particle

§ 123. Free Vibrations Neglecting Resisting Forces

The study of vibrations is essential for a number of physical and engineering fields. Although the vibrations studied in such different fields as mechanics, radio engineering, and acoustics are of different physical nature, the fundamental laws hold for all of them. The study of mechanical vibrations is therefore of importance not only because they are frequently encountered in engineering but also because the results obtained in investigating mechanical vibrations can be used

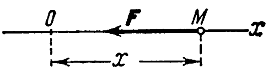

Fig. 280

in studying and understanding vibration phenomena in other fields.

We shall start with examining free vibration of a particle, neglecting resisting forces. Consider a particle M (Fig. 280) moving rectilinearly under the action of a single *restoring force* $\boldsymbol{F}$ directed towards a fixed centre O and proportional to the distance from that centre. The projection of $\boldsymbol{F}$ on axis Ox is

$$F_x = -cx. \tag{60}$$

We see that the force $\boldsymbol{F}$ tends to return the particle to its position of equilibrium O, where $F = 0$, which is why it is called a "restoring" force. Examples of such a force are an elastic force (§ 113, Fig. 255) and the force of attraction analysed in Problem 99 (§ 105).

Let us derive the equation of motion of particle M. Writing the differential equation of motion (6), we obtain:

$$m\frac{d^2x}{dt^2} = -cx.$$

Dividing both sides of the equation by m and introducing the notation

$$\frac{c}{m} = k^2, \tag{61}$$

we reduce the equation to the form

$$\frac{d^2x}{dt^2} + k^2x = 0. \tag{62}$$

Eq. (62) is the *differential equation of free vibrations without resistance*. The solution of this linear homogeneous differential equation of the second order is sought in the form $x = e^{nt}$. Assuming $x = e^{nt}$ in Eq. (62), we obtain for the determination of n the so-called characteristic equation, which in the present case has the form $n^2 + k^2 = 0$. As the solutions of this equation are purely imaginary ($n_{1,2} = \pm ik$), from the theory of differential equations the general solution of Eq. (62) has the form

$$x = C_1 \sin kt + C_2 \cos kt, \tag{63}$$

where C_1 and C_2 are constants of integration.

If we replace C_1 and C_2 by constants a and α, such that $C_1 = a \cos \alpha$ and $C_2 = a \sin \alpha$, we obtain $x = a$ ($\sin kt \cdot \cos \alpha + \cos kt \sin \alpha$), or

$$x = a \sin (kt + \alpha). \tag{64}$$

This is another form of the solution of Eq. (62) in which the constants of integration appear as a and α and which is more convenient for general analysis.*)

The velocity of a particle in this type of motion is

$$v_x = \dot{x} = ak \cos (kt + \alpha). \tag{65}$$

The vibration of a particle described by Eq. (64) is called *simple harmonic motion*. Its graph for $\alpha = \frac{\pi}{2}$ is shown in Fig. 152*c* (§ 69).

All the characteristics of this type of motion lend themselves to visual kinematic interpretation. Consider a particle B moving uniformly along a circle of radius a from a point B_0 defined by the angle $DOB_0 = \alpha$ (Fig. 281), and let the constant angular velocity of radius OB be k. Then, at any instant t angle $\varphi = \angle DOB = \alpha + kt$ and, it will be readily noticed, the projection M of point B on the diameter perpendicular to DE moves according to the law $x = a \sin (kt + \alpha)$, where $x = OM$, i.e., the projection performs harmonic motion.

---

*) That the expressions (63) or (64) are solutions of Eq. (62) can also be verified by directly substituting these values of x into Eq. (62).

The quantity a, which is the maximum distance of M from the centre of vibration, is called the *amplitude of vibration*. The quantity $\varphi = kt + \alpha$ is called the *phase of vibration*. Unlike the coordinate x, the phase φ defines both the position of the particle at any given time and the direction of its subsequent motion. For example, from position M at phase φ the particle will move to the right, at phase $(\pi - \varphi)$ it will move to the left. Phases differing by 2π are considered identical (the little circles in Fig. 152*c*, p. 181, indicate identical phases). The quantity α defines the *initial phase*, with which the motion begins. For example, at $\alpha = 0$ the motion is according to the sine law (it begins at O and the velocity is directed to the right); and at $\alpha = \frac{\pi}{2}$ the motion is according to the cosine law (starting from point $x = a$ with a velocity $v_0 = 0$). The quantity k which coincides with the angular velocity of the rotating radius OB in Fig. 281 is called the *angular*, or *circular*, *frequency of vibration*.

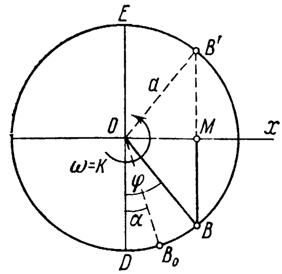

Fig. 281

The time T (or τ) in which the moving particle makes one complete vibraton is called the *period of vibration*. In one period the phase changes by 2π. Consequently, we must have $kT = 2\pi$, whence the period

$$T = \frac{2\pi}{k}. \tag{66}$$

The quantity ν, which is the inverse of the period and specifies the number of oscillations per second is called the *frequency of vibration*:

$$\nu = \frac{1}{T} = \frac{k}{2\pi}. \tag{67}$$

It can be seen from this that the quantity k differs from ν only by a constant multiplier 2π. Usually we shall speak of the quantity k as of frequency.

The values of a and α are determined from the initial conditions. Assuming that, at $t = 0$, $x = x_0$ and $v_x = v_0$, we obtain from Eqs. (64) and (65) $x_0 = a \sin \alpha$ and $\frac{v_0}{k} = a \cos \alpha$. By first squaring and adding these equations and then dividing them, we obtain

$$a = \sqrt{x_0^2 + \frac{v_0^2}{k^2}}, \quad \tan \alpha = \frac{kx_0}{v_0}. \tag{68}$$

Note the following properties of free vibration without resistance:

(1) the amplitude and initial phase depend on the initial conditions;

(2) the frequency k, and consequently the period T, *do not depend* on the initial conditions [see Eqs. (61) and (66)] and are invariable characteristics for a given vibrating system.

It follows, in particular, that if a problem requires that only the period (or frequency) of vibration be determined, it is necessary to write a differential equation of motion in the form (62). Then T is found immediately from Eq. (66) without integrating.

Effect of a Constant Force on the Free Vibration of a Particle. Let the particle M in Fig. 282 be subject, in addition to the restoring force F directed towards the centre O, to a force P constant in magnitude and direction. The value of force F continues to be proportional

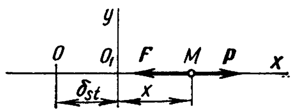

Fig. 282

to the distance from the centre O, i.e., $F = c \cdot OM$. Obviously, in that case the equilibrium point is O_1 at a distance $OO_1 = \delta_{st}$ from O, given by the equation $c\delta_{st} = P$, or

$$\delta_{st} = \frac{P}{c}. \tag{69}$$

We shall call δ_{st} the *static deflection* of the particle.

Placing the origin of the reference system at O_1, direct axis O_1x in the direction of force P. Then $F_x = -c\,(x + \delta_{st})$, and $P_x = P$. Writing the differential equation of motion (6) and taking into account that, by Eq. (69), $c\delta_{st} = P$, we have:

$$m\frac{d^2x}{dt^2} = -cx, \quad \text{or} \quad \frac{d^2x}{dt^2} + k^2x = 0.$$

The obtained equation, in which k is given by Eq. (61), is the same as Eq. (62). Hence we conclude that a constant force P does not affect the character of the vibrations of a particle under the action of a restoring force F and only displaces the centre of vibration in the direction of P by the amount of the static deflection δ_{st}.

Let us express the period of vibration in terms of δ_{st}. From (61) and (69), we have $k^2 = P/m\delta_{st}$. Then Eq. (66) gives:

$$T = 2\pi\sqrt{\frac{m}{P}\,\delta_{st}}. \tag{70}$$

Thus, *the period of vibration is in proportion to the square root of the static deflection* δ_{st}.

In particular, if P is the force of gravity, as in the case of vibration of a load on a vertical spring (see Fig. 283), then $P = mg$, and Eq. (70) takes the form

$$T = 2\pi\sqrt{\frac{\delta_{st}}{g}}. \qquad (70')$$

Problem 117. A weight attached to end B of the vertical spring AB in Fig. 283 is released from rest. Determine the law of motion of the weight if the elongation of the spring in the equilibrium position is δ_{st} (the static elongation of the spring).

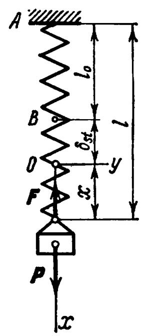

Fig. 283

Solution. Place the origin O of coordinate axes in the position of static equilibrium of the system and direct axis Ox vertically down. The elastic force $F = c\,|\,\Delta l\,|$. In our case $\Delta l = \delta_{st} + x$, hence

$$F_x = -c(\delta_{st} + x).$$

Writing the differential equation of motion, we obtain:

$$m\frac{d^2x}{dt^2} = -c(\delta_{st} + x) + P.$$

But from the conditions of the problem the gravitational force $P = mg = c\delta_{st}$ (in the position of equilibrium force P is balanced by the elastic force $c\delta_{st}$). Introducing the notation $\frac{c}{m} = \frac{g}{\delta_{st}} = k^2$, we reduce the equation to the form

$$\frac{d^2x}{dt^2} + k^2x = 0,$$

whence immediately we find the period of vibration in the form (70')

$$T = \frac{2\pi}{k} = 2\pi\sqrt{\frac{\delta_{st}}{g}}.$$

Thus, the period of vibration is proportional to the square root of the static elongation of the spring.

The solution of the obtained differential equation is

$$x = C_1 \sin kt + C_2 \cos kt.$$

From the initial conditions, at $t = 0$, $x = -\delta_{st}$, and $v_x = 0$. As

$$v_x = \frac{dx}{dt} = kC_1 \cos kt - kC_2 \sin kt,$$

substituting the initial conditions, we obtain $C_2 = -\delta_{st}$, $C_1 = 0$. Hence, the amplitude of vibration is δ_{st} and the motion is according to the law

$$x = -\delta_{st}\cos kt.$$

We see that the maximum elongation of the spring in this motion is $2\delta_{st}$. The same result was obtained in a different way in Problem 107, where a beam played the role of the spring.

This solution shows that a constant force P does not change the type of motion under the action of an elastic force F but only shifts the centre of the vibrations in the direction of the action of the force by the quantity δ_{st} (without the force P the vibration would, evidently, be about B).

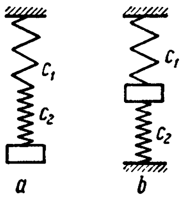

Fig. 284

Problem 118. Determine the period of vibration of a load of weight P attached to two springs of stiffness c_1 and c_2 as shown in Fig. 284*a*.

Solution. In the static position both springs are subjected to a tensile force P. Therefore, the static elongations are $\delta_{1\,st} = \frac{P}{c_1}$, $\delta_{2\,st} = \frac{P}{c_2}$, and the total elongation is

$$\delta_{st} = \delta_{1\,st} + \delta_{2\,st} = \frac{P(c_1 + c_2)}{c_1 c_2},$$

and

$$c_{eq} = \frac{c_1 c_2}{c_1 + c_2},$$

where c_{eq} is the equivalent spring constant of the two given springs. In particular, at $c_1 = c_2$ we have $c_{eq} = c/2$.

From formula (70′) the period of vibration is

$$T = 2\pi\sqrt{\frac{\delta_{st}}{g}} = 2\pi\sqrt{\frac{P}{g}\frac{c_1 + c_2}{c_1 c_2}}.$$

Problem 119. Solve the preceding problem assuming the load to be suspended as in Fig. 284*b*.

Solution. In this case, obviously, the static elongations (compressions) of both springs are the same. The upper spring is subject to a tensile force $P - c_2\delta_{st}$. In equilibrium, $c_1\delta_{st} = P - c_2\delta_{st}$, whence

$$\delta_{st} = \frac{P}{c_1 + c_2}.$$

Here $c_{eq} = c_1 + c_2$, and the period of vibration

$$T = 2\pi\sqrt{\frac{\delta_{st}}{g}} = \sqrt{\frac{P}{g(c_1 + c_2)}}.$$

§ 124. Free Vibration with a Resisting Force Proportional to Velocity (Damped Vibration)

Let us see how the resistance of a surrounding medium affects vibrations, assuming the resisting force proportional to the first power of the velocity: $\boldsymbol{R} = -\mu \boldsymbol{v}$ (the minus indicates that force $\boldsymbol{R}$ is opposite to $\boldsymbol{v}$). Let a moving particle be acted upon by a restoring force $\boldsymbol{F}$ and a resisting force $\boldsymbol{R}$ (Fig. 285). Then $F_x = -cx$ and $R_x = -\mu v_x = -\mu \frac{dx}{dt}$, and the differential equation of motion is

$$m\frac{d^2x}{dt^2} = -cx - \mu\frac{dx}{dt}.$$

Dividing both sides by m, we obtain:

$$\frac{d^2x}{dt^2} + 2b\frac{dx}{dt} + k^2x = 0, \tag{71}$$

where

$$\frac{c}{m} = k^2, \quad \frac{\mu}{m} = 2b. \tag{72}$$

It is easy to verify that k and b have the same dimension (s^{-1}), which makes it possible to compare them.

Eq. (71) is the *differential equation of free vibrations with a resisting force proportional to the velocity*. Its solution, as in the case of

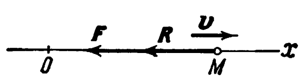

Fig. 285

Eq. (62), is sought in the form $x = e^{nt}$. Substituting into Eq. (71), we obtain the characteristic equation $n^2 + 2bn + k^2 = 0$, the roots of which are:

$$n_{1,2} = -b \pm \sqrt{b^2 - k^2}. \tag{73}$$

Let us consider the case when $k > b$, i.e., when the resistance is small as compared with the restoring force. Introducing the notation

$$k_1 = \sqrt{k^2 - b^2}, \tag{74}$$

from (73) we obtain $n_{1,2} = -b \pm ik_1$, i.e., the solutions of the characteristic equation are complex. In that case the general solution of Eq. (71) differs from the solution of Eq. (62) only by the multiplier e^{-bt}, i.e., it has the form

$$x = e^{-bt}(C_1 \sin k_1t + C_2 \cos k_1t), \tag{75}$$

or, by analogy with Eq. (64),

$$x = ae^{-bt}\sin(k_1 t + \alpha). \tag{76}$$

The quantities a and α are constants of integration and are determined by the initial conditions.

Vibrations according to the law (76) are called *damped* because, due to the multiplier e^{-bt}, the value of $x = OM$ decreases with time and tends to zero. A graph of such vibrations is given in Fig. 286 [the curve lies between the broken curves $x = ae^{-bt}$ and $x = -ae^{-bt}$, as $\sin(k_1 t + \alpha)$ cannot exceed unity].

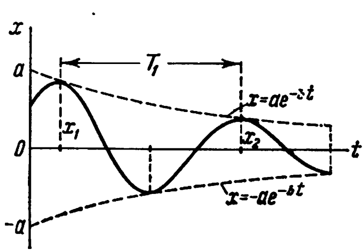

Fig. 286

The time T_1, equal to the period of $\sin(k_1 t + \alpha)$, i.e., the quantity

$$T_1 = \frac{2\pi}{k_1} = \frac{2\pi}{\sqrt{k^2 - b^2}} \tag{77}$$

is conventionally called the *period of damped vibration*. In the course of one period the particle performs a complete vibration, e.g., having begun moving from position $x = 0$ to the right (see Fig. 285) it arrives at the same position, again moving to the right. Taking Eq. (66) into account, Eq. (77) can be written in the form

$$T_1 = \frac{2\pi}{k\sqrt{1 - \frac{b^2}{k^2}}} = \frac{T}{\sqrt{1 - \frac{b^2}{k^2}}} \approx T\left(1 + \frac{1}{2}\frac{b^2}{k^2}\right). \tag{77'}$$

From the equations we see that $T_1 > T$, i.e., that resistance to vibration tends to increase the period of vibration. When, however, the resistance is small ($b \ll k$), the quantity b^2/k^2 can be neglected in comparison with unity, and we can assume $T_1 \approx T$. Thus, a small resistance has no practical effect on the period of vibration.

The time interval between two successive displacements of an oscillating particle to the right or to the left is also equal to T_1*). Hence, if the first (maximum) displacement x_1 to the right takes place at time t_1, the second displacement x_2 will be at time $t_2 = t_1 + T_1$, etc. Then, by Eq. (76) and taking into account that $k_1 T_1 = 2\pi$, we have:

$$x_1 = ae^{-bt_1}\sin(k_1 t_1 + \alpha),$$
$$x_2 = ae^{-b(t_1+T_1)}\sin(k_1 t_1 + k_1 T_1 + \alpha) = x_1 e^{-bT_1}.$$

*) The instants when x is maximum or minimum are found from the equation $\frac{dx}{dt} = ae^{-bt}[k_1\cos(k_1 t + \alpha) - b\sin(k_1 t + \alpha)] = 0$. If at some instant $t = t_1$ the expression in the brackets becomes zero, then, apparently, at times $t_1 + T_1$, $t_1 + 2T_1$, etc., it will again become zero since $k_1 T_1 = 2\pi$.

Similarly, for any displacement x_{n+1} we shall have $x_{n+1} = x_n e^{-bT_1}$. Thus we find that the amplitude of vibration decreases in geometric progression. The ratio of this progression e^{-bT_1} is called the *decrement*, and the modulus of its logarithm, i.e., the quantity bT_1, the *logarithmic decrement.*

It follows from these results that a small resistance has practically no effect on the period of vibration but gradually damps it by virtue of the amplitude of vibration decreasing according to a law of geometric progression.

Fig. 287

Let us consider the case when $b > k$, i.e., the resistance is large as compared with the restoring force. Introducing the notation $b^2 - k^2 = r^2$, we find that in this case the solutions of the characteristic equation (73) are $n_{1,2} = -b \pm \pm r$, i.e., both are real and negative (as $r < b$). Consequently, when $b > k$ the solution of Eq. (71) describing the law of motion of the particle has the form

$$x = C_1 e^{-(b+r)t} + C_2 e^{-(b-r)t}.$$

Since with time the function e^{-at}, where $a > 0$, decreases gradually, tending to zero, the particle no longer vibrates but instead, under the influence of the restoring force, gradually approaches the position of equilibrium $x = 0$. A graph of such motion (if at $t = 0$, $x = x_0$ and $v_0 > 0$) has the form shown in Fig. 287.

§ 125. Forced Vibration. Resonance

Let us consider an important case of vibration where, in addition to a restoring force $\boldsymbol{F}$, a particle is also subjected to a force $\boldsymbol{Q}$, varying periodically with time, whose projection on axis Ox is

$$Q_x = Q_0 \sin pt. \tag{78}$$

This force is called a *disturbing force*, and the vibration caused by it is called *forced.* The quantity p in Eq. (78) is called the *frequency of the disturbing force.*

A disturbing force may vary with time according to other laws, but we shall consider only the case of Q_x defined by Eq. (78). This type of disturbing force is called a *periodic force.* An example involving such a force is given in Problem 120 on p. 351.

(1) Undamped Forced Vibration*). Consider the motion of a particle on which, besides the restoring force F, is acting only a disturbing force Q (Fig. 280). The differential equation of motion will be

$$m\frac{d^2x}{dt^2} = -cx + Q_0 \sin pt.$$

Divide both sides of the equation by m and assume

$$\frac{Q_0}{m} = P_0. \tag{79}$$

Then, taking into account the expression (61), the equation takes the form

$$\frac{d^2x}{dt^2} + k^2x = P_0 \sin pt. \tag{80}$$

Eq. (80) is the *differential equation of undamped forced vibration* of a particle. From the theory of differential equations, its solution is $x = x_1 + x_2$, where x_1 is the general solution of the equation without the right side, i.e., the solution of Eq. (62) as given by Eq. (64), and x_2 is a particular solution of the complete equation (80).

Assuming $p \neq k$, let us find the solution of x_2 in the form

$$x_2 = A \sin pt,$$

where A is a constant such that Eq. (80) becomes an identity. Substituting the expression of x_2 and its second derivative into Eq. (80), we have:

$$-p^2A \sin pt + k^2A \sin pt = P_0 \sin pt.$$

This equation is satisfied at any t, if $A(k^2 - p^2) = P_0$, or

$$A = \frac{P_0}{k^2 - p^2}.$$

Thus, the required particular solution is

$$x_2 = \frac{P_0}{k^2 - p^2} \sin pt. \tag{81}$$

As $x = x_1 + x_2$ and the expression for x_1 is given by Eq. (64), the general solution of Eq. (80) takes the final form

$$x = a \sin (kt + \alpha) + \frac{P_0}{k^2 - p^2} \sin pt, \tag{81'}$$

where a and α are constants of integration determined by the initial conditions.

Solution (81′) shows that in the present case the vibration of a particle consists of (1) *free vibrations* of amplitude a (depending on the

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*) The results obtained in this item can be obtained as a special case of item (2).

initial conditions) and frequency k called *natural vibrations* and (2) *forced vibrations* of amplitude A (not depending on the initial conditions) and frequency p.

In practice, due to the inevitable presence of various damping forces, the natural vibrations rapidly disappear. Therefore in this type of motion the forced vibrations defined by Eq. (81) are of primary importance.

The frequency p of forced vibrations is, evidently, equal to the frequency of the disturbing force. The amplitude of this vibration is

$$A=\frac{P_0}{|k^2-p^2|}=\frac{\delta_0}{\left|1-\left(\frac{p}{k}\right)^2\right|}, \tag{82}$$

where, according to Eqs. (61) and (79), $\delta_0=\frac{P_0}{k^2}=\frac{Q_0}{c}$, i.e., δ_0 is the magnitude of the static deflection of the particle under the action of force Q_0. Thus, A depends on the ratio of the frequency p of the disturbing force to the frequency k of the natural vibrations. A graph of this dependence is given in Fig. 289*a* (p. 348) (the curve marked $h=0$; the other curves in the diagram show the dependence of A on p/k with a resisting force).

It can be seen from the graph [or from Eq. (82)] that forced vibrations of different amplitudes can be induced by choosing different ratios p/k. When $p=0$ (or $p \ll k$), the amplitude is equal to δ_0 (or is very close to it). When p is almost equal to k, the amplitude becomes very large. Finally, when $p \gg k$, the amplitude A is almost zero (vibration virtually stops).

Note also that at $p<k$, as can be seen from a comparison of Eqs. (78) and (81), the phases of the forced vibrations and the disturbing force always coincide (both are equal to pt). If, however, $p>k$, by bringing the minus under the sine we can write Eq. (81) in the form

$$x_2=\frac{P_0}{p^2-k^2}\sin(pt-\pi).$$

Thus, at $p>k$, the phase shift between the forced vibrations and the disturbing force is π (when force Q is maximum and is directed to the right, the vibrating particle is in its extreme left position, etc.).

Resonance. When $p=k$, i.e., when the frequency of the disturbing force equals the frequency of the natural vibrations, the phenomenon known as *resonance* occurs. The case is not covered by Eqs. (81) and (82), but it can be proved that when resonance takes place, the amplitude of forced vibration increases indefinitely, as shown below in Fig. 290. The general properties of forced vibrations, and resonance in particular, are discussed in greater detail at the end of this section [item (3)].

At $p = k$, Eq. (80) does not contain the particular solution $x_2 = A \sin pt$, and the solution must be sought in the form

$$x_2 = Bt \cos pt.$$

Then $\frac{d^2x_2}{dt^2} = -2Bp \sin pt - p^2Bt \cos pt$, and, taking into account that $p = k$, substitution into Eq. (80) yields $-2Bp \sin pt = P_0 \sin pt$, whence $B = -P_0/2p$. From this we obtain the law of undamped forced vibrations when resonance occurs:

$$x_2 = -\frac{P_0}{2p} t \cos pt, \quad \text{or} \quad x_2 = \frac{P_0}{2p} t \sin \left(pt - \frac{\pi}{2}\right). \tag{83}$$

We see that the amplitude of forced vibration during resonance does increase in proportion to time, and the law of vibration has the form shown in Fig. 290. The phase shift in resonance is $\pi/2$.

(2)* Damped Forced Vibration. Consider the motion of a particle on which are acting a restoring force $\boldsymbol{F}$, a damping force $\boldsymbol{R}$ proportional to the velocity (see § 124), and a disturbing force $\boldsymbol{Q}$ given by Eq. (78). The differential equation of this motion has the form

$$m \frac{d^2x}{dt^2} = -cx - \mu \frac{dx}{dt} + Q_0 \sin pt.$$

Dividing the equation by m and taking into account the expressions (72) and (79), we obtain:

$$\frac{d^2x}{dt^2} + 2b \frac{dx}{dt} + k^2x = P_0 \sin pt. \tag{84}$$

Eq. (84) is the *differential equation of damped forced vibration* of a particle. Its general solution, as is known, has the form $x = x_1 + x_2$, where x_1 is the general solution of the equation without the right side, i.e., of Eq. (71) [at $k > b$ this solution is given by Eq. (76)], and x_2 is a particular solution of the complete equation (84). Let us find the solution x_2 in the form

$$x_2 = A \sin (pt - \beta),$$

where A and β are constants so chosen that Eq. (84) becomes an identity. Differentiating, we obtain:

$$\frac{dx_2}{dt} = Ap \cos (pt - \beta), \quad \frac{d^2x_2}{dt^2} = -Ap^2 \sin (pt - \beta).$$

Substituting these expressions of the derivatives and x_2 into the left side of Eq. (84) and introducing for the sake of brevity the notation $pt - \beta = \psi$ (or $pt = \psi + \beta$), we obtain:

$$A(-p^2 + k^2) \sin \psi + 2bp A \cos \psi = P_0 (\cos \beta \sin \psi + \sin\beta \cos\psi).$$

For this equation to be satisfied at any value of ψ, i.e., at any instant of time, the factors of sin ψ and cos ψ in the left and right sides should be separately equal. Hence,

$$A(k^2 - p^2) = P_0 \cos\beta, \quad 2bpA = P_0 \sin\beta.$$

First squaring and adding these equations (they are also used to determine β uniquely) and then dividing one by the other, we obtain:

$$A = \frac{P_0}{\sqrt{(k^2-p^2)^2+4b^2p^2}}, \quad \tan\beta = \frac{2bp}{k^2-p^2}. \tag{85}$$

As $x = x_1 + x_2$, and the expression for x_1 (when $k > b$) is given by Eq. (76), we have the final solution of Eq. (84) in the form

$$x = ae^{-bt}\sin(k_1t+\alpha) + A\sin(pt-\beta), \tag{86}$$

where a and α are constants of integration determined by the initial conditions, and the expressions for A and β are given by Eqs. (85) and do not depend on the initial conditions. For $b = 0$ the solution (86) is just (81) and (81′) for the case without resisting forces.

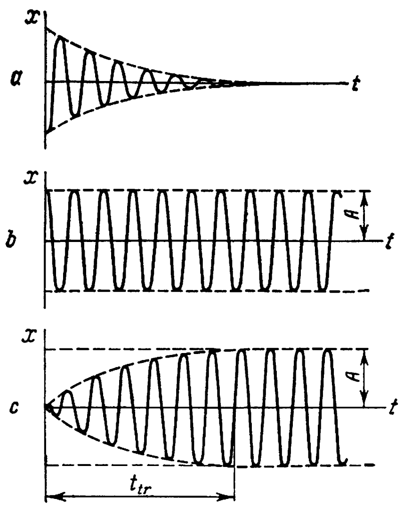

Fig. 288

These vibrations are compounded of *natural* vibration [the first term in Eq. (86); Fig. 288*a*] and *forced* vibration [the second term in Eq. (86); Fig. 288*b*]. The natural vibration of the particle in such a case was discussed in § 124. It was established that it is transient and is damped fairly quickly, and after a certain interval of time t_{tr}, called the *transient period*, can be neglected.

If, for example, we assume that free vibrations can be neglected from the moment when their amplitude is less than 0.01 A, then the value of t_{tr} can be determined from the equation $ae^{-bt} = 0.01\,A$,

$$t_{tr} = \frac{1}{b}\ln\frac{100a}{A}. \tag{87}$$

We see, thus, that the less the resistance (i.e., the less the value of b), the greater the transient period.

A possible picture of transient vibration according to the law (86) and starting from rest, is shown in Fig. 288*c*. Given other initial conditions and ratios of the frequencies p and k_1, the character of the vibrations in the time interval $0 < t < t_{tr}$ can be quite different.

However, in all cases, after the transient period elapses the natural vibrations will, for all practical purposes, cease and the particle will vibrate according to the law

$$x = A \sin (pt - \beta). \tag{88}$$

This is *steady-state forced vibration*, a sustained periodic motion with an amplitude A defined by Eq. (85) and a frequency p equal to the frequency of the disturbing force. The quantity β characterises the phase shift of forced vibration with respect to the disturbing force.

Let us investigate the results obtained. First let us introduce the notation

$$\frac{p}{k} = \lambda, \quad \frac{b}{k} = h, \quad \frac{P_0}{k^2} = \frac{Q_0}{c} = \delta_0, \tag{89}$$

where λ is the frequency ratio, h a quantity characterising the damping effect, δ_0 the magnitude of the static deflection of a particle

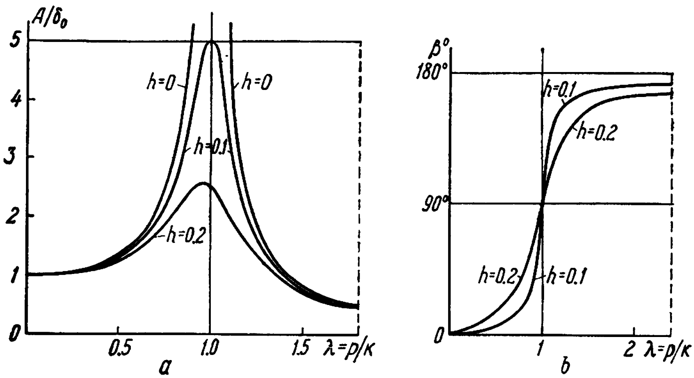

Fig. 289

under the action of force Q_0 (for instance, in the oscillation of a load on a spring δ_0 is equal to the static elongation of the spring caused by Q_0).

Then, dividing the numerator and denominator of Eq. (85) by k^2, we obtain:

$$A = \frac{\delta_0}{\sqrt{(1-\lambda^2)^2 + 4h^2\lambda^2}}, \quad \tan \beta = \frac{2h\lambda}{1-\lambda^2}. \tag{90}$$

It can be seen from Eq. (90) that A and β depend on two dimensionless parameters λ and h. Graphs of this relation for certain values of h are given in Fig. 289. The quantity A/δ_0 along the ordinate axis in the first graph is called the *dynamic coefficient*. The values of δ_0, λ, and h can be computed for each specific problem from its conditions, and the values of A and β determined from the respective graphs or Eqs. (90). These graphs (and equations) also show that by altering the frequency ratio λ we can induce forced vibrations of different amplitude.

When the resistance is very small (as ordinarily in the atmosphere) and λ is not close to unity, it is possible in Eqs. (90) to assume approximately $h \approx 0$.

In this case we obtain the results of item (1), namely

$$A \approx \frac{\delta_0}{|1-\lambda^2|}; \quad \beta \approx 0 \quad (\text{at } \lambda < 1), \ \beta \approx 180^\circ \ (\text{at } \lambda > 1). \quad (91)$$

Let us consider also the following special cases.

(1) If the frequency ratio λ is very small ($p \ll k$), then, assuming as an approximation $\lambda \approx 0$, we obtain from Eq. (90) $A \approx \delta_0$. The vibration in this case has an amplitude equal to the static deflection δ_0, and the phase shift is $\beta = 0$.

(2) If the frequency ratio λ is very large ($p \gg k$), A becomes very small. This case is of special interest for the absorption of vibrations in structures, instruments, etc. Assuming the resistance to be small and neglecting $2h\lambda$ and 1 as compared with λ^2 in Eq. (90), we obtain for computing A an approximate formula:

$$A = \frac{\delta_0}{\lambda^2} = \frac{P_0}{p^2}. \quad (91')$$

(3) In all cases of practical interest h is very small. Then, from Eq. (90), if λ is almost unity, the amplitude of forced vibrations becomes very large. This phenomenon is called *resonance**).

At resonance the amplitude of forced vibrations and phase shift can be computed according to approximate formulas derived from Eqs. (90), assuming $\lambda = 1$:

$$A_r = \frac{\delta_0}{2h}, \quad \beta_r = \frac{\pi}{2}. \quad (91'')$$

We see that when h is small, A_r can become very large.

The resonance condition in which the amplitude of vibration is A_r, like forced vibration in general, does not develop immediately. The development of steady-state vibration is similar to that shown in Fig. 288*c*. The less the resistance, i.e., the smaller b or h are, the larger the magnitude of A_r, but also the larger the transient period t_{tr} [see Eq. (87)].

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*) It can be seen from Eq. (90) that $A = A_{max}$ when the term in the denominator $f(\xi) = (1-\xi)^2 + 4h^2\xi$ (where $\xi = \lambda^2$) is at its minimum. Solving the equation $f'(\xi) \equiv -2(1-\xi-2h^2) = 0$, we obtain that A is maximum at $\xi = 1 - 2h^2$, i.e., at $\lambda_r = \sqrt{1-2h^2}$. Thus, resonance sets in when λ is a little less than unity. In practice, however, neglecting h^2, we can assume that $\lambda_r = 1$.

When there is no resistance, i.e., $b = h = 0$, as was established before, the law of forced vibration under resonance is given by Eq. (83), and the vibration graph has the form shown in Fig. 290. Thus, with no damping force the vibration amplification process in resonance is unlimited and the amplitude increases indefinitely. When the damping forces are very small the picture is similar.

(3) General Properties of Forced Vibrations. It follows from the results obtained above that forced vibration has the following important properties, which distinguish it from the natural vibration of a particle.

(a) The amplitude of forced vibration does not depend on the initial conditions. (b) Forced vibration does not die out in the presence of resistance. (c) The frequency of forced vibration is equal to the

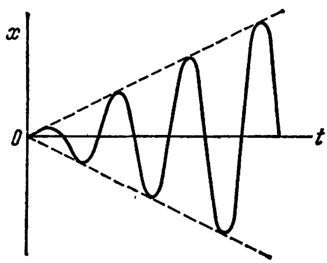

Fig. 290

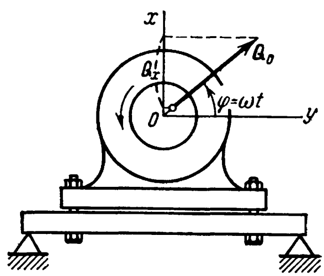

Fig. 291

frequency of the disturbing force and does not depend on the characteristics of the vibrating system (the disturbing force "impresses" its own vibration frequency on the system). (d) Even when the disturbing force Q_0 is small, large forced vibration can be induced if the resistance is small and the frequency p is almost equal to k (resonance). (f) Even if the disturbing force is large, forced vibration can be damped if the frequency p is much larger than k.

Forced vibration, and resonance in particular, plays an important part in many branches of physics and engineering. Lack of balance in working machines and motors, for example, usually causes forced vibration to appear in the machine or its foundation.

The variation of the amplitude of forced vibration can be traced by making a motor for which $p = \omega$, where ω is the angular velocity of rotation (see Problem 120), rotate with different speeds. As ω increases, the amplitude A of the vibrating part (or foundation) grows. When $\omega = k$, resonance appears and the amplitude of the forced vibration is maximum. When ω increases further, the amplitude A decreases, and at $\omega \gg k$, the value of A is virtually zero. Resonance is most undesirable in engineering structures and should be prevented by choosing the frequency ratio p/k so that forced vibration would tend to zero ($p \gg k$).

In radio engineering the reverse is true. Resonance is extremely useful and is used to separate the signals of one radio station from those of all others (tuning).

The design of such instruments as vibrographs, which measure the displacements of vibrating bodies (foundations, machine parts, etc.) and, in particular, seismographs, which record the vibrations of the earth's crust, are based on the theory of forced vibration.

Problem 120. The deflection caused in a beam by the weight of a motor mounted as shown in Fig. 291 is $\delta_{st} = 1$ cm. At how many rpm of the shaft will resonance appear?

Solution. From Eq. (70′) it follows that the period of natural vibration of the beam is

$$T = 2\pi \sqrt{\frac{\delta_{st}}{g}}.$$

If the centre of gravity of the shaft is not concentric with its axis, a centrifugal force Q_0 will develop (Fig. 291). Its component $Q_x = Q_0 \sin \omega t$ (where ω is the angular velocity of the shaft) is the disturbing force acting on the beam; its frequency is $p = \omega$. Hence, the period of the forced vibration is $T_f = \frac{2\pi}{\omega}$.

Resonance will appear when $T_f = T$, i.e., at

$$\omega_{cr} = \sqrt{\frac{g}{\delta_{st}}} = 31.3\ s^{-1}.$$

Hence the critical speed

$$n_{cr} = \frac{30\omega_{cr}}{\pi} = 300 \text{ rpm}.$$

The working speed of the motor should be much greater than n_{cr}.

Problem 121. Analyse the forced vibration of a load attached to a spring (Problem 117) if the upper end A of the spring oscillates vertically according to the law $\xi = a_0 \sin pt$.

Solution. Draw axis Ox as in Problem 117 (see Fig. 283). If we imagine the upper end of the spring displaced from point A downwards by a quantity ξ, the length of the spring will be $l = l_0 - \xi + \delta_{st} + x$. Then $F_x = -c\Delta l = -c(\delta_{st} + x - \xi)$, and the differential equation of motion, neglecting the resistance of the air and taking into account that $P = c\delta_{st}$, will be

$$m\frac{d^2x}{dt^2} = -c(\delta_{st} + x - \xi) + P = -cx + c\xi.$$

Introducing, as in Problem 117, the notation $\frac{c}{m} = k^2$, we obtain:

$$\frac{d^2x}{dt^2} + k^2x = k^2a_0 \sin pt.$$

Consequently, the load will experience forced vibration, since, if we assume $b = 0$ and $P_0 = k^2 a_0$, the equation coincides with Eq. (80) or (84). It can be seen from Eq. (89) that in the present case $\delta_0 = a_0$ and $h = 0$. The amplitude of forced vibration and the phase shift are determined by Eq. (91).

If $p \ll k$ (the top end of the spring oscillates very slowly), then $\lambda \approx 0$ and $A \approx a_0$ and the phase shift $\beta = 0$. The load will oscillate as if the spring were a rigid rod, which physically corresponds to the condition $k \gg p$. At $p = k$ resonance appears and the amplitude increases sharply. If the frequency p becomes larger than k ($\lambda > 1$) the load will vibrate in such a way that it will move down when the end of the spring moves up and vice versa (a phase shift of $\beta = 180°$), and the larger the value of p the smaller the amplitude. Finally, when p is much greater than k ($\lambda \gg 1$), the amplitude $A \approx 0$. The load will remain in the position of static equilibrium (point O) even though the top end of the spring will oscillate with an amplitude a_0 (th frequency of this vibration is so large that the load, as it were, is unable to keep up with it).

Chapter 22*

Motion of a Body in the Earth's Gravitational Field

§ 126. Motion of a Particle Thrown at an Angle to the Horizon in the Earth's Gravitational Field

The problem of motion in the earth's gravitational field arises when the motion of long-range missiles and artificial earth satellites and questions of interplanetary flight are considered. In these cases, when range and height of the path are comparable with the radius of the earth, it is necessary (unlike the problem in § 108) to take into account the change in gravitational force with distance.

We shall consider a moving particle of mass m. Let the particle at the initial time be at a point M_0 on the surface of the earth (Fig. 292) and let it have a velocity v_0 directed at an angle α to the horizon. Neglecting the resistance of the air (for the heights considered this is quite permissible in the first approximation) and assuming the earth to be fixed, we have acting on the moving particle only the gravitational force F directed towards the centre of the earth. By the law of gravitation [see § 113, item (4)]

$$F = mg\frac{R^2}{r^2}, \qquad (92)$$

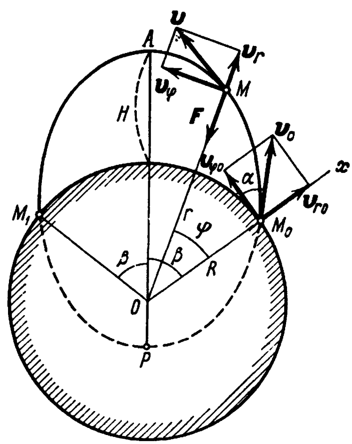

Fig. 292

where $r = OM$ is the distance of the particle from the centre of the earth, $R = OM_0$ is the value of r for the point of emergence M_0, and g is the acceleration of gravity at point M_0 *).

*) In Eq. (92) R can have any value larger than the earth's radius. When M_0 is at the surface of the earth, we shall usually consider R to be the radius of the earth's equator $R_0 = 6\ 378$ km, and $g = g_0 = 9.81$ m/s² (in all computations g is the absolute acceleration of gravity, not the relative acceleration, which takes into account the earth's rotation; see § 121).

As F is a central force (§ 117), the path of the particle is a plane curve, and the motion can be described in polar coordinates $r = OM$ and φ, with the origin (pole) O at the centre of the earth. We shall set the direction of the polar axis Ox along OM_0. Let us now write the differential equations of motion of the particle.

From the law of areas (§ 117) it follows that, in motion under the action of a central force, the moment of the velocity vector v with respect to a centre O (twice the areal velocity of the particle) is a constant quantity. Thus $m_O(v) = c$. But from the diagram we see that if vector v is resolved into a radial component v_r and a lateral component v_φ (§ 71), then

$$m_O(v) = m_O(v_\varphi) = rv_\varphi, \quad \text{where } v_\varphi = r\frac{d\varphi}{dt}.$$

Hence, we obtain the first equation:

$$r^2\frac{d\varphi}{dt} = c \tag{93}$$

The value of the constant c is determined from the conditions at the point of emergence M_0 where, it will be readily noticed, $m_O(v_0) = Rv_0 \cos\alpha$, whence

$$c = Rv_0 \cos\alpha. \tag{94}$$

The second equation is obtained from the theorem of the change in kinetic energy in its differential form (§ 114):

$$d\left(\frac{mv^2}{2}\right) = dA$$

But from Eq (41) in § 113

$$dA = -F\,dr = -mgR^2\frac{dr}{r^2},$$

and the second equation is

$$d\left(\frac{v^2}{2}\right) = gR^2 d\left(\frac{1}{r}\right), \tag{95}$$

where (see § 71)

$$v^2 = v_r^2 + v_\varphi^2 = \left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\varphi}{dt}\right)^2. \tag{96}$$

By solving the differential equations (93) and (95), we can determine r and φ as functions of time t, i.e., develop the equation of motion of the particle. Let us, instead, find its path directly. To simplify the computation we introduce a new variable u, assuming

$$u = \frac{1}{r}, \quad \frac{du}{d\varphi} = -\frac{1}{r^2}\frac{dr}{d\varphi}. \tag{97}$$

Taking into account Eqs. (97) and (93), we obtain:

$$\frac{dr}{dt} = \frac{dr}{d\varphi}\frac{d\varphi}{dt} = -r^2\frac{du}{d\varphi}\frac{c}{r^2} = -c\frac{du}{d\varphi}, \quad r\left(\frac{d\varphi}{dt}\right) = \frac{c}{r} = cu.$$

Substituting these values into Eq. (96), we have:

$$v^2 = c^2\left[u^2 + \left(\frac{du}{d\varphi}\right)^2\right],$$

and Eq. (95) after dividing it by $d\varphi$ and computing the derivative of v^2 takes the form

$$c^2\left[u\frac{du}{d\varphi} + \frac{du}{d\varphi}\frac{d^2u}{d\varphi^2}\right] = gR^2\frac{du}{d\varphi}.$$

Substituting the expression for c from Eq. (94) and eliminating $du/d\varphi$, we finally obtain the differential equation of the path:

$$\frac{d^2u}{d\varphi^2} + u = \frac{g}{v_0^2\cos^2\alpha}, \quad \text{or} \quad \frac{d^2u}{d\varphi^2} + u = \frac{1}{p}, \tag{98}$$

where

$$p = \frac{v_0^2\cos^2\alpha}{g}. \tag{99}$$

The solution of this equation consists of the general solution of the equation without the right side [which coincides with the solution of Eq. (62) for $k = 1$] and the particular solution of the equation with the right side. Consequently, $u = u_1 + u_2$, where u_1 has the form (63) or (64) at $k = 1$, and $u_2 = 1/p$, which can be verified by substitution. Thus the solution of Eq. (98) is

$$u = c_1\sin(\varphi + c_2) + \frac{1}{p}, \quad \text{or} \quad u = \frac{1 + c_1p\sin(\varphi + c_2)}{p},$$

where c_1 and c_2 are constants of integration. Assuming $c_1p = -e$ and $c_2 = \pi/2 - \beta$, where e and β are new constants, and passing from u to r, we finally obtain the equation of the path in the form:

$$r = \frac{p}{1 - e\cos(\varphi - \beta)} \tag{100}$$

We know from analytical geometry that Eq. (100) is the equation of a conic section (ellipse, parabola, hyperbola), where p is the focal parameter and e the eccentricity, expressed in polar coordinates whose pole O is one of the foci. The geometrical meaning of the constant β can be visualised from the fact that at $\varphi = \beta$ the denominator in Eq. (100) is at minimum and, consequently, the quantity $r = OM$ is at maximum. Thus, angle β defines the position of the path's axis of symmetry (AP in Fig. 292) with respect to OM_0 or the point of release M_0.

To determine the values of the integration constants e and β, for the initial position $\varphi = 0$, i.e., at point M_0, we must know in addition to r (or u) also the derivative of r (or of u) with respect to φ. Eqs. (32) obtained in § 71 and the last of the Eqs. (97) yield

$$\frac{v_r}{v_\varphi} = \frac{1}{r}\frac{dr}{d\varphi}, \quad \text{or} \quad -\frac{1}{r}\frac{v_r}{v_\varphi} = \frac{du}{d\varphi}.$$

But as Fig. 292 shows, at point M_0, $r = R$ and $v_r/v_\varphi = \tan\alpha$. Consequently, the initial conditions for u have the form:

$$\text{at } \varphi=0,\quad u=\frac{1}{R} \quad\text{and}\quad \frac{du}{d\varphi}=-\frac{1}{R}\tan\alpha.$$

From Eq. (100), and passing again from r to u, we obtain:

$$u=\frac{1}{p}[1-e\cos(\varphi-\beta)],\quad \frac{du}{d\varphi}=\frac{e}{p}\sin(\varphi-\beta).$$

Substituting the initial values, we have:

$$\frac{p}{R}=1-e\cos\beta,\quad -\frac{p}{R}\tan\alpha=-e\sin\beta,$$

or, replacing p by its expression from (99),

$$e\cos\beta=1-\frac{v_0^2\cos^2\alpha}{gR},\quad e\sin\beta=\frac{v_0^2\sin 2\alpha}{2gR}. \tag{101}$$

By dividing these equations and then squaring and adding them, we finally obtain:

$$\tan\beta=\frac{v_0^2\sin 2\alpha}{2(gR-v_0^2\cos^2\alpha)}, \tag{102}$$

$$e=\sqrt{1+\frac{v_0^2\cos^2\alpha}{g^2R^2}(v_0^2-2gR)}. \tag{103}$$

Eq. (102) defines angle β, i.e., the position of the axis of symmetry of the path with respect to the point of emergence M_0. Eq. (103) gives the eccentricity of the path. It shows that the path of a particle is

(a) an ellipse ($e<1$) if $v_0<\sqrt{2gR}$,

(b) a parabola ($e=1$) if $v_0=\sqrt{2gR}$,

(c) a hyperbola ($e>1$) if $v_0>\sqrt{2gR}$.

The velocity $v_p=\sqrt{2gR}$ is called the *parabolic*, or *escape*, *velocity*. Assuming $R=R_0=6\,378$ km and $g=g_0=9.81$ m/s^2, we obtain $v_p\approx 11.2$ km/s. Thus, a body projected from the surface of the earth at any angle to the horizon with an initial velocity $v_0 \geqslant 11.2$ km/s will move in a parabola or hyperbola (at $\alpha=90°$, in a straight line), receding infinitely away from the earth. Velocities of this order are essential for interplanetary travel*). If the velocity is less than the escape velocity, the body will either fall back to earth or become an artificial satellite.

The equation of motion of a particle on its path, i.e., its location at any instant, can be derived by substituting for r in Eq. (93) its expression (100) and integrating the new equation.

*) The velocity necessary for a spaceship to escape the combined pull of the earth and the sun is greater than $\sqrt{2g_0R_0}$, and for some direction v_0 is about 16.7 km/s.

§ 127. Artificial Earth Satellites. Elliptical Paths

At $v_0 < \sqrt{2gR}$, the path of a body projected from the earth's surface is an ellipse whose axis PA, at angle β to Ox, is the axis of symmetry (see Fig. 292). If the initial conditions at point M_0 are such that angle $\beta \neq \pi$, the path intersects the surface of the earth at point M_1 located symmetrically with respect to axis PA, i.e., the body will fall to the earth. Consequently, a body can become an earth satellite only given initial conditions that yield $\beta = \pi$. But as Eqs. (101) show, $\beta = \pi$ only at $\alpha = 0$ (or $\alpha = \pi$) and $v_0^2 \geqslant gR$, as at $\beta = \pi$ and $v_0^2 < gR$ the first of the equations (101) yields $e < 0$, which is impossible as e is a positive quantity.

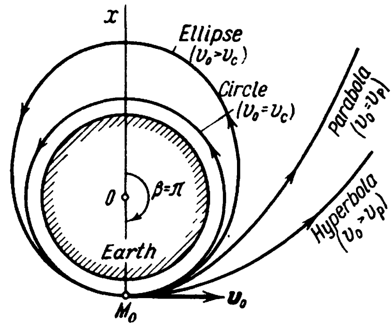

Fig. 293

Consequently, for a body projected from the earth's surface to become an artificial satellite two conditions must be satisfied:

$$\alpha = 0, \quad \sqrt{2g_0R_0} > v_0 \geqslant \sqrt{g_0R_0}. \tag{104}$$

From Eqs. (101) we find that the eccentricity of a satellite orbit at $\alpha = 0$ and $\beta = \pi$ will be

$$e = \frac{v_0^2}{gR} - 1. \tag{105}$$

The velocity $v_c = \sqrt{gR}$, at which $e = 0$ and the satellite follows a circular orbit of radius R, is called the *circular velocity*. For a body projected from the surface of the earth assuming $R = R_0 = 6\,378$ km and $g = g_0 = 9.81$ m/s^2, the circular velocity $v_c \approx 7\,910$ m/s. At $v_0 > v_c$, a satellite orbit is an ellipse whose eccentricity will increase as v_0 increases (Fig. 293).

A body projected from the surface of the earth can never become an artificial satellite if the angle of elevation $\alpha \neq 0$, whatever the initial velocity v_0 (even neglecting the resistance of the air). That is why, for instance, it is impossible to launch an artificial earth satellite by shooting one from a cannon. A guided rocket capable of lifting a satellite to a required altitude and imparting to it at a point M_0 (Fig. 293) the required velocity v_0 at an angle $\alpha \approx 0$ to the horizon must be used. This was just how the world's first artificial satellites, the Soviet sputniks, were launched.

Finally, the greater the height H of point M_0 above the surface of the earth the less the resistance of the atmosphere and the longer a satellite's lifetime. It will also be noticed that it becomes possible to project a satellite into orbit at $\alpha \neq 0$.

The greater the elevation H, the less the circular velocity $v_c = \sqrt{gR}$ as $g = g_0 R_0^2/R^2$ and $R = R_0 + H$, and consequently,

$$v_c = \sqrt{gR} = \sqrt{\frac{g_0 R_0^2}{R}} = \sqrt{g_0 R_0}\sqrt{\frac{R_0}{R_0+H}}.$$

For example, at $H = 0$, $v_c = 7\,910$ m/s; at $H = 500$ km, $v_c = 7\,620$ m/s; at $H = 1\,000$ km, $v_c = 7\,360$ m/s, etc. The total energy needed to launch a satellite, though, increases with H. For, denoting the energy per unit of mass by the symbol T, the energy required to boost a satellite to point M_0 (neglecting the resistance of the air) will be $\widetilde{T}_1 = 0.5v_{in}^2 = 0.5\frac{2g_0R_0H}{R_0+H}$ (see § 115, Problem 108), and the energy needed to impart the orbital velocity will be $\widetilde{T}_2 = 0.5v_c^2$. Hence, the total expenditure of energy per unit mass will be

$$\widetilde{T} = 0.5\left(\frac{g_0R_0^2}{R_0+H} + \frac{2g_0R_0H}{R_0+H}\right) = 0.5g_0R_0\left(1 + \frac{1}{1+R_0/H}\right)$$

and will increase as H increases.

Elliptical Paths. At $\alpha > 0$ and $v_0 < \sqrt{2g_0R_0}$, a body projected from the surface of the earth will describe an elliptical arc and fall to the ground. Such elliptical paths are travelled by long-range (intercontinental) ballistic missiles. Let us find the principal characteristics of such paths.

Since axis PA (see Fig. 292) is the axis of symmetry of the path, the point of impact will be M_1 and the range S will be equal to the arc $\widehat{M_0M_1}$, whence

$$S = 2R_0\beta, \tag{106}$$

where β is defined by Eq. (102) and by convention R_0 is the mean radius of the earth.

The greatest height H of the path is, evidently, $[(r)_{\varphi=\beta} - R_0]$ or, according to Eqs. (99) and (100),

$$H = \frac{v_0^2\cos^2\alpha}{(1-e)\,g_0} - R_0, \tag{107}$$

where e is defined by Eq. (103).

The time of flight τ can be found from Eq. (93), which, together with Eq. (94), gives

$$dt = \frac{r^2}{c}d\varphi = \frac{r^2}{R_0v_0\cos\alpha}\,d\varphi.$$

Substituting the expression for r from Eqs. (99) and (100) and integrating, we obtain:

$$\tau = \frac{v_0^3 \cos^3 \alpha}{R_0 g_0^2} \int_{-\beta}^{\beta} \frac{d\psi}{(1 - e\cos\psi)^2},$$

where $\psi = \varphi - \beta$. We finally have:

$$\tau = \frac{2v_0^3 \cos^3 \alpha}{R_0 g_0^2 (\sqrt{1-e^2})^3} (z + e \sin z), \tag{108}$$

where

$$z = 2 \arctan \left(\sqrt{\frac{1+e}{1-e}} \tan \frac{\beta}{2} \right). \tag{109}$$

With these formulas and knowing v_0 and the angle of elevation α, we can find the range S, the highest point of the path H, and the time of flight τ.

From the practical point of view it is important to determine the minimum velocity v_0^{min} and the optimum angle of elevation α_{opt} at which the required range $S = 2R_0\beta$ can be obtained.

For this we compute the value of v_0 from Eq. (102):

$$v_0 = \sqrt{\frac{2R_0 g_0 \tan \beta}{\sin 2\alpha + 2\cos^2 \alpha \tan \beta}} \tag{110}$$

For a given range (or given angle β) the required velocity depends on the angle of elevation α. Angle α enters only into the denominator of Eq .(110); hence, v_0 is minimum when the denominator is maximum. Equating the derivative of the denominator with respect to α to zero, we find

$$\cos 2\alpha - \sin 2\alpha \tan \beta = 0$$

whence $\cot 2\alpha_{opt} = \tan \beta$, and the optimum angle of elevation is

$$\alpha_{opt} = 45° - \frac{\beta}{2}. \tag{111}$$

That this value of α_{opt} gives v_0^{min} can be readily verified by the sign of the second derivative. Substituting the value of α_{opt} into Eq. (110) and taking into account that $2\cos^2\alpha = 1 + \cos 2\alpha$, we obtain:

$$v_0^{min} = \sqrt{2R_0 g_0 \frac{\sin \beta}{1 + \sin \beta}} \tag{112}$$

Eqs. (111) and (112) give the least initial velocity and the optimum angle of elevation for a required range. The height of the path and the time of flight are computed from Eqs. (107) and (108), in which v_0 and α are substituted by their expressions from Eqs. (111) and (112). Table 1 below gives the characteristics of some optimum elliptical paths computed from these formulas, assuming $R_0 = R_{mean} =$

= 6 370 km (the accuracy of all quantities in the table is up to 5 units of the last digit).

Remember that these calculations refer to motion in a vacuum and do not take into account the rotation of the earth. In conclusion it should be noted that at small range (small angles β) the elliptical

Table 1

Angle β	Range S in km	Necessary initial velocity v_0^{min} in m/s	Optimum angle of elevation α_{opt}	Height of path H in km	Time of flight τ
10°	2 220	4 300	40°	500	12 min 30 s
20°	4 450	5 650	35°	900	19 min 10 s
30°	6 670	6 460	30°	1 170	24 min 50 s
40°	8 900	7 000	25°	1 300	30 min 00 s
70°	15 570	7 780	10°	900	40 min 10 s
90°	20 020	7 910	0°	0	42 min 10 s

arc described by a projectile approaches a parabolic arc. If we take $\sin\beta \approx \beta$ and $2R_0\beta = X$ and in the other equations neglect β in comparison with unity, all these equations will develop in the limit into the corresponding equations of parabolic paths (see § 108). In particular, from Eqs. (111) and (112) we immediately obtain $\alpha_{opt} = \alpha^* = 45°$ and $v_0^{min}\sqrt{g_0 X^*}$.

§ 128. Weightlessness

When a body is at rest on a horizontal plane near the surface of the earth, the forces of gravity acting on it are balanced by the reaction of the plane. These external forces give rise to internal stresses in the body in the form of reciprocal actions and reactions among its component particles. When such internal stresses appear in a body, it can be said to be in a state of "*weightness*". The sensation of "weightness" a person feels on the earth is due to just such reciprocal actions of the parts of his body on each other.

But when a body is *moving* in the earth's gravitational field, internal stresses due to external actions or the type of motion may not appear. In that case the body is said to be in a state of *weightlessness*. Thus, *weightlessness in a gravitational field is a state in which neither the external forces acting on a body nor its motion give rise to internal stresses of forces of any kind*. The state is similar to that a body would be in when at rest outside the gravitational fields of celestial bodies*).

*) Note the difference between *weightness* and *weightlessness* as characterising states in which a body may be and the concept of *weight*, which is a quantitative characteristic (see footnote on p. 148).

In considering the stresses generated in a body by external forces it is necessary to distinguish between so-called mass and surface forces. *Mass forces* are forces that act on all the particles of a body and are proportional in magnitude to the masses of those particles; an example of a mass force is gravity. *Surface forces* are forces applied to points on the surface of a body; examples are the reactions of constraints, tractive forces, the resistance of a surrounding medium, etc. In determining the law of motion (or conditions of equilibrium) of a body, the physical nature of the applied forces is immaterial; only their magnitudes and directions matter. Not so in the case of internal stresses in a body. The reason is that mass forces act directly on all its particles; the action of surface forces is transmitted to them by the pressures of neighbouring particles.

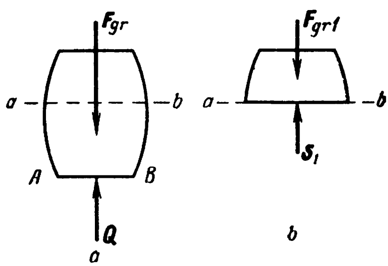

Fig. 294

Consider a body of mass m in translational, not necessarily rectilinear, motion (the paths of the body's particles may be curves) in the earth's gravitational field. The dimensions of the body may be so small in comparison with the radius of the earth that we can neglect the differences in the distances of the various particles from the centre of the earth and assume that the gravitational forces impart to them the same acceleration g. In that case the resultant of the forces of gravity acting on the body is

$$\boldsymbol{F}_{\text{gr}} = m\boldsymbol{g}. \tag{113}$$

In particular, in the case of a body moving (or at rest) near the surface of the earth, it can for all practical purposes be assumed that $\boldsymbol{F}_{\text{gr}} = m\boldsymbol{g}_0 = \boldsymbol{P}$, where in magnitude $\boldsymbol{P}$ equals the body's weight.

Assume that acting on the body besides gravity are surface forces applied along a surface AB and having a resultant $\boldsymbol{Q}$ (Fig. 294*a*). Force $\boldsymbol{Q}$ may be the reaction of the floor of an elevator (or airplane or spaceship cabin) on which the body rests, or a tractive or resisting force, etc. The direction of $\boldsymbol{Q}$ can be arbitrary, notably as shown in Fig. 294*a*.

From § 99, it follows that a body in translational motion can be treated as a material particle. Writing the equation of motion in vector form (which is valid for any direction of $\boldsymbol{Q}$), we obtain:

$$m\boldsymbol{w} = \boldsymbol{F}_{\text{gr}} + \boldsymbol{Q}. \tag{114}$$

From this, taking into account Eq. (113), we find the acceleration of the body:

$$\boldsymbol{w} = \boldsymbol{g} + \frac{\boldsymbol{Q}}{m}. \tag{115}$$

Now let us compute the internal stresses due to forces $\boldsymbol{F}_{gr}$ and $\boldsymbol{Q}$ at any section ab perpendicular to vector $\boldsymbol{Q}$, i.e., the forces with which the parts of the body separated by the section act on each other. For this consider the motion of one of the parts, say the upper, and denote its mass m_1. Acting on it are the forces of gravity, whose resultant, by Eq. (113), is $\boldsymbol{F}_{gr1} = m_1\boldsymbol{g}$, and the forces of the other part of the body, whose resultant is $\boldsymbol{S}_1$ (Fig. 294*b*). Since the part of the body is, like the whole, in translational motion with acceleration $\boldsymbol{w}$, we have $m_1\boldsymbol{w} = \boldsymbol{F}_{gr1} + \boldsymbol{S}_1$, or $m_1\boldsymbol{w} = m_1\boldsymbol{g} + \boldsymbol{S}_1$, whence $\boldsymbol{S}_1 = m_1(\boldsymbol{w} - \boldsymbol{g})$. Substituting the expression of $\boldsymbol{w}$ [(Eq. (115)], we finally obtain:

$$\boldsymbol{S}_1 = \frac{m_1}{m}\boldsymbol{Q}. \tag{116}$$

As the force of gravity $\boldsymbol{F}_{gr}$ does not appear in the equation it is, obviously, valid for motion in a gravitational field of several celestial bodies (e.g., the earth, the moon, etc.). The only requirement is that the moving body be small in comparison with the distance of its particles from the centres of the respective attracting bodies.

It follows from the vector equation (116) that at $\boldsymbol{Q} \neq 0$ the body is not weightless. This is also true of a body at rest, as the acceleration $\boldsymbol{w}$ does not appear in Eq. (116) either. If the body rests on a horizontal plane motionless relative to the earth, $\boldsymbol{Q}$ is the reaction of the plane and in magnitude $Q = F_{gr} = P$, where P is the weight of the body. Thus, at $Q = P$, Eq. (116) gives the internal stresses in a body at rest on a horizontal plane. In a moving body, if the surface force $Q < P$, the stresses in any section of the body are less than at rest (negative loading); if $Q > P$ (e.g., $\boldsymbol{Q}$ is the thrust of a rocket launched vertically), the internal stresses are greater than at rest (g-loading).

Eq. (116) leads to the following important conclusion: in a body at rest or in translational motion in the earth's gravitational field the internal stresses due to external forces, and hence the state of weightness, occur only when the acting forces are *surface forces*. When the only external force acting on a body moving in a gravitational field is gravity ($Q = 0$), as Eq. (116) shows, no internal forces due to external action appear, and the body is in the state of weightlessness.*) And, according to Eq. (115), all particles of the body are moving with the acceleration $\boldsymbol{w} = \boldsymbol{g}$.

Physically, this result is explained by the fact that the force of gravity (a mass force) acts directly on all particles of the body, imparting, as we established, accelerations equal to $\boldsymbol{g}$. But that is the acceleration every particle would have had if it were free. Consequently, in translational motion under the influence of only gravita-

*) Note that throughout we are dealing with internal stresses caused by external forces. Other types of internal stresses, such as temperature stresses in structural elements or the strains in the human body caused by physical exercise, can occur in conditions of weightlessness.

tional forces each particle of a body moves like a free body and exerts no pressures on the other particles. Hence, the body is in the state of weightlessness.

Note that a body can be weightless only if it is in translational motion. That is because in non-translational motion all the particles, which we treat as material points, move with different accelerations, that of the centre of gravity C being $\boldsymbol{w}_C = \boldsymbol{g}$. For any other particle B the acceleration is, according to Eq. (72) in § 88,

$$\boldsymbol{w}_B = \boldsymbol{w}_C + \boldsymbol{w}_{BC} = \boldsymbol{g} + \boldsymbol{w}_{BC}. \tag{117}$$

Write the vector equation of motion of particle B: $m_B \boldsymbol{w}_B = m_B \boldsymbol{g} + \boldsymbol{N}_B$, where m_B is the mass of the particle, $m_B \boldsymbol{g}$ is the gravitational force acting on it, and $\boldsymbol{N}_B$ is the resultant of the pressures of neighbouring particles. Substituting the value of $\boldsymbol{w}_B$ from Eq. (117), we have $m_B (\boldsymbol{g} + \boldsymbol{w}_{BC}) = m_B \boldsymbol{g} + \boldsymbol{N}_B$, whence

$$\boldsymbol{N}_B = m_B \boldsymbol{w}_{BC}. \tag{118}$$

As in non-translational motion $\boldsymbol{w}_{BC} \neq 0$, it follows that $\boldsymbol{N}_B \neq 0$, i.e., the particles do interact and the body is not weightless. Weightlessness is possible only when $\boldsymbol{w}_{BC} = 0$, i.e., when a body is in translational motion.

It is important in all the above reasoning that the dimensions of the body concerned be small as compared with the radius of the earth (or the distance to the attracting body), as only then can the accelerations $\boldsymbol{g}$ be assumed the same for all its particles.

Thus, any body in free (i.e., subject only to the forces of gravity) translational motion in the earth's gravitational field is in the state of weightlessness provided the body's dimensions are small as compared with its distance to the centre of the earth. The above is true for motion in the gravitational field of any celestial bodies.

Consequently, if the resistance of the air is small enough to be neglected, any body falling to the earth or thrown from its surface and in translational motion is in the state of weightlessness. Notably, all artificial earth satellites or space vehicles moving beyond the earth's atmosphere without rotating are weightless together with all bodies within them.

Weightlessness is an important factor to be reckoned with in space flight as it changes the operating conditions of instruments and devices to the extent that some, such as those incorporating physical pendulums or employing free flow of liquid, become simply useless. Molecular forces, which in terrestrial conditions are small compared with the reciprocal pressures due to weightness, become a significant factor in weightlessness, affecting certain phenomena. For example, molecular forces will cause a wetting liquid in a closed vessel to spread evenly over the walls of the vessel, with any air inside occupying the centre of the vessel. A nonwetting liquid will collect into a sphere*).

A reference frame moving together with bodies in free translational motion in a gravitational field (i.e., in a state of weightlessness) possesses an interesting property. Let us call such a frame a *local reference frame* and investigate the motion of a particle of mass m relative to it, assuming the domain in which the motion takes place so small that for all practical purposes $\boldsymbol{g} =$ const. Then, as the reference system is in translational motion, in the equation of relative motion of a particle [Eq. (51) in § 120] the transport force of inertia, $\boldsymbol{F}^{\mathrm{i}}_{\mathrm{tr}} = -m\boldsymbol{w}_{\mathrm{tr}} = -m\boldsymbol{g}$, will be balanced by the force of gravity, $\boldsymbol{F}_{\mathrm{gr}} = m\boldsymbol{g}$, and $\boldsymbol{F}^{\mathrm{i}}_{\mathrm{Cor}} = 0$. Consequently, Eq. (51) of § 120 acquires the same form as in an inertial frame of reference, i.e.,

$$m\boldsymbol{w} = \boldsymbol{F}, \tag{119}$$

where $\boldsymbol{F}$ is the resultant of all forces acting on the particle except the *force of gravity*. Thus, a local frame of reference is, in small domains, for all practical purposes inertial, even though it is moving with acceleration relative to a stellar frame of reference. As the force of gravity $\boldsymbol{F}_{\mathrm{gr}}$ is balanced by $\boldsymbol{F}^{\mathrm{i}}_{\mathrm{tr}}$ and does not enter the equations of motion, they are written as though the local reference frame lies outside the gravitational field.

Note that, if not for its diurnal rotation, the earth in its rotation about the sun would be in a state of "solar weightlessness". Therefore for motions in domains that are small as compared with the distance to the sun a frame of reference with its origin at the centre of the earth and axes directed towards the fixed stars (i.e., not participating in the diurnal rotation of the earth) is for all practical purposes inertial, and the equations of motion in it have the form (119). A reference frame rigidly connected with the earth and taking part in its diurnal motion is, accordingly, not inertial (see § 121.)

Another example is a frame of reference connected with a spaceship moving freely without rotation in a gravitational field. In it the equations of motion also have the form (119) and, as said before, all mechanical phenomena take place as though the ship were outside a gravitational field.

It follows from Eq. (119) that at $\boldsymbol{F} = 0$, i.e., when gravity is the only acting force, a body will remain freely "floating" in any part of the vehicle, a manifestation of weightlessness that has been expe-

*) Weightlessness affects the functioning of a number of organs of the human body, e.g., the organ of equilibrium, and adaptation to weightlessness requires appropriate training. It has therefore been suggested that for long flights in space stations be built in the shape of a large wheel with rooms in the rim. The rotation of the spaceship would make the particles of a body in it act on each other with forces described by Eq. (118), thereby creating an artificial state of weightness (artificial gravity).

rimentally observed*). In itself, however, the fact of "free suspension" does not necessarily imply weightlessness. It can be observed on the earth, for example, with a body submerged in a liquid of the same specific gravity. But there will be *surface forces* of pressure from the liquid acting on the body, creating internal stresses, and it will be in the state of weightness.

Weightlessness can occur when several mass forces are acting on a body, because of the absence of internal stresses. Nor does the phenomenon depend on the frame of reference in which the motion is examined, and the observed effects will be the same as seen by an observer in any reference system. For example, a nonwetting liquid will be seen as gathering into a sphere by an astronaut in a spaceship and an observer on the earth equipped with a sufficiently powerful telescope.

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*) A terrestrial observer would explain this as being due to the fact that the spaceship and the "floating" body in its cabin are moving around the earth along virtually identical orbits and therefore are motionless relative to each other.

Part 4

DYNAMICS OF A SYSTEM AND A RIGID BODY

Chapter 23

Introduction to the Dynamics of a System. Moments of Inertia of Rigid Bodies

§ 129. Mechanical Systems. External and Internal Forces

A mechanical system is defined as such a collection of material points (particles) or bodies in which the position or motion of each particle or body of the system depends on the position and motion of all the other particles or bodies. We shall regard a body as a system of its particles.

A classical example of a mechanical system is the solar system, all the component bodies of which are connected by the forces of their mutual attraction. Other examples of mechanical systems are machines, or any mechanism whose members are connected with pins, rods, cables, belts, etc., by holonomic constraints. In this case the bodies of the system are subjected to the reciprocal compressive or tensile forces transmitted through the constraints.

A collection of bodies not connected by interacting forces does not comprise a mechanical system (e.g., a group of flying aircraft). In this book we shall consider only mechanical systems, calling them just "systems" for short. It follows from the above that the forces acting on the particles or bodies of a system can be subdivided into external and internal forces.

External forces are defined as the forces exerted on the members of a system by particles or bodies not belonging to the given system. *Internal forces* are defined as the forces of interaction between the members of the same system. We shall denote external forces by the symbol $\boldsymbol{F}^e$, and internal forces by the symbol $\boldsymbol{F}^i$.

Both external and internal forces can be either active forces or reactions of constraints. The division of forces into external and internal is purely relative, and it depends on the extent of the system whose motion is being investigated. In considering the motion of the solar system as a whole, for example, the gravitational attraction of the sun acting on the earth is an internal force; in investigating the earth's motion about the sun, the same force is external.

Internal forces possess the following properties:

(1) The geometrical sum (the principal vector) of all the internal forces of a system is zero. This follows from the third law of dynamics, which states that any two particles of a system (Fig. 295) act on each other with equal and oppositely directed forces F^i_{12} and F^i_{21}, the sum of which is zero. Since the same is true for any pair of particles of a system,

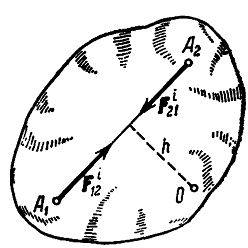

Fig. 295

$$\sum F^i_k = 0.$$

(2) The sum of the moments (the principal moment) of all the internal forces of a system with respect to any centre or axis is zero. For if we take an arbitrary centre O, it is apparent from Fig. 295 that $m_O(F^i_{12}) + m_O(F^i_{21}) = 0$. The same result holds for the moments about any axis. Hence, for the system as a whole we have:

$$\sum m_O(F^i_k) = 0 \quad \text{or} \quad \sum m_x(F^i_k) = 0$$

It does not follow from the above, however, that the internal forces are mutually balanced and do not affect the motion of the system, for they are applied to *different* particles or bodies and may cause their mutual displacement. The internal forces will be balanced only when a given system is a rigid body (see § 3).

§ 130. Mass of a System. Centre of Mass

The motion of a system depends, besides the acting forces, on its total mass and the distribution of this mass. The *mass of a system* is equal to the arithmetical sum of the masses of all the particles or bodies comprising it *):

$$M = \sum m_k.$$

*) We shall conventionally denote mass by the same letter M as the moment of a force. Any possibility of confusion is precluded by the fact that when the symbol M denotes moment of a force, it is always provided with a subscript (e.g., M_C, M_x, M_t). In Chapter 30, however, where the symbols of moment and mass intermingle frequently, we shall denote the mass of a system by the symbol $\mathscr{M}$.

In a homogeneous field of gravity, where g = const., the weight of every particle of a body is proportional to its mass, hence the distribution of mass can be judged according to the position of the centre of gravity. Let us rewrite the equations defining the coordinates of the centre of gravity [see § 54, Eqs. (74)] in a form manifestly including mass. For that in Eqs. (74) assume that $p_k = m_k g$ and $P = Mg$. Cancelling out g, we obtain:

$$x_C = \frac{\sum m_k x_k}{M}, \quad y_C = \frac{\sum m_k y_k}{M}, \quad z_C = \frac{\sum m_k z_k}{M}. \tag{1}$$

The equations include only the masses m_k of the material points (particles) of the body and their coordinates x_k, y_k, z_k. Hence, the position of point C (x_C, y_C, z_C) gives the distribution of mass in the body or in any mechanical system, where m_k and x_k, y_k, z_k are the masses and coordinates of the system's respective points.

The geometric point C whose coordinates are given by Eqs. (1) is called the *centre of mass*, or *centre of inertia* of a mechanical system.

If the position of the centre of mass is defined by its radius vector $\boldsymbol{r}_C$, we can obtain from Eqs. (1) the following expression:

$$\boldsymbol{r}_C = \frac{\sum m_k \boldsymbol{r}_k}{M}, \tag{1'}$$

where $\boldsymbol{r}_k$ is the radius vector of particle k of the system.

Although in a homogeneous gravitational field the centres of mass and gravity coincide, the two concepts are not identical. The concept of centre of gravity, as the point through which the resultant of the forces of gravity passes, has meaning only for a rigid body in a uniform field of gravity. The concept of centre of mass, as a characteristic of the distribution of mass in a system, on the other hand, has meaning for any system of particles or bodies, regardless of whether a given system is subjected to the action of forces or not.

§ 131. Moment of Inertia of a Body About an Axis. Radius of Gyration

The position of centre of mass does not characterise completely the distribution of mass in a system. For if in the system in Fig. 296 the distance h of each of two identical spheres A and B from the axis Oz is increased by the same quantity, the location of the centre of mass will not change, though the distribution of mass will change and influence the motion of the system (all other conditions remaining the same, the rotation about axis Oz will be slower).

Accordingly, another characteristic of the distribution of mass, called the moment of inertia, is introduced in mechanics. *The moment* *of inertia of a body (system) with respect to a given axis Oz (or the axial moment of inertia) is defined as the quantity equal to the sum of the masses of the particles of the body (system) each multiplied by the square of its perpendicular distance from the axis*:

$$J_z = \sum m_k h_k^2. \tag{2}$$

It follows from the definition that the moment of inertia of a body (or system) with respect to any axis is always positive.

It will be shown further on that axial moment of inertia plays the same part in the rotational motion of a body as mass does in translational motion, i.e., *moment of inertia is a measure of a body's inertia in rotational motion.*

By Eq. (2), the moment of inertia of a body is equal to the sum of the moments of inertia of all its parts with respect to the same axis.

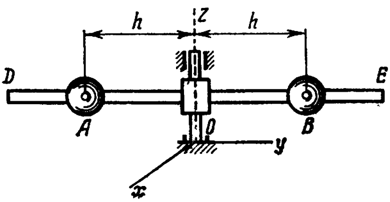

Fig. 296

For a material point located at a distance h from an axis, $J_z = mh^2$. The unit for the moment of inertia in the SI system is 1 kg-m², and in the mkg(f)s system 1 kgf-m-s².

In computing the axial moments of inertia the distances of the points from the axes can be expressed in terms of their coordinates x_k, y_k, z_k (e.g., the square of the distance from axis Ox is $y_k^2 + z_k^2$, etc.). Then the moments of inertia about the axes $Oxyz$ will be given by the following equations:

$$J_x = \sum m_k (y_k^2 + z_k^2), \quad J_y = \sum m_k (z_k^2 + x_k^2), \quad J_z = \sum m_k (x_k^2 + y_k^2). \tag{3}$$

The concept of the *radius of gyration* is often employed in calculations. The radius of gyration of a body with respect to an axis Oz is a linear quantity ρ_g defined by the equation

$$J_z = M\rho_g^2, \tag{4}$$

where M is the mass of the body.

It follows from the definition that geometrically the radius of gyration is equal to the distance from the axis Oz to a point, such that if the mass of the whole body were concentrated in it, the moment of inertia of the point would be equal to the moment of inertia of the whole body.

Knowing the radius of gyration, we can obtain the moment of inertia of a body from Eq. (4) and vice versa.

Eqs. (2) and (3) are valid for both rigid bodies and systems of material points. In the case of a solid body, dividing it into elementary parts, we find that in the limit the sum in Eq. (2) becomes an integral. Hence, taking into account that $dm = \rho\, dV$, where ρ is the density and V the volume, we obtain:

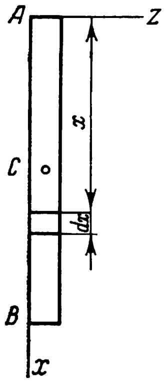

Fig. 297

$$J_z = \int_{(V)} h^2\, dm, \quad \text{or} \quad J_z = \int_{(V)} \rho h^2\, dV. \tag{5}$$

The integral extends over the whole volume V of the body, and the density ρ and distance h depend on the coordinates of the points of the body. Similarly, for solid bodies Eqs. (3) take the form:

$$J_x = \int_{(V)} \rho\, (y^2 + z^2)\, dV, \ \text{etc.} \tag{5'}$$

Eqs. (5) and (5′) are useful in calculating the moments of inerita of homogeneous bodies of geometric shape. As in that case the density ρ is constant, it can be taken out of the integral sign.

Let us determine the moments of inertia of some homogeneous bodies.

(1) Thin Homogeneous Rod of Length l and Mass M. Let us find its moment of inertia with respect to an axis Az perpendicular to the rod (Fig. 297). If we lay off a coordinate axis Ax along AB, for any line element of length dx we have $h = x$ and its mass $dm = \rho_1 dx$, where $\rho_1 = M/l$ is the mass o fa unit length of the rod, and Eq. (5) gives*):

$$J_A = \int_0^l x^2\, dm = \rho_1 \int_0^l x^2\, dx = \rho_1 \frac{l^3}{3}.$$

Substituting the expression for ρ_1, we obtain finally

$$J_A = \frac{1}{3} M l^2. \tag{6}$$

(2) Thin Circular Homogeneous Ring of Radius R and Mass M. Let us find its moment of inertia with respect to an axis Cz perpendicular to the plane of the ring through its centre (Fig. 298*a*). As all the points of the ring are at a distance $h_k = R$ from axis Cz, Eq. (2) gives

$$J_C = \sum m_k R^2 = \left(\sum m_k\right) R^2 = M R^2.$$

*) Here and further on, J_A denotes the moment of inertia with respect to an axis through A perpendicular to the plane of the cross section in the diagram.

Hence, for a ring

$$J_C = MR^2. \tag{7}$$

It is evident that the same result is obtained for the moment of inertia of a cylindrical shell of mass M and radius R with respect to its axis.

(3) Circular Homogeneous Disc or Cylinder of Radius R and Mass M. Let us compute the moment of inertia of a circular disc with respect to an axis Cz perpendicular to it through its centre (Fig. 298*b*).

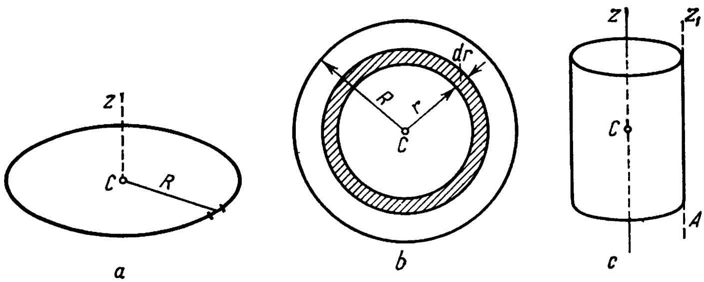

Fig. 298

Consider an elemental ring of radius r and width dr. Its area is $2\pi r\,dr$, and its mass $dm = \rho_2 2\pi r\,dr$, where $\rho_2 = \frac{M}{\pi R^2}$ is the mass of a unit area of the disc. From Eq. (7) we have for the elemental ring

$$dJ_C = r^2 dm = 2\pi\rho_2 r^3\,dr,$$

and for the whole disc

$$J_C = 2\pi\rho_2 \int_0^R r^3\,dr = \frac{1}{2}\pi\rho_2 R^4.$$

Substituting the expression for ρ_2, we obtain finally

$$J_C = \frac{1}{2} MR^2. \tag{8}$$

It is evident that the same formula is obtained for the moment of inertia J_z of a homogeneous circular cylinder of mass M and radius R with respect to its axis Cz (Fig. 298*c*).

(4) Rectangular Lamina, Cone, and Sphere. Omitting the computations, here are the equations of the moments of inertia of several bodies (the student is invited to deduce these formulas independently).

(a) uniform rectangular lamina of mass M with sides of length a and b (axis x is coincident with side a, axis y with side b):

$$J_x = \frac{1}{3} Mb^2, \qquad J_y = \frac{1}{3} Ma^2;$$

(b) uniform right circular cone of mass M and base radius R (axis z is coincident with the axis of the cone):

$$J_z = 0.3MR^2;$$

(c) uniform sphere of mass M and radius R (axis z is coincident with a diameter):

$$J_z = 0.4MR^2.$$

The moments of inertia of non-homogeneous and composite bodies can be determined experimentally with the help of appropriate instruments. One such method is given in § 155.

§ 132. Moments of Inertia of a Body About Parallel Axes. The Parallel Axis (Huygens') Theorem

In the most general case, the moments of inertia of the same body with respect to different axes are different. Let us see how to determine the moment of inertia of a body with respect to any axis if its moment of inertia with respect to a parallel axis through the body is known.

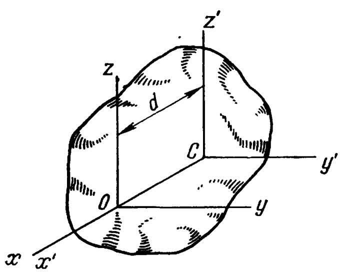

Fig. 299

Draw through the centre of mass of a body C arbitrary axes $Cx'y'z'$, and through an arbitrary point O on axis Cx' axes $Oxyz$, so that $Oy \parallel Cy'$ and $Oz \parallel Cz'$ (Fig. 299). Denoting the distance between axes Cz' and Oz by d, from Eqs. (3)

$$J_{Oz} = \sum m_k (x_k^2 + y_k^2),$$

$$J_{Cz'} = \sum m_k (x_k'^2 + y_k'^2).$$

But it is apparent from the drawing that for any point of the body $x_k = x_k' - d$, or $x_k^2 = x_k'^2 + d^2 - 2x_k'd$, and $y_k = y_k'$. Substituting these expressions for x_k and y_k into the expression for J_{Oz} and taking the common multipliers d^2 and $2d$ outside the parentheses, we obtain:

$$J_{Oz} = \sum m_k (x_k'^2 + y_k'^2) + \left(\sum m_k\right) d^2 - 2d \left(\sum m_k x_k'\right).$$

The first summation in the right member of the equation is equal to $J_{Cz'}$, and the second to the mass M of the body. Let us find the value of the third summation. From Eq. (1) we know that, for the coordinates of the centre of mass, $\sum m_k x_k' = Mx_C'$. But since in our case point C is the origin, $x_C' = 0$, and consequently $\sum m_k x_k' = 0$.

We finally obtain

$$J_{Oz} = J_{Cz'} + Md^2. \tag{9}$$

Eq. (9) expresses the **parallel axis theorem** enunciated by Huygens[*]: *The moment of inertia of a body with respect to any axis is equal to the moment of inertia of the body with respect to a parallel axis through the centre of mass of the body plus the product of the mass of the body and the square of the distance between the two axes.*

It follows from Eq. (9) that $J_{Oz} > J_{Cz'}$. Consequently, of all the axes of same direction, the moment of inertia is least with respect to the one through the centre of mass.

The parallel axis theorem can also be used to determine the moment of inertia of a body with respect to a given axis Oz_1 if its moment of inertia with respect to any parallel axis Az_2 and the distances d_1 and d_2 of each axis from the body's centre of mass are known. Hence, knowing J_{Az_2} and d_2, we obtain $J_{Cz'}$ from Eq. (9) and, applying the same formula, determine the required moment of inertia J_{Oz_1}.

Problem 122. Determine the moment of inertia of a thin rod with respect to an axis Cz perpendicular to it through its centre of mass.

Solution. Draw an axis Az through end A of the rod (see Fig. 297; axis Cz is not shown in it). Then, by Eq. (9),

$$J_C = J_A - Md^2.$$

In our case $d = \frac{l}{2}$, where l is the length of the rod, and J_A is found from Eq. (6). Hence,

$$J_C = \frac{1}{3}Ml^2 - \frac{1}{4}Ml^2 = \frac{1}{12}Ml^2.$$

Problem 123. Determine the moment of inertia of a cylinder with respect to an axis Az_1 through its generator (see Fig. 298*c*).

Solution. From the parallel axis theorem, $J_{Az_1} = J_{Cz} + Md^2$. In our case $d = R$ and from Eq. (8) $J_{Cz} = \frac{1}{2}MR^2$.

Substituting these expressions, we obtain:

$$J_{Az_1} = \frac{1}{2}MR^2 + MR^2 = \frac{3}{2}MR^2.$$

[*] Christian Huygens (1629-1695), celebrated Dutch mechanic, physicist and astronomer. He developed the first pendulum clock. In this connection he studied the vibrations of the compound pendulum (see § 155) and introduced the concept of moment of inertia of a body.

§ 133*. Product of Inertia. Principal Axes of Inertia of a Body

The moment of inertia of a body with respect to an axis also does not completely characterise the distribution of mass of the system. For example, if the rod DE in Fig. 296 is turned in plane Oyz so as to make other than a right angle with axis Oz and the distance h of spheres A and B from the axis is kept the same by moving them outward, neither the location of the centre of mass nor the moment of inertia of the spheres with respect to axis Oz will change. Yet the distribution of mass will have changed (the symmetry with respect to axis Oz being disturbed), and this will affect the system's rotation about the axis (additional lateral stresses will appear in the bearing).

Accordingly, the concept of the product of inertia is introduced as characterising such asymmetry in the distribution of mass. Drawing coordinate axes $Oxyz$ through point O, the *products of inertia* with respect to those axes are the quantities J_{xy}, J_{yz}, J_{zx} given by the following equations:

$$J_{xy}=\sum m_k x_k y_k, \quad J_{yz}=\sum m_k y_k z_k, \quad J_{zx}=\sum m_k z_k x_k, \tag{10}$$

where m_k is the mass of the points and x_k, y_k, z_k are their coordinates. Obviously, $J_{xy} = J_{yx}$, etc.

For solid bodies, Eqs. (10) by analogy with (5′) take the form:

$$J_{xy}=\int\limits_{(V)} \rho xy\,dV, \text{ etc.} \tag{10′}$$

Unlike axial moments of inertia, products of inertia can be either positive, negative or, in selected coordinates, zero.

Consider a homogeneous body having an axis of symmetry. Draw the coordinate axes $Oxyz$ so that axis z is directed along the axis of symmetry (see, for example, Fig. 342). By virtue of symmetry, to each point of mass m_k with coordinates (x_k, y_k, z_k) there corresponds another point of equal mass with coordinates $(-x_k, -y_k, z_k)$. Consequently, $\sum m_k x_k z_k = 0$ and $\sum m_k y_k z_k = 0$, and taking into account Eqs. (10), we obtain:

$$J_{xz}=0, \quad J_{yz}=0. \tag{11}$$

Thus, symmetry in the distribution of mass with respect to axis Oz is characterised by two centrifugal moments of inertia, J_{xz} and J_{yz}, becoming zero. *Axis Oz with respect to which the products of inertia J_{xz}, J_{yz}, whose subscripts contain the notation of that axis, are zero is called the principal axis of inertia of the body with respect to point O*

It follows from what has been said that if a body has an axis of symmetry, that axis is the principal axis of inertia with respect to any of its points.

The principal axis of inertia is not necessarily the axis of symmetry. Consider a homogeneous body having a plane of symmetry. Drawing axes Oxy in that plane and axis Oz perpendicular to them, by virtue of symmetry, to every point of mass m_k with coordinates (x_k, y_k, z_k) there corresponds a point of same mass with coordinates $(x_k, y_k, -z_k)$. Consequently, as in the previous case, we find that $\sum m_k x_k z_k = 0$ and $\sum m_k y_k z_k = 0$, or $J_{xz} = 0$ and $J_{yz} = 0$.

Thus, if a body has a plane of symmetry, any axis perpendicular to that plane is the principal axis of inertia of the body with respect to the point O at which the axis intersects the plane.

Eqs. (11) express the conditions that axis Oz is the principal axis of inertia of a body with respect to point O (the origin of the coordinate system). Similarly, if $J_{xy} = 0$ and $J_{xz} = 0$, axis Ox will be the principal axis of inertia with respect to point O, etc. Consequently, if all the products of inertia are zero, i.e.,

$$J_{xy} = 0, \quad J_{yz} = 0, \quad J_{zx} = 0, \tag{12}$$

each of the coordinate axes $Oxyz$ is a principal axis of inertia with respect to point O (the origin of the coordinate system).

For example, all three axes $Oxyz$ in Fig. 342 are principal axes of inertia with respect to point O (Oz as the axis of symmetry and Ox and Oy as perpendicular to the planes of symmetry).

The moments of inertia with respect to the principal axes of inertia are called the *principal moments of inertia of a body*. The principal axes of inertia constructed relative to a body's centre of mass are called the *principal central axes of inertia*. It follows from what has been said that if a body has an axis of symmetry, that axis is one of the body's principal central axes of inertia, as the centre of mass lies on it (see § 56). If a body has a plane of symmetry, the axis perpendicular to it through the centre of mass is also one of the principal central axes of inertia of the body.

The above examples considered symmetrical bodies. However, it can be proved that through any point of a body at least three mutually perpendicular axes can be drawn for which Eqs. (12) are satisfied, i.e., which are the principal axes of inertia of the body with respect to that point (the proof is presented below).

The concept of principal axes of inertia plays an important part in rigid-body dynamics. If the coordinate axes $Oxyz$ are drawn along them, all the products of inertia become zero, which greatly simplifies the relevant equations and formulas (see § 158). The concept is also employed in solving problems of dynamic balancing of masses (§ 169), centre of impact (§ 165), and others.

Let us show that principal axes of inertia exist at any point of a body. For this let us first prove the following **theorem:** *If the moment of inertia with respect to an axis Oz is greater or smaller than*

the moment of inertia with respect to any neighbouring axis through O, that axis (Oz) is the principal axis of inertia of the body with respect to O. Thus, we must prove that if in any elementary rotation of axis Oz about point O, $dJ_z = 0$ (extremum condition), that axis is the principal axis of inertia with respect to point O.

Proof. Draw the xyz axes through O and turn them (without altering the position of the body) about axis Ox through an angle $d\theta_1$ (Fig. 300). From analytic geometry, in such a rotation the coordinate y_k of any point B_k of the body transforms according to the formula

$$y'_k = y_k \cos d\theta_1 + z_k \sin d\theta_1,$$

or, since $d\theta_1$ is an elementary angle, $y'_k = y_k + z_k\, d\theta_1$, which is also apparent from the drawing. Hence, $y'^2_k = y^2_k + 2y_k z_k d\theta_1 + z^2_k d\theta^2_1$. Neglecting infinitesimals of higher orders, i.e., the members containing $d\theta^2_1$, and taking into account $x'_k = x_k$, we obtain:

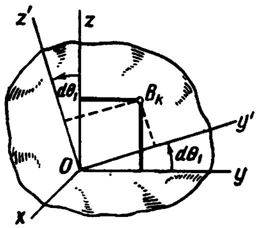

Fig. 300

$$J_{z'} = \sum m_k (x'^2_k + y'^2_k) = \sum m_k (x^2_k + y^2_k) + 2\sum m_k y_k z_k\, d\theta_1.$$

Taking the common multiplier outside the parentheses, we obtain:

$$J_{z'} = J_z + 2J_{yz}\, d\theta_1, \quad \text{or} \quad (dJ_z)_1 = J_{z'} - J_z = 2J_{yz}\, d\theta_1,$$

where $(dJ_z)_1$ is the elementary increment of J_z in the rotation considered. Similarly, in an elementary rotation about axis Oy through angle $d\theta_2$ we obtain $(dJ_z)_2 = 2J_{xz} d\theta_2$. Since any elementary displacement of axis Oz is compounded of two such rotations, the extremum condition of J_z takes the form

$$(dJ_z)_1 = 0, \quad (dJ_z)_2 = 0.$$

Hence, if this condition is satisfied, as $d\theta_1 \neq 0$, $d\theta_2 \neq 0$, we should have

$$J_{xz} = 0, \quad J_{yz} = 0,$$

i.e., axis Oz is the principal axis of inertia with respect to point O, and the theorem has been proved.

Note that if the direction of Oz is changed continuously, the axial moment of inertia J_z also changes continuously. But J_z cannot increase or decrease indefinitely, as for real bodies $0 < J_z < \infty$. Hence, from the corresponding theorems of calculus it follows that out of all the orientations of axis Oz there is at least one for which J_z has a maximum (compared with its value for neighbour axes), and one for which it is minimum. It thus follows that through any point in a body there pass at least two principal axes of inertia. It can also be shown that they are mutually perpendicular*). But by definition [Eqs. (11)], if two mutually perpendicular axes drawn through point O (e.g., Ox and Oy) are principal axes with respect to it, then $J_{xz} = 0$, $J_{yz} = 0$, and the third axis perpendicular to them (Oz) is also a principal axis of inertia with respect to O. Consequently, through any point of a body there pass at least three mutually perpendicular principal axes of inertia of the body with respect to that point.

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*) Let these principal axes be Oz and Oz' and the angle between them $\alpha \neq 0$. Draw in plane zOz' axes $Oy \perp Oz$ and $Oy' \perp Oz'$ (see Fig. 300, assuming angle $d\theta_1 = \alpha$). Then $y'_k = y_k \cos\alpha + z_k \sin\alpha$, $z'_k = z_k \cos\alpha - y_k \sin\alpha$ and $J_{y'z'} = \sum m_k y'_k z'_k = \sum m_k (z_k^2 - y_k^2) \sin\alpha \cos\alpha$, as $\sum m_k y_k z_k = J_{yz} = 0$ (Oz is a principal axis). Using Eqs. (3), it will be readily found by subtracting that $J_y - J_z = \sum m_k (z_k^2 - y_k^2)$; consequently, $J_{y'z'} = 0.5\,(J_y - J_z) \sin 2\alpha$. But Oz' is also a principal axis, consequently we should have $J_{y'z'} = 0$, or $(J_y - J_z) \sin 2\alpha = 0$. Hence, if $J_y \neq J_z$, $\alpha = 90°$, Q.E.D. If $J_y = J_z$, then any axis in plane zOz' is a principal axis, including the one perpendicular to Oz.

Chapter 24
Theorem of the Motion of the Centre of Mass of a System

§ 134. The Differential Equations of Motion of a System

Suppose we have a system of n particles. Choosing any particle of mass m_k, belonging to the system, let us denote the resultant of all the external forces acting on the particle (both active forces and the forces of reaction) by the symbol $\boldsymbol{F}_k^e$, and the resultant of all the internal forces by $\boldsymbol{F}_k^i$. If the particle has an acceleration $\boldsymbol{w}_k$, then, by the fundamental law of dynamics,

$$m_k\boldsymbol{w}_k = \boldsymbol{F}_k^e + \boldsymbol{F}_k^i.$$

Similar results are obtained for any other particle, whence, for the whole system, we have:

$$\left.\begin{aligned} m_1\boldsymbol{w}_1 &= \boldsymbol{F}_1^e + \boldsymbol{F}_1^i, \\ m_2\boldsymbol{w}_2 &= \boldsymbol{F}_2^e + \boldsymbol{F}_2^i, \\ &\cdots\cdots\cdots \\ m_n\boldsymbol{w}_n &= \boldsymbol{F}_n^e + \boldsymbol{F}_n^i. \end{aligned}\right\} \qquad (13)$$

These equations, from which we can develop the law of motion of any particle of the system, are called the *differential equations of motion of a system in vector form*. Eqs. (13) are differential because $\boldsymbol{w}_k = \frac{d\boldsymbol{v}_k}{dt} = \frac{d^2\boldsymbol{r}_k}{dt^2}$. In the most general case the forces in the right side of the equations depend on time, coordinates of the particles of the system, and velocities (see §§ 99 and 107).

By projecting Eqs. (13) on coordinate axes, we can obtain the differential equations of motion of a given system in terms of the projections on these axes.

The complete solution of the principal problem of dynamics for a system would be to develop the equation of motion for each particle of the system from the given forces by integrating the corresponding differential equations. For two reasons, however, this solution is not usually employed. Firstly, the solution is too involved and will almost inevitably lead into insurmountable mathematical difficulties. Secondly, in solving problems of mechanics it is usually sufficient to know certain overall characteristics of the motion of a system, without investigating the motion of each particle. These overall characteristics can be found with the help of the *general theorems of system dynaimcs*, which we shall now study.

The main application of Eqs. (13) or their corollaries will be to develop the respective general theorems.

§ 135. Theorem of Motion of Centre of Mass

In many cases the nature of the motion of a system (especially of a rigid body) is completely described by the law of motion of its centre of mass. To develop this law, let us take the equations of motion of a system (13) and add separately their left and right sides. We obtain:

$$\sum m_k \boldsymbol{w}_k = \sum \boldsymbol{F}_k^e + \sum \boldsymbol{F}_k^i. \tag{14}$$

Let us transform the left side of the equation. For the radius vector of the centre of mass we have, from Eq. (1′),

$$\sum m_k \boldsymbol{r}_k = M\boldsymbol{r}_C.$$

Taking the second derivative of both sides of this equation with respect to time, and noting that the derivative of a sum equals the sum of the derivatives, we find that

$$\sum m_k \frac{d^2\boldsymbol{r}_k}{dt^2} = M\frac{d^2\boldsymbol{r}_C}{dt^2},$$

or

$$\sum m_k \boldsymbol{w}_k = M\boldsymbol{w}_C, \tag{15}$$

where $\boldsymbol{w}_C$ is the acceleration of the centre of mass of the system. As the internal forces of a system give (see § 129) $\sum \boldsymbol{F}_k^i = 0$, by substituting all the developed expressions into Eq. (14), we obtain finally:

$$M\boldsymbol{w}_C = \sum \boldsymbol{F}_k^e. \tag{16}$$

Eq. (16) states the theorem of the motion of the centre of mass of a system. Its form coincides with that of the equation of motion of a particle [§ 100, Eq. (3)] of mass $m = M$ where the acting forces are equal to $\boldsymbol{F}_k^e$. We can therefore formulate the **theorem of the motion of the centre of mass** as follows: *The centre of mass of a system moves as if it were a particle of mass equal to the mass of the whole system to which are applied all the external forces acting on the system.*

Projecting both sides of Eq. (16) on the coordinate axes, we obtain:

$$M\frac{d^2x_C}{dt^2}=\sum F^e_{kx},\quad M\frac{d^2y_C}{dt^2}=\sum F^e_{ky},\quad M\frac{d^2z_C}{dt^2}=\sum F^e_{kz}. \qquad (16')$$

These are the *differential equations of motion of the centre of mass* in terms of the projections on the coordinate axes.

The theorem is valuable for the following reasons:

(1) It justifies the use of the methods of particle dynamics. It follows from Eqs. (16′) that the solutions developed on the assumption that a given body is equivalent to a particle define the law of motion of the centre of mass of that body. Thus, these solutions have a concrete meaning.

In particular, if a body is in translational motion, its motion is completely specified by the motion of its centre of mass, and consequently, a body in translatory motion can always be treated as a particle of mass equal to the mass of the body. In all other cases, a body can be treated as a particle only when the position of its centre of mass is sufficient to specify the position of the body

(2) The theorem makes it possible, in developing the equation of motion for the centre of mass of any system, to ignore all unknown internal forces. This is of special practical value.

§ 136. The Law of Conservation of Motion of Centre of Mass

The following important corollaries arise from the theorem of the motion of centre of mass:

(1) Let the sum of the external forces acting on a system be zero:

$$\sum F^e_k=0.$$

It follows, then, from Eq. (16) that $w_C = 0$ or $v_C = \text{const}$.

Thus, if the sum of all the external forces acting on a system is zero, the centre of mass of that system moves with velocity of constant magnitude and direction, i.e., uniformly and rectilinearly. In particular, if the centre of mass was initially at rest, it will remain at rest. The action of the internal forces, we see, does not affect the motion of the centre of mass.

(2) Let the sum of the external forces acting on a system be other than zero, but let the sum of their projections on one of the coordinate axes (axis x, for instance) be zero:

$$\sum F^e_{kx}=0.$$

The first of Eqs. (16′) then gives

$$\frac{d^2x_C}{dt^2}=0 \quad \text{or} \quad \frac{dx_C}{dt}=v_{Cx}=\text{const}.$$

Thus, if the sum of the projections on an axis of all the external forces acting on a system is zero, the projection of the velocity of the centre of mass of the system on that axis is a constant quantity. In particular, if at the initial moment $v_{Cx}=0$, it will remain zero at any subsequent instant, i.e., the centre of mass of the system will not move along axis x (x_C = const.).

The above results express the *law of conservation of motion of the centre of mass of a system.* Let us examine some cases of its application.

(a) Motion of the Centre of Mass of the Solar System. Since in practice the attraction of the stars can be neglected, we may assume that there are no external forces acting on the solar system. Hence, in the first approximation, its centre of mass is in uniform rectilinear motion through space.

(b) Action of a Force Couple on a Body (see, for example, Fig. 42). If a force couple ($\boldsymbol{F}$, $\boldsymbol{F}'$) starts acting on a body, the geometrical sum of these external forces will be zero ($\boldsymbol{F}+\boldsymbol{F}'=0$). Consequently, if the centre of mass C of a system was initially at rest, it must remain at rest after the couple is applied. Thus, no matter to what part of a free rigid body a couple is applied, the body will start rotating about its centre of mass (see § 19).

(c) Motion on a Horizontal Plane. In the absence of friction a person would be unable to walk along a horizontal plane with the help of his muscular effort (internal forces) alone, as the sum of the projections on any horizontal axis Ox of all the external forces acting on the person (the force of gravity and reaction of the plane) would be zero and his centre of mass would not move parallel to the plane (x_C = const.).

If, for example, he made a step forward with his right foot, his left one would slip back and his centre of mass would remain at rest. With friction, though, the slipping of the left foot would be opposed by a frictional force directed *forward.* This would be the external force making it possible for the person to move in the direction of its action, i.e., forward.

Something similar takes place in the motion of a locomotive or motor car. The pressure of the boiler steam or engine gases is an internal force and cannot by itself displace the centre of mass of the system. Motion is made possible because the engine transmits a *torque* to the so-called driving wheels. The point of contact B of the driving wheel (Fig. 301) tends to slip to the left, generating a frictional force directed to the right. This external force makes the centre of mass of the steam engine or motor car move to the right. When this force is absent, or when it is not sufficent to overcome the opposition to the rotation of the driven wheels*), motion does not take place. The driving wheels will slip without any forward motion.

(d) Braking. Braking is effected by pressing a brake shoe to a drum connected rigidly with a rolling wheel. The frictional force developing between the shoe and drum is an internal force, and alone it

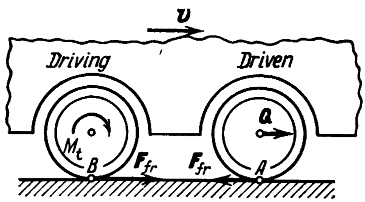

Fig. 301

cannot alter the motion of the centre of mass, i.e., it cannot stop, say, a train or car. The friction of the shoe on the drum, however, retards the rotation of the wheel around its axis and thereby increases the friction of the wheel on the rail or road, which is directed against the motion. This external force will retard the motion of the centre of mass of the vehicle, i.e., produce the braking effect (see Problem 148, § 156).

§ 137. Solution of Problems

The theorem of the motion of the centre of mass is useful to develop the equation of motion of the mass centre if the external forces are known, and conversely, to determine the principal vector of the extrenal forces acting on a system if the equation of motion of the mass centre is known. The first problem was already studied in particle dynamics. Examples of solutions of the second problem are examined below.

The theorem makes it possible to exclude all the internal forces from consideration. This suggests that in investigating a system it should be so chosen as to make some of the immediately unknown forces *internal* with respect to the system.

Whenever the law of conservation of motion of the centre of mass applies, the theorem makes it possible to determine the displace-

*) The torque does not act on the driven wheel. Acting on it is a force applied at its axis, which tends to move the wheel forward together with its point of contact A with the ground. The frictional force acting on this wheel is directed in the opposite direction and opposes the motion.

ment of any part of the system if the displacement of another part is known.

We have proved that when $\sum F^e_{kx} = 0$ and at the initial moment $v_{Cx} = 0$, $x_C = \text{const.}$ throughout the motion of the system. Let us have a system of three bodies of masses m_1, m_2, m_3, whose initial coordinates of the centres of mass are x_1, x_2, x_3. If under the action of any internal (or external) forces the bodies perform absolute displacements, the projections of which on axis x are ξ_1, ξ_2, ξ_3, the respective coordinates will be $x_1 + \xi_1$, $x_2 + \xi_2$, $x_3 + \xi_3$. The coordinate of the centre of mass x_C of the whole system in the initial and final positions will then be:

$$x_{C_0} = \frac{m_1x_1 + m_2x_2 + m_3x_3}{M},$$

$$x_{C_1} = \frac{m_1(x_1+\xi_1) + m_2(x_2+\xi_2) + m_3(x_3+\xi_3)}{M}.$$

As $x_C = \text{const.}$, $x_{C_0} = x_{C_1}$ and consequently,

$$m_1\xi_1 + m_2\xi_2 + m_3\xi_3 = 0, \tag{17}$$

or

$$p_1\xi_1 + p_2\xi_2 + p_3\xi_3 = 0. \tag{17'}$$

Thus, when the law of conservation of motion of the centre of mass holds good for displacements along an axis Ox, the algebraic sum of the products of the masses (or weights) of the bodies of the system on the projections of the absolute displacements of their centres of mass must be zero if at the initial moment $v_{Cx} = 0$. In computing ξ_1, ξ_2, etc., attention should always be paid to their signs.

Problem 124. Two men of weights p_A and p_B are seated in the bow and stern of a boat of weight P at a distance l from each other (Fig. 302). Neglecting the resistance of the water, determine the direction and size of the displacement of the boat if the men change places.

Solution. In order to exclude the unknown forces of friction of their shoe soles on the bottom of the boat and the muscular efforts of the men, we consider the boat and the men in it as one system, and the mentioned forces become internal. The external forces acting on the boat are the vertical forces $\boldsymbol{P}$, $\boldsymbol{p}_A$, $\boldsymbol{p}_B$, and $\boldsymbol{N}$. Hence $\sum F^e_{kx} = 0$, and, as at the initial moment $v_{Cx} = 0$, $x_C = \text{const.}$ Consequently, the absolute displacements of all the bodies are governed by the formula (17).

Denoting the boat and the men in their initial and final positions, we find that the displacement of the boat $\xi_b = x$*). Furthermore, the

*) To avoid confusing signs in the equations the boat should be depicted in the displaced position so that the x coordinate is positive (see Fig. 302). Then, if the computations yield a negative result for x, it means the displacement is in the opposite direction.

absolute displacement of the first man is $\xi_A = x + l$, the absolute displacement of the second man is BB_1, and the projection of the latter displacement on axis x is $\xi_B = -(l - x)$. Hence, from Eq. (17′),

$$Px + p_A(x + l) + p_B[-(l - x)] = 0,$$

whence we find the displacement of the boat:

$$x = \frac{p_B - p_A}{P + p_A + p_B} l.$$

If $p_B > p_A$, then $x > 0$, i.e., the boat will move to the right; if $p_B < p_A$, the boat will move to the left. If $p_B = p_A$, the boat will remain at rest.

The procedure mentioned above is important enough to repeat the formulation: *in solving problems of this type, the system to be investigated must be chosen so that the immediately unknown forces would be internal.*

Problem 125. The centre of gravity of the shaft of the motor in Fig. 303 is located at a distance $AB = a$ from the axis of rotation.

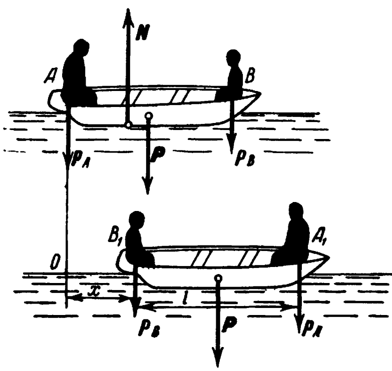

Fig. 302

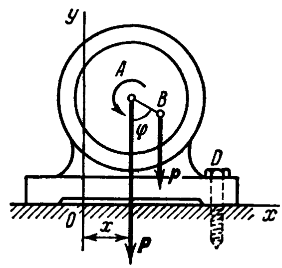

Fig. 303

The shaft is of mass m_1, and the mass of the other part of the motor is m_2. Deduce the law of motion of the motor on a smooth horizontal surface if the shaft rotates with a uniform angular velocity ω. Also determine the maximum stress that will be developed in a bolt D fastening the motor to the surface.

Solution. In order to eliminate the forces rotating the shaft by making them internal, consider the motor with the shaft as a single system.

(1) For the motor standing freely on the plane, all the forces acting on it ($P = m_2 g$, $p = m_1 g$, and the reactions of the surface) are vertical and, as in the previous problem, the law of conservation of the motion of the centre of mass parallel to axis Ox will apply.

Depict the motor in an arbitrary position, assuming as initial the position in which points B and A are on the same vertical (on axis y). Then in the arbitrary position $\xi_A = x$, $\xi_B = x + a \sin \varphi$. Hence, taking into account that $\varphi = \omega t$, we find from Eq. (17)

$$m_2 x + m_1 (x + a \sin \omega t) = 0,$$

whence

$$x = -\frac{m_1 a}{m_2 + m_1} \sin \omega t.$$

Thus, the motor will perform simple harmonic motion with an angular frequency ω.

(2) When the motor is fastened, the horizontal reaction R_x of the bolt, by the first of Eqs. (16′), will be

$$R_x = M \frac{d^2 x_C}{dt^2}, \quad \text{where} \quad x_C = \frac{m_2 x_A + m_1 x_B}{M}.$$

In this case point A is fixed, and $x_A = h$ (h = const.) and $x_B = h \quad a \sin \omega t$. Differentiating the expression of x_C and multi-

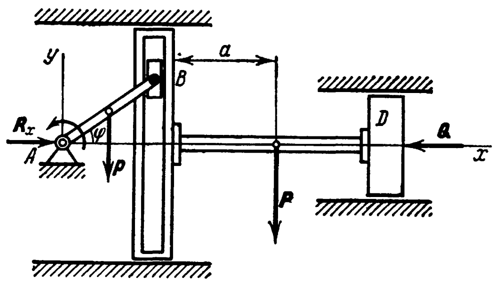

Fig. 304

plying it by M, where M is the mass of the whole system, we obtain

$$R_x = M \frac{d^2 x_C}{dt^2} = m_1 \frac{d^2 x_B}{dt^2} = -m_1 a \omega^2 \sin \omega t.$$

The pressure on the bolt is equal to $|R_x|$ in magnitude and opposite in direction. Its maximum value will be $m_1 a \omega^2$.

Problem 126. Crank AB of length r and weight p of the mechanism in Fig. 304 rotates with a constant angular velocity ω and actuates the slotted bar and the piston D connected to it. The total weight of the bar and piston is P. Acting on the piston during the motion is a constant force Q. Neglecting friction, determine the maximum horizontal pressure of the crank on its axle A.

Solution. In order to eliminate the forces rotating the crank and the pressure exerted on it by the slotted bar, consider the motion of the system as a whole. Denoting the horizontal reaction of the axle A by $\boldsymbol{R}_x$, we have from the first of Eqs. (16′)

$$M\frac{d^2x_C}{dt^2}=R_x-Q,$$

where, by Eqs. (1), $Mx_c = m_1x_1 + m_2x_2$.
In our case

$$m_1=\frac{p}{g},\quad x_1=\frac{r}{2}\cos\omega t,\quad m_2=\frac{P}{g},\quad x_2=a+r\cos\omega t,$$

as $\varphi = \omega t$. We finally obtain

$$R_x=Q+M\frac{d^2x_C}{dt^2}=Q-\frac{r\omega^2}{g}\left(\frac{p}{2}+P\right)\cos\omega t.$$

The pressure on the shaft is equal in magnitude to $|R_x|$ and oppositely directed. The maximum pressure will be at $\varphi = 180°$ and will be equal to

$$Q+\frac{r\omega^2}{g}\left(\frac{p}{2}+P\right).$$

Chapter 25
Theorem of the Change in the Linear Momentum of a System

§ 138. Linear Momentum of a System

The linear momentum, or simply the momentum, of a system is defined as the vector quantity Q equal to the geometric sum (the principal vector) of the momenta of all the particles of the system (Fig. 305):

$$Q = \sum m_k v_k. \tag{18}$$

It can be seen from the diagram that, irrespective of the velocities of the particles (provided that they are not parallel) the momentum

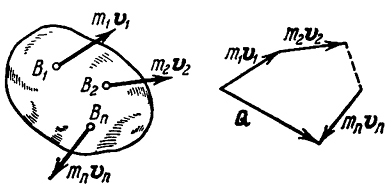

Fig. 305

vector can take any value, or even be zero when the polygon constructed with the vectors $m_k v_k$ as its sides is closed. Consequently, the quantity Q does not characterise the motion of the system completely.

Let us develop a formula with which it is much more convenient to compute Q and also to explain its meaning. It follows from Eq. (1′) that

$$\sum m_k r_k = M r_C.$$

Differentiating both sides with respect to time, we obtain:

$$\sum m_k \frac{dr_k}{dt} = M \frac{dr_C}{dt} \quad \text{or} \quad \sum m_k v_k = M v_C,$$

whence we find that

$$Q = M v_C, \tag{19}$$

i.e., *the momentum of a system is equal to the product of the mass of the whole system and the velocity of its centre of mass.* This equation is especially convenient in computing the momentum of rigid bodies.

It follows from Eq. (19) that if the motion of a body (or a system) is such that the centre of mass remains motionless, the momentum of the body is zero. Thus, the momentum of a body rotating about a fixed axis through its centre of mass is zero (the polygon in Fig. 305 is closed).

If, on the other hand, a body is in relative motion, the quantity $\boldsymbol{Q}$ will not characterise the rotational component of the motion about the centre of mass. Thus, for a rolling wheel, $\boldsymbol{Q} = M\boldsymbol{v}_C$, regardless of how the wheel rotates about its centre of mass C.

We see, therefore, that momentum characterises only the translatory motion of a system, which is why it is often called *linear momentum.* In relative motion, the quantity $\boldsymbol{Q}$ characterises only the translatory component of the motion of a system together with its centre of mass.

§ 139. Theorem of the Change in Linear Momentum

Consider a system of n particles. Writing the differential equations of motion (13) for this system and adding them, we obtain:

$$\sum m_k \boldsymbol{w}_k = \sum \boldsymbol{F}_k^e + \sum \boldsymbol{F}_k^i.$$

From the property of internal forces the last summation is zero. Furthermore,

$$\sum m_k \boldsymbol{w}_k = \frac{d}{dt}\left(\sum m_k \boldsymbol{v}_k\right) = \frac{d\boldsymbol{Q}}{dt},$$

and we finally have

$$\frac{d\boldsymbol{Q}}{dt} = \sum \boldsymbol{F}_k^e. \tag{20}$$

Eq. (20) states the **theorem of the change in the linear momentum of a system in differential form:** *The derivative of the linear momentum of a system with respect to time is equal to the geometrical sum of all the external forces acting on the system.*

In terms of projections on cartesian axes we have

$$\frac{dQ_x}{dt} = \sum F_{kx}^e, \quad \frac{dQ_y}{dt} = \sum F_{ky}^e, \quad \frac{dQ_z}{dt} = \sum F_{kz}^e. \tag{21}$$

Let us develop another expression for the theorem. Let the momentum of a system be $\boldsymbol{Q}_0$ at time $t = 0$, and at time t_1 let it be $\boldsymbol{Q}_1$.

Multiplying both sides of Eq. (20) by dt and integrating, we obtain:

$$Q_1 - Q_0 = \sum \int_0^{t_1} F_k^e \, dt,$$

or

$$Q_1 - Q_0 = \sum S_k^e, \tag{22}$$

as the integrals to the right give the impulses of the external forces.

Eq. (22) states the **theorem of the change in the linear momentum of a system in integral form**: *The change in the linear momentum of a system during any time interval is equal to the sum of the impulses of the external forces acting on the body during the same interval of time.*

In terms of projections on cartesian axes we have

$$\left.\begin{aligned} Q_{1x} - Q_{0x} &= \sum S_{kx}^e, \\ Q_{1y} - Q_{0y} &= \sum S_{ky}^e, \\ Q_{1z} - Q_{0z} &= \sum S_{kz}^e. \end{aligned}\right\} \tag{23}$$

Let us show the connection between this theorem and the theorem of the motion of centre of mass. As $Q = Mv_C$, by substituting this expression into Eq. (20) and taking into account that $\frac{dv_C}{dt} = w_C$, we obtain $Mw_C = \sum F_k^e$, i.e., Eq. (16).

Consequently, the theorem of the motion of centre of mass and the theorem of the change in the momentum of a system are, in effect, two forms of the same theorem. Whenever the motion of a rigid body (or system of bodies) is being investigated, both theorems may be used, though Eq. (16) is usually more convenient.

For a continuous medium (a fluid), however, the concept of centre of mass of the whole system is virtually meaningless, and the theorem of the change in the momentum of a system is used in the solution of such problems. This theorem is also very useful in investigating the theory of impact (Chapter 29) and jet propulsion (§ 142).

The practical value of the theorem is that it enables us to exclude from consideration the immediately unknown internal forces (for instance, the reciprocal forces acting between the particles of a liquid).

§ 140. The Law of Conservation of Linear Momentum

The following important corollaries arise from the theorem of the change in the momentum of a system:

(1) Let the sum of all the external forces acting on a system be zero:

$$\sum F_k^e = 0.$$

It follows from Eq. (20) that in this case $Q = \text{const}$. Thus, if the sum of all the external forces acting on a system is zero, the momentum vector of the system is constant in magnitude and direction.

(2) Let the external forces acting on a system be such that the sum of their projections on any axis Ox is zero:

$$\sum F_{kx}^{e} = 0.$$

It follows from Eqs. (21) that in this case $Q_x = \text{const}$. Thus, if the sum of the projections on any axis of all the external forces acting on a system is zero, the projection of the momentum of that system on that axis is a constant quantity.

These results express the *law of conservation of the linear momentum of a system*. It follows from the above that internal forces are incapable of changing the total momentum of a system. Let us consider some examples.

(a) **Recoil.** If a rifle and the bullet in its barrel are considered as one system, the pressure of the gases during the shot will be an internal force incapable of changing the total momentum of the system. Therefore, as the gases acting on the bullet impart to it a certain momentum in the direction of the muzzle, they must at the same time impart an identical momentum to the rifle in the opposite direction, and the rifle "kicks". This is the phenomenon of recoil well known in artillery.

(b) **Propeller Propulsion.** A rotating propeller or screw throws back a certain mass of air (or water) along its axis. If this backward-moving mass and the moving aircraft (or ship) are considered as one system, the forces of interaction between the propeller and the medium are internal and cannot change the total momentum of the system. Therefore, the flowback of air (water) causes the aircraft (or the vessel) to receive a corresponding forward velocity, such that the total momentum of the system remains zero, since it was zero before motion began.

The action of oars and paddle-wheels is of the same nature.

(c) **Reaction Propulsion.** In a reaction-propelled vehicle (e.g., a rocket), the gaseous products of combustion of a fuel are ejected with a high speed from the rear end (or the nozzle of a jet engine). The pressure caused by the gases is an internal force and it cannot change the total momentum of the rocket-combustion products system. But as the ejected gases possess a certain backward-directed momentum, the rocket receives a corresponding forward velocity. The magnitude of this velocity will be determined in § 142.

Note that in propeller propulsion motion is imparted to, say, an aircraft by throwing back the particles of the fluid in which it is moving. In a vacuum such propulsion is impossible. In reaction propulsion, motion is imparted by the ejection of a working fluid produced in the engine itself (the products of combustion), and motion is possible both in the air and in a vacuum.

§ 141. Solution of Problems

The theorem of the change in linear momentum is usually employed in studying the motion of a medium (a liquid or gas). Application of the theorem eliminates all the internal forces from the consideration. Therefore, the investigated system should be so chosen that all or part of the immediately unknown forces would be internal.

The law of conservation of momentum is conveniently used in cases when the velocity of one part of a system has to be determined from the change in the translational velocity of another part of the system. In particular, the law is widely used in the theory of impact.

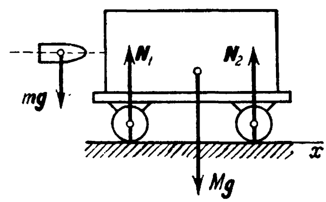

Fig. 306

Problem 127. A bullet of mass m fired horizontally with a velocity u hits a box of sand standing on a truck (Fig. 306). What velocity will the truck receive as a result of the impact if its mass together with the box of sand is M?

Solution. Consider the bullet and the truck as one system. This enables us to exclude the forces generated when the bullet hits the sand. The sum of the projections of the external forces on the horizontal axis Ox is zero. Consequently, $Q_x = \text{const.}$, or $Q_{0x} = Q_{1x}$, where Q_0 is the momentum of the system before the impact, and Q_1, after the impact. As the truck was motionless before the impact, $Q_{0x} = mu$.

After the impact the truck and bullet are moving with the same velocity v. Then

$$Q_{1x} = (m+M)\,v,$$

and equating the right sides of the expressions for Q_{0x} and Q_{1x}, we obtain:

$$v = \frac{m}{m+M}\,u.$$

Problem 128. Determine the recoil of a gun if its barrel is horizontal, the weight of the recoiling parts is P, the weight of the shell is p, and the muzzle velocity of the shell is u.

Solution. To exclude the unknown forces developed by the pressure of the gases, consider the shell and the gun as one system.

Neglecting the resistance to the recoil during the motion of the shell in the bore, we find that the sum of the projections of the applied external forces on the axis Ox is zero (Fig. 307). Hence, $Q =$

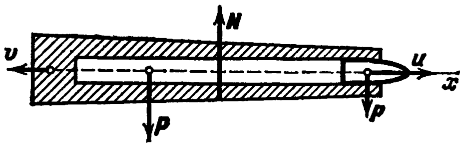

Fig. 307

$=$ const. and $Q_x =$ const. and, since before the shot the system was motionless ($Q_0 = 0$), for any moment of time $Q_x = 0$.

If the velocity of the recoiling parts at the final instant is v, then the absolute velocity of the shell at that moment is $u + v$. Consequently,

$$Q_x = \frac{P}{g} v_x + \frac{p}{g}(u_x + v_x) = 0, \qquad \text{(a)}$$

whence we find:

$$v_x = -\frac{p}{p+P} u_x.$$

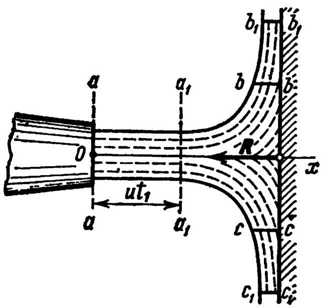

Fig. 308

If we knew the absolute muzzle velocity u_a of the shell, we could have substituted u_{ax} for $u_x + v_x$ in equation (a), whence

$$v_x = -\frac{p}{P} u_{ax}.$$

The minus sign in both cases means that v is in the opposite direction of u.

Note that in calculating the total momentum of a system the *absolute* velocities of its parts should be considered.

Problem 129. Force of a Jet of Water. A jet of water of diameter $d = 4$ cm is discharged from a nozzle with a velocity $u = 10$ m/s and impinges normally against a fixed vertical wall (Fig. 308). Neglecting the compression in the jet, determine the force of the water on the wall.

Solution. To exclude the internal forces of interaction of the water particles between each other at the time of impact, apply the first of Eqs. (23)

$$Q_{1x} - Q_{0x} = \sum S_{kx}^e \qquad \text{(a)}$$

to the part of the jet filling the volume abc at the given instant. Let us calculate for this volume the difference $Q_{1x} - Q_{0x}$ for a certain time interval t_1. During this interval, the volume of water will occupy configuration $a_1b_1c_1$, and the value of Q_x will decrease by mu, where

m is the mass of volume aa_1. The liquid filling volumes bb_1 and cc_1 moves normally to axis Ox and therefore does not increase the value of Q_x. As we have only Q_x decreasing,

$$Q_{1x}-Q_{0x}=-mu.$$

Reaction $\boldsymbol{R}$ of the wall will be the only external force acting on the given volume and giving a projection on axis x. Assuming $R =$ = const., we obtain:

$$\sum S_{kx}^e=R_x t_1=-Rt_1,$$

and equation (a) gives:

$$mu = Rt_1. \tag{b}$$

Now compute m. As the displacement $aa_1 = ut_1$,

$$m=\rho\frac{\pi d^2}{4}ut_1,$$

where ρ is the mass of a unit volume of the liquid, i.e., its density. Substituting this value into equation (b) and taking into account that for water $\rho = 1\,000$ kg/m^3, we finally obtain:

$$R=\rho\frac{\pi d^2}{4}u^2=125.6 \text{ N} \approx 12.8 \text{ kgf}.$$

The pressure of the water on the wall is equal to this value.

§ 142*. Bodies Having Variable Mass. Motion of a Rocket

In classical mechanics, the mass of every particle or point of a moving system is considered to be a constant. In some cases, however, the number of particles in a given system or body may change with time (particles may be discharged from a body or added to it from outside). As a result the total weight of a given body will change. We have already examined problems involving the addition or subtraction of finite masses (Problems 127 and 128 in the previous section, and Problem 94 in § 103). In this section we shall examine another case of considerable practical importance, when the process of separation of particles from, or their addition to, a body takes place continuously. A body whose mass M changes continuously with time, thanks to the addition of particles to it or their ejection from it, is said to have a *variable mass*.*) For a body having variable mass

$$M = F(t),$$

where $F(t)$ is a continuous function of time.

*) Variable mass as treated here has nothing in common with variable mass as treated in the mechanics of the theory of relativity. Here it is due to the change in the number of particles in a given body.

To such bodies belong rockets and jet aircraft, whose masses decrease more or less continuously as the fuel burns out.

If we can neglect the size of a moving body as compared with the distance travelled, it can be considered as a particle having variable mass.

Let us develop the equation of motion for a rocket whose mass continually decreases, considering it as a particle having variable mass in the sense described above. Let u denote the *velocity relative to the rocket* of the burned gas issued from the rocket. To exclude the forces ejecting the gas and to make them internal, let us consider at any instant t the system consisting of the rocket itself and a particle issuing from it during a time interval dt (Fig. 309). The mass μ

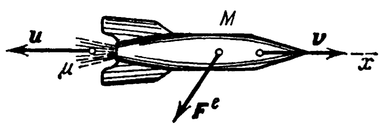

Fig. 309

of this particle is equal in magnitude to the quantity dM by which the mass of the rocket changes during the interval dt. As M is a decreasing quantity, $dM < 0$, and consequently $\mu = |dM| = -dM$.

For this system, Eq. (20) may be written in the form

$$dQ = F^e \, dt, \tag{24}$$

where F^e is the geometrical sum of the external forces acting on the rocket.

If the velocity v of the rocket changes by dv during the time interval dt, the momentum of the system will receive an increment $M\,dv$. The ejected particle will receive during this time an additional velocity u, and the momentum of the system will increase by $u\mu = -u\,dM$. Consequently, $dQ = M\,dv - u\,dM$. Substituting this expression into Eq. (24) and dividing through by dt, we obtain:

$$M\frac{dv}{dt} = F^e + u\frac{dM}{dt}. \tag{25}$$

Eq. (25) states in vector form the *differential equation of motion of a particle having variable mass*, known also as *Meshchersky's equation*.*)

Since the dimension of the second component in the right side of Eq. (25) is that of force, denoting it by the symbol Φ, we may

*) I. V. Meshchersky (1859-1935). An outstanding Russian scholar of mechanics. Eq. (25) was first developed by him in a work written in 1897.

write the equation in the form

$$M\frac{dv}{dt}=F^e+\Phi. \tag{26}$$

Thus, the so-called reaction effect is produced by an additional force Φ, called the *reaction force*, or *thrust*, acting on a moving rocket.

The quantity dM/dt is equal in magnitude to the mass of the fuel burned out in a unit of time, i.e., the per-second expenditure of fuel G_{sec}. Thus, taking into account the sign,

$$\frac{dM}{dt}=-G_{sec},$$

whence

$$\Phi=-uG_{sec}, \tag{27}$$

i.e., the thrust is equal to the product of the per-second expenditure of fuel and the relative velocity of the exhaust gases and is directed opposite to that velocity.

Note. These results would be exact if the ejected particles did not interact with each other (as if they were pellets fired in rapid succession for example). Actually, though, the exhaust gases are discharged in a continuous stream and the particles exert reciprocal forces on each other. Therefore, in atmospheric flight a rocket will additionally be subjected to a force $p_g\sigma$ in the direction of its motion and a force $p_{atm}\,\sigma$ in the opposite direction, where σ is the area of the exit nozzle, p_g is the pressure of the exit gases over that area, and p_{atm} is the atmospheric pressure. As $p_g > p_{atm}$, the thrust will be larger than given by Eq. (27) by the amount $(p_g - p_{atm})\,\sigma$ (for flight in a vacuum, $p_{atm} = 0$). This is taken into account by the introduction in place of u of some effective velocity u_{eff} greater than u (for instance, at $u = 1\,900$ m/s, $u_{eff} = 2\,200$ m/s).

Let us develop the equation of motion of a rocket subjected to the thrust force alone, considering $F^e = 0$ and the escape velocity u of the gases to be constant. If we direct axis x in the direction of the motion (see Fig. 309), then $v_x = v$, $u_x = -u$, and, assuming $F^e = 0$, Eq. (25) in terms of the projections on axis x takes the form

$$M\frac{dv}{dt}=-u\frac{dM}{dt},\quad \text{or}\quad dv=-u\frac{dM}{M}.$$

Integrating and assuming that at the initial moment the mass $M = M_0$ and the velocity $v = v_0$ and is directed along Ox, we obtain:

$$v=v_0+u\ln\frac{M_0}{M}. \tag{28}$$

Let us denote the mass of the rocket with all its equipment (the payload) by M_r and the mass of the fuel M_f. Then, evidently, $M_0 = M_r + M_f$, and after the fuel is burned out the total mass of the rocket will be equal to M_r. Substituting these values into Eq. (28), we obtain *Tsiolkovsky's formula**) for the velocity of a rocket when the fuel is completely burned out (the velocity at the end of the so-called *boost stage*):

$$v_1 = v_0 + u \ln\left(1 + \frac{M_f}{M_r}\right). \tag{29}$$

This result is strictly accurate for conditions of vacuum and absence of any force field. It follows from Eq. (29) that the final velocity of a rocket depends on (1) its initial velocity v_0; (2) the relative velocity u of the exhaust gases; and (3) the relative fuel stock M_f/M_r, known as *Tsiolkovsky's number*. It is very interesting to note that the velocity of a rocket at the end of the boost stage does not depend on the performance of the rocket motor, i.e., on the speed with which the fuel is burned. Table 1 offers an idea of the dependence of v_1/u on M_f/M_r (at $v_0 = 0$).

Table 1

M_f/M_r	v_1/u
1	0.69
2	1.10
3	1.39
4	1.61
5	1.79
10	2.40
20	3.00

The practical importance of Tsiolkovsky's formula is that it points to the possible ways and means of developing the high velocities necessary for space flight. For this we must increase M_f/M_r, u, and v_0, the more effective being the increase of u and v_0. The increase of u and M_f/M_r is connected with the type of fuel and the rocket design (for big liquid-propellant rockets $M_f/M_r = 3$ to 4, $u = 2\,000$ to 2 500 m/s). It is possible to increase v_0 by using a compound (multistage) rocket. As each stage burns out its fuel, it is separated automatically from the last stage, which thus receives an additional (initial) velocity.

Such multistage rockets are used in all space launchings.

*) Konstantin Eduardovich Tsiolkovsky (1857-1935), a celebrated Russian scientist and inventor. Eq. (29) first appeared in a work of his published in May 1903 in the journal *Nauchnoye Obozreniye* (*Scientific Review*).

Chapter 26
Theorem of the Change in the Angular Momentum of a System

§ 143. Total Angular Momentum of a System

The concept of the angular momentum, or moment of momentum, of a material particle was introduced in § 116. *The total angular momentum of a system with respect to a centre O is defined as the quantity* $\boldsymbol{K}_O$ *equal to the geometrical sum of the angular momenta of all the particles of the system with respect to that centre*[*].

$$K_O = \sum \boldsymbol{m}_O (m_k \boldsymbol{v}_k). \qquad (30)$$

The angular momenta of a system with respect to each of three rectangular coordinate axes are found similarly:

$$K_x = \sum m_x (m_k \boldsymbol{v}_k), \quad K_y = \sum m_y (m_k \boldsymbol{v}_k),$$
$$K_z = \sum m_z (m_k \boldsymbol{v}_k). \qquad (31)$$

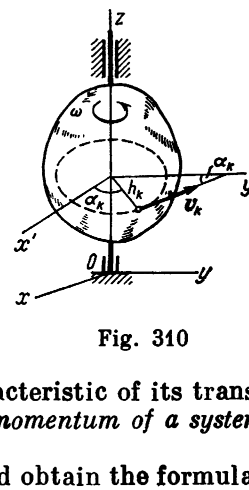

Fig. 310

By the theorem proved in § 44, K_x, K_y, and K_z are the respective projections of vector $\boldsymbol{K}_O$ on the coordinate axes.

Just as the momentum of a system is a characteristic of its translational motion (see § 138), the *total angular momentum of a system* is a characteristic of its rotational motion.

To understand the physical meaning of $\boldsymbol{K}_O$ and obtain the formulas necessary for problem solutions, let us compute the angular momentum of a body rotating about a fixed axis (Fig. 310). As usual, we shall determine vector $\boldsymbol{K}_O$ in terms of its projections K_x, K_y, and K_z.

First, let us find the formula for determining K_z, i.e., the angular momentum of a rotating body with respect to the axis of rotation.

The linear velocity of any particle of the body at a distance h_k from the axis is ωh_k. Consequently, for that particle $m_z (m_k \boldsymbol{v}_k) =$

[*] We shall usually call the total angular momentum of a system just the angular momentum of that system.

$= m_k v_k h_k = m_k \omega h_k^2$. Then, taking the common multiplier ω outside of the parentheses, we obtain for the whole body:

$$K_z = \sum m_z (m_k v_k) = (\sum m_k h_k^2)\,\omega.$$

The quantity in the parentheses is the moment of inertia of the body with respect to axis z (§ 131). We finally obtain:

$$K_z = J_z \omega. \qquad (32)$$

Thus, *the angular momentum of a rotating body with respect to the axis of rotation is equal to the product of the moment of inertia of the body and its angular velocity.*

If a system consists of several bodies rotating about the same axis, then, apparently,

$$K_z = J_{1z}\omega_1 + J_{2z}\omega_2 + \ldots + J_{nz}\omega_n. \qquad (33)$$

The analogy between Eqs. (19) and (32) will be readily noticed: the momentum of a body is the product of its mass (the quantity characterising the body's inertia in translational motion) and its velocity; the angular momentum of a body is equal to the product of its moment of inertia (the quantity characterising a body's inertia in rotational motion) and its angular velocity.

Let us now compute the quantities K_x and K_y. As in the determination of the moment of a force, to determine $m_x (m_k v_k)$ we must project vector $m_k v_k$ on plane Oyz, i.e., on axis y', and find the moment of the projection with respect to point O (see § 43). We obtain $m_x (m_k v_k) = -(m_k v_k \cos \alpha_k)\, z_k$. But $v_k \cos \alpha_k = \omega h_k \cos \alpha_k = \omega x_k$, as from Fig. 310 it is apparent that $h_k \cos \alpha_k = x_k$. Consequently, taking the common multiplier outside of the parentheses, we find that

$$K_x = \sum m_x (m_k v_k) = -(\sum m_k x_k z_k)\,\omega.$$

The sum in the parentheses is the product of inertia J_{xz} (§ 133). A similar expression is obtained for K_y, with y_k substituted for x_k. Finally, we obtain:

$$K_x = -J_{xz}\omega, \quad K_y = -J_{yz}\omega. \qquad (34)$$

Thus, the angular momentum of a rotating body with respect to a centre O on the axis of rotation Oz is a vector $\mathbf{K}_O$ whose projections on the x, y, z axes are given by the formulas (32) and (34). It will be observed that in the most general case vector $\mathbf{K}_O$ is not directed along the axis of rotation Oz. But if the axis of rotation is, for point O, the principal axis of inertia of the body (in particular, the axis of symmetry), then $J_{xz} = J_{yz} = 0$, and $K_x = K_y = 0$ and $K_O = K_z$. Consequently, if a body rotates about an axis that is its principal axis of inertia with respect to point O (or about its axis of symmetry), then vector $\mathbf{K}_O$ is directed along the axis of rotation and is equal in magnitude to K_z, i.e., to $J_z\omega$.

§ 144. Theorem of the Change in the Total Angular Momentum of a System (the Principle of Moments)

The principle of moments, which was proved for a single particle (§ 116), is valid for all the particles of a system. If, therefore, we consider a particle of mass m_k and velocity $\boldsymbol{v}_k$ belonging to a system, we have for that particle:

$$\frac{d}{dt}[\boldsymbol{m}_O(m_k\boldsymbol{v}_k)] = \boldsymbol{m}_O(\boldsymbol{F}_k^e) + \boldsymbol{m}_O(\boldsymbol{F}_k^i),$$

where $\boldsymbol{F}_k^e$ and $\boldsymbol{F}_k^i$ are the resultants of all the external and internal forces acting on the particle.

Writing such equations for all the particles of the system and adding them, we obtain:

$$\frac{d}{dt}\left[\sum \boldsymbol{m}_O(m_k\boldsymbol{v}_k)\right] = \sum \boldsymbol{m}_O(\boldsymbol{F}_k^e) + \sum \boldsymbol{m}_O(\boldsymbol{F}_k^i).$$

But from the properties of the internal forces of a system (§ 129), the last summation vanishes. Hence, taking into account Eq. (30), we obtain finally:

$$\frac{d\boldsymbol{K}_O}{dt} = \sum \boldsymbol{m}_O(\boldsymbol{F}_k^e). \tag{35}$$

This equation states the following **principle of moments** for a system: *The derivative of the total angular momentum of a system about any fixed centre with respect to time is equal to the sum of the moments of all the external forces acting on that system about that centre.*

Projecting both sides of Eq. (35) on a set of fixed axes *Oxyz* and taking into account the theorem proved in § 44, we obtain:

$$\frac{dK_x}{dt} = \sum m_x(\boldsymbol{F}_k^e), \quad \frac{dK_y}{dt} = \sum m_y(\boldsymbol{F}_k^e), \quad \frac{dK_z}{dt} = \sum m_z(\boldsymbol{F}_k^e). \tag{36}$$

Equations (36) express the principle of moments with respect to any fixed axis.

The theorem just proved is widely used in studying the rotation of a body about a fixed axis, and also in the theory of gyroscopic motion and the theory of impact. This, however, is not all. It was proved in the course of kinematics that the most general motion of a body is a combination of a translation together with some pole and a rotation about that pole. If the pole is located in the centre of mass, the translational component of the motion can be investigated by applying the theorem of the motion of the centre of mass, and the rotational component, by the theorem of moments. This indicates the theorem's importance in studying the motion of free bodies (a flying aircraft, shell or missile; see § 158) and, in particular, in studying plane motion (§ 156).

The principle of moments is also convenient in investigating the rotation of a system, because, analogous to the theorem of the change in linear momentum, it makes it possible to exclude from consideration all immediately unknown internal forces.

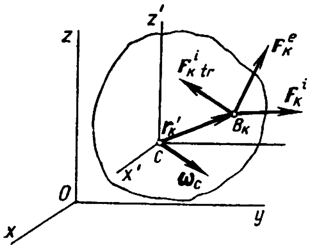

Fig. 311

Theorem of Moments With Respect to a Centre of Mass.*) To use the principle of moments in the study of plane motion or motion of a free rigid body it is necessary to formulate the principle for the case of motion of a system with respect to its centre of mass. Let $Oxyz$ be fixed axes in which a mechanical system is moving (Fig. 311), and let $Cx'y'z'$ be axes in translational motion together with the system's centre of mass C with an acceleration $\boldsymbol{w}_C$. It was shown in § 120 that all the equations of dynamics can be written for the axes $Cx'y'z'$ the same as for fixed axes by adding to the forces $\boldsymbol{F}^e_k$ and $\boldsymbol{F}^i_k$ acting on every point of the system the transport inertia force $\boldsymbol{F}^i_{k\,\mathrm{tr}}$ (the Coriolis forces are zero as the axes are in translational motion). Hence, for the axes $Cx'y'z'$ Eq. (35) takes the form

$$\frac{d\boldsymbol{K}_C}{dt}=\sum \boldsymbol{m}_C(\boldsymbol{F}^e_k)+\sum \boldsymbol{m}_C(\boldsymbol{F}^i_{k\,\mathrm{tr}}). \tag{37}$$

since the sum of the moments of internal forces with respect to any centre is zero. The value of $\boldsymbol{K}_C$ is given by the formula

$$\boldsymbol{K}_C=\sum \boldsymbol{m}_C(m_k\boldsymbol{v}'_k), \tag{38}$$

where $\boldsymbol{v}'_k$ is the velocities of the points of the system with respect to the axes $Cx'y'z'$.

Let us determine the value of the latter sum in Eq. (37). By definition, $\boldsymbol{F}^i_{k\,\mathrm{tr}}=-m_k\boldsymbol{w}_{k\,\mathrm{tr}}$. As axes $Cx'y'z'$ are in translational motion, for any point B_k of the system $\boldsymbol{w}_{k\,\mathrm{tr}}=\boldsymbol{w}_C$; consequently, $\boldsymbol{F}^i_{k\,\mathrm{tr}}=-m_k\boldsymbol{w}_C$ and, using Eq. (49) from § 42, we have $\boldsymbol{m}_C(\boldsymbol{F}^i_{k\,\mathrm{tr}})=$ $=\boldsymbol{r}'_k\times(-m_k\boldsymbol{w}_C)=-m_k\boldsymbol{r}'_k\times\boldsymbol{w}_C$. Then, taking the common multiplier $\boldsymbol{w}_C$ outside of the parentheses and having in mind that by Eq. (1') $\sum m_k\boldsymbol{r}'_k=M\boldsymbol{r}'_C$, we obtain:

$$\sum \boldsymbol{m}_C(\boldsymbol{F}^i_{k\,\mathrm{tr}})=-\left(\sum m_k\boldsymbol{r}'_k\right)\times\boldsymbol{w}_C=-M\boldsymbol{r}'_C\times\boldsymbol{w}_C. \tag{39}$$

*) Note that $\boldsymbol{F}^i_k$ denotes internal forces, and $\boldsymbol{F}^i_{k\,tr}$ the inertia transport force.

But point C is the origin of the coordinate system $Cx'y'z'$, and so $\boldsymbol{r}'_C = 0$ and the expression (39) becomes zero. As a consequence, Eq. (37) yields

$$\frac{d\boldsymbol{K}_C}{dt} = \sum \boldsymbol{m}_C (\boldsymbol{F}_k^e). \tag{40}$$

Comparing this with Eq. (35), we conclude that, *for axes in translational motion together with the centre of mass of a system, the theorem of moments with respect to the centre of mass has the same form as with respect to a fixed centre*. Similarly, for moments with respect to axes $Cx'y'z'$, formula (40) yields equations analogous to Eqs. (36).

Note, that in any other moving frame of reference either $\boldsymbol{r}'_C \neq 0$ or the Coriolis forces are not zero, and the equation of moments will not be of the form (35).

§ 145. The Law of Conservation of the Total Angular Momentum

The following important corollaries can be derived from the principle of moments.

(1) Let the sum of the moments of all the external forces acting on a system with respect to a centre O be zero:

$$\sum \boldsymbol{m}_O (\boldsymbol{F}_k^e) = 0.$$

It follows, then, from Eq. (35) that $\boldsymbol{K}_O = \text{const}$. Thus, *if the sum of the moments of all external forces acting on a system taken with respect to any centre is zero, the total angular momentum of the system with respect to that centre is constant in magnitude and direction.* Application of this result for the case of planetary motion was considered in § 117.

(2) Let the external forces acting on a system be such that the sum of their moments with respect to any fixed axis Oz is zero:

$$\sum m_z (\boldsymbol{F}_k^e) = 0.$$

It follows, then, from Eqs. (36) that $K_z = \text{const}$. Thus, *if the sum of the moments of all the external forces acting on a system with respect to any axis is zero, the total angular momentum of the system with respect to that axis is constant.*

These conclusions express the *law of conservation of the total angular momentum of a system*. It follows from them that internal forces cannot change the total angular momentum of a system.

Rotating Systems. Consider a system rotating about an axis Oz which is fixed or passes through the centre of mass. By Eq. (32),

$K_z = J_z\omega$, and if $\sum m_z (F_k^e) = 0$, then

$$J_z\omega = \text{const}.$$

This leads us to the following conclusions:

(a) If a system is *non-deformable* (a rigid body), then $J_z = \text{const.}$, whence $\omega = \text{const.}$ That is, a rigid body will rotate about a fixed axis with a constant angular velocity.

(b) If a system is *deformable*, it will have particles which, under the action of internal (or external) forces, may move away from the axis, thereby increasing J_z, or approach the axis, thereby decreasing J_z. But as $J_z\omega = \text{const.}$, the angular velocity of the system will decrease as the moment of inertia increases, and increase as the moment of inertia decreases. Thus, the action of internal forces can change the angular velocity of a rotating system, as the constancy of K_z does not, in the general case, mean the constancy of ω.

Let us consider a few examples.

Experiments with Zhukovsky's platform. The law of conservation of angular momentum can be demonstrated visually by means of a simple instrument known as "Zhukovsky's platform". It represents a circular horizontal disc supported by step ball bearings so that it is free to rotate with negligible friction about a vertical axis z. For a person standing on the platform, $\sum m_z (F_k^e) = 0$, and consequently, $J_z\omega = \text{const.}$ If, now, the person stretches out his arms and twists himself to start rotating about the vertical axis and then lowers his arms, J_z will decrease and, consequently, the angular velocity will increase. This trick of increasing angular velocity is widely employed by ballet dancers, in turning somersaults, etc.

A person standing motionless on the platform ($K_z = 0$) can turn in any direction by merely extending one arm horizontally and sweeping it in the opposite direction. His angular velocity will be such that the total quantity K_z for the system will remain zero.

Riding a swing. When a person rides a swing, the pressure of his feet (an internal force) cannot by itself make him swing higher. For this, in the upper left-hand position A_0 of the swing, the person must squat down. When the swing passes through the vertical, he stands up quickly. This brings the mass closer to the axis of rotation z, the quantity J_z decreases, and the angular velocity ω jumps. The increase in ω will carry the swing higher than the initial height A_0. In the upper right-hand position, when $\omega = 0$, the person again squats (which, evidently, will not influence the value of ω), and passing through the vertical he stands up again, etc. As a result the amplitude of the swing will increase with each swing.

The forced vibration of the swing is called *parametric*, as it is induced not by a periodically alternating force (§ 125) but as a result of a change in the parameters of the system: its moment of inertia and the position of its centre of gravity.

Rotation of a gun shell in the barrel. If we consider a gun barrel and a shell in it as a single system, the pressure of the gases when the gun is fired will be an internal force which cannot change the angular momentum of the system, which was zero before the shot. Therefore, if the grooves in the bore make the shell rotate, say, to the right, the barrel will tend to rotate to the left so that at any instant $J_{\text{shell}}\omega_{\text{shell}} = J_{\text{barrel}}\omega_{\text{barrel}}$. This rotation is opposed by the trunnions of the gun carriage, and an additional force is brought to act on them.

Reaction moment of a propeller. The rotor of a helicopter not only drives the air downward [see § 140, example (*b*)], but also imparts a rotation to the mass being swept down. The total angular momentum of the mass of the driven air and the helicopter must be zero, as initially the system was motionless and the forces of interaction between the rotor and the air are internal. Therefore, the helicopter will start rotating in the opposite direction of the rotor. The turning moment acting on the helicopter is called the *reaction moment*, or *reaction torque*.

In order to prevent the reaction rotation of a single-rotor helicopter, it is provided with an auxiliary torque-control rotor mounted on the tail boom. In a multiple-rotor helicopter the rotors are made to be revolved.

The reaction moment can be used for the experimental determination of the torque developed by an aircraft engine, since the two quantities are equal in magnitude and the reaction moment can be measured by mounting the engine on a suitable balance.

§ 146. Solution of Problems

The principle of moments is convenient in studying the rotation of bodies (§§ 154 and 158) or the motion of systems including rotating and translating bodies (Problem 133).

The law of conservation of angular momentum makes it possible to determine the change of the angular velocity (or the angle of rotation) of any part of a system if the displacement or the velocity of the other portion of the system is known. In solving such problems, all immediately unknown internal forces and external forces intersecting with, or parallel to, the axis of rotation can be ignored.

Problem 130. Two discs having moments of inertia J_1 and J_2 are mounted on a shaft as shown in Fig. 312. The shaft is twisted and then released. Find the dependence between the angular velocities and the angle of turn of the discs in the torsional vibrations. Neglect the mass of the shaft and consider the moments of inertia J_1 and J_2 relative to axis x known.

Solution. To exclude the unknown elastic forces which cause the discs to vibrate, consider both discs and the shaft as a single system. The external forces (the reactions of the bearings and the force of gravity) intersect with axis x, whence $\sum m_x(F_k^e) = 0$, and $K_x = \text{const}$. But since at the initial moment $K_x = 0$, during the whole of the vibration we must have $K_x = J_1\omega_1 + J_2\omega_2 = 0$ (the angular momentum of the system with respect to axis x equals the sum of the angular momenta of each disc with respect to the same axis). We find from this that

$$\omega_1 = -\frac{J_2}{J_1}\omega_2 \quad \text{and} \quad \varphi_1 = -\frac{J_2}{J_1}\varphi_2,$$

where φ_1 and φ_2 are the angles through which the discs were twisted, measured from the initial position (the latter result is obtained by integrating the first equation).

Thus, the vibrations will be in opposite directions, and the angular amplitudes will be inversely proportional to the moments of inertia

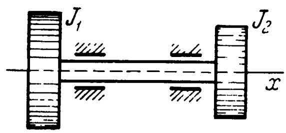

Fig. 312

of the discs. The stationary, or nodal, cross section of the shaft lies closer to the disc whose moment of inertia is larger.

Problem 131. A governor AB with a moment of inertia J_z consists of two symmetrically placed weights of mass m, each attached to two springs as shown in Fig. 313, and it rotates about a vertical axis Oz. At time $t_0 = 0$, the governor receives an angular velocity ω_0, and each weight starts to oscillate in damped vibration about its respective centre C at a distance l from axis Oz. Neglecting friction and considering the weights as particles, determine the dependence of the angular velocity ω of the governor on the position of the weights.

Solution. To exclude the unknown elastic forces of the springs, consider the governor and the weights as one system. Then $\sum m_z(F_k^e) = 0$, and $K_z = \text{const}$. At time $t_0 = 0$, the displacement $x = 0$ and $K_{z0} = (J_z + 2ml^2)\omega_0$. At any arbitrary instant t, $K_z = [J_z + 2m(l + x)^2]\omega$. As $K_z = K_{z0}$,

$$\omega = \frac{J_z + 2ml^2}{J_z + 2m(l+x)^2}\omega_0.$$

Consequently, when $x > 0$, we have $\omega < \omega_0$, and when $x < 0$, we have $\omega > \omega_0$, i.e., the angular velocity changes about a mean value ω_0. When the vibrations of the weights dampen with time, x tends to zero and ω to ω_0.

Problem 132. A track is laid along the circumference of a disc of radius R and weight P. Standing on the track is a toy spring-wound cart of weight p. The disc rotates together with the cart about a vertical axis Oz with an angular velocity ω_0 (Fig. 314). Determine

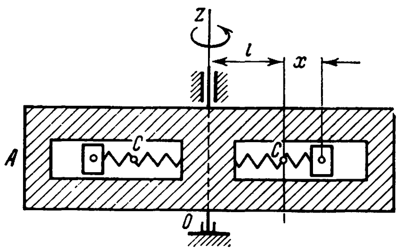

Fig. 313

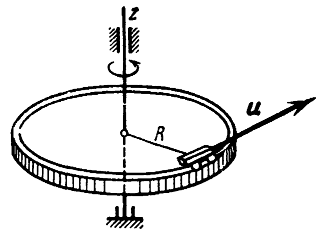

Fig. 314

how the angular velocity of the disc will change if at some instant the cart will start moving in the direction of the rotation with a velocity u relative to the disc.

Solution. To exclude the unknown frictional forces between the wheels of the cart and the disc, consider both as one system. The moments of the external forces acting on the system with respect to axis z are zero. Consequently, K_z = const. Considering the disc to be homogeneous ($J_z = 0.5MR^2$) and the cart as a particle, we have:

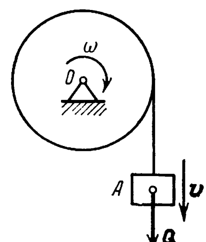

Fig. 315

$$K_{z0} = \left(0.5\frac{P}{g}R^2 + \frac{p}{g}R^2\right)\omega_0.$$

When the cart starts moving, its absolute velocity will be $v_a = u + \omega R$, where ω is the new angular velocity of the disc. The angular momentum of the cart about axis z will be $mv_aR = m(uR + \omega R^2)$, and for the whole system we have:

$$K_{z1} = 0.5\frac{P}{g}R^2\omega + \frac{p}{g}(uR + R^2\omega).$$

As K_z = const., $K_{z1} = K_{z0}$, whence $\omega = \omega_0 - \frac{p}{0.5P + p}\frac{u}{R}$.

The angular velocity of the disc, we see, decreases. If the cart travels in the opposite direction, ω will increase. Note that in calculating K_z the absolute velocities of all moving points of the system were taken.

Problem 133. Wound on a drum of weight P and radius r (Fig. 315) is a string carrying a load A of weight Q. Neglecting the mass of the string and friction, determine the angular acceleration of the drum when the load falls, if the radius of gyration of the drum with respect to its axis is ρ.

Solution. Applying the theorem of moments with respect to axis O, we have:

$$\frac{dK_O}{dt} = \sum m_O(\boldsymbol{F}_k^e). \tag{a}$$

The moving system consists of two bodies, consequently

$$K_O = K_{\text{drum}} + K_{\text{load}}.$$

The load is in translational motion, and we consider it as a particle, its velocity is $v = \omega r$. The drum rotates about a fixed axis, consequently,

$$K_{\text{load}} = \frac{Q}{g}vr = \frac{Q}{g}r^2\omega, \qquad K_{\text{drum}} = J_O\omega = \frac{P}{g}\rho^2\omega,$$

and

$$K_O = (Qr^2 + P\rho^2)\frac{\omega}{g}.$$

Substituting this expression for K_O into equation (a), we obtain:

$$\frac{Qr^2 + P\rho^2}{g}\frac{d\omega}{dt} = Qr,$$

whence

$$\varepsilon = \frac{Qrg}{Qr^2 + P\rho^2}.$$

Chapter 27
Theorem of the Change in the Kinetic Energy of a System

§ 147. Kinetic Energy of a System

The kinetic energy of a system is defined as a scalar quantity T equal to the arithmetical sum of the kinetic energies of all the particles of the system:

$$T=\sum \frac{m_k v_k^2}{2}. \tag{41}$$

Kinetic energy is a characteristic of both the translational and rotational motion of a system, which is why the theorem of the change in kinetic energy is so frequently used in problem solutions. The main difference between T and the previously introduced characteristics $\boldsymbol{Q}$ and $\boldsymbol{K}_O$ is that kinetic energy is a scalar quantity, and essentially a positive one. It, therefore, does not depend on the directions of the absolute motions of parts of a system and does not characterise the changes in these directions.

Another important point should be noted. Internal forces act on the parts of a system in mutually opposite directions. For this reason, as we have seen, they do not change the vector parameters $\boldsymbol{Q}$ and $\boldsymbol{K}_O$. But if, under the action of internal forces, the speeds of the particles of a system change, the quantity T will change too. Consequently, the kinetic energy of a system differs further from the quantities $\boldsymbol{Q}$ and $\boldsymbol{K}_O$ in that it is affected by the action of both external and internal forces.

If a system consists of several bodies, its kinetic energy is, evidently, equal to the sum of the kinetic energies of all the bodies:

$$T=\sum T_k.$$

Let us develop the equations for computing the kinetic energy of a body in different types of motion.

(1) Translational Motion. In this case all the points of a body have the same velocity, which is equal to the velocity of the centre of mass.

Therefore, for any point k we have $v_k = v_C$, and Eq. (41) gives:

$$T_{\text{trans}} = \sum \frac{m_k v_C^2}{2} = \frac{1}{2}\left(\sum m_k\right) v_C^2,$$

or

$$T_{\text{trans}} = \frac{1}{2} M v_C^2. \tag{42}$$

Thus, in translational motion, the kinetic energy of a body is equal to half the product of the body's mass and the square of the velocity of the centre of mass. The value of T does not depend on the direction of motion.

(2) Rotational Motion. The velocity of any point of a body rotating about an axis Oz (see Fig. 310) is $v_k = \omega h_k$, where h_k is the distance of the point from the axis of rotation and ω is the angular velocity of the body. Substituting this expression into Eq. (41) and taking the common multipliers outside of the parentheses, we obtain:

$$T_{\text{rotation}} = \sum \frac{m_k \omega^2 h_k^2}{2} = \frac{1}{2}\left(\sum m_k h_k^2\right)\omega^2.$$

The term in the parentheses is the moment of inertia of the body with respect to axis z. Thus, we finally obtain:

$$T_{\text{rotation}} = \frac{1}{2} J_z \omega^2, \tag{43}$$

i.e., in rotational motion, the kinetic energy of a body is equal to half the product of the body's moment of inertia with respect to the axis of rotation and the square of its angular velocity. The value of T does not depend on the direction of the rotation.

(3) Plane Motion*). In plane motion, the velocities of all the points of a body are at any instant directed as if the body were rotating about an axis perpendicular to the plane of motion and passing through the instantaneous centre of zero velocity P (Fig. 316). Hence, by Eq. (43),

$$T_{\text{plane}} = \frac{1}{2} J_P \omega^2, \tag{43'}$$

where J_P is the moment of inertia of the body with respect to the instantaneous axis of rotation, and ω is the angular velocity of the body.

The quantity J_P in Eq. (43′) is variable, as the position of the centre P continuously changes with the motion of the body. Let us introduce instead of J_P a constant moment of inertia J_C with respect to an axis through the centre of mass C of the body. By the parallel axis theorem (§ 132), $J_P = J_C + Md^2$, where $d = PC$. Substituting this expression for J_P into Eq. (43′) and taking into

*) This case can be developed as a particular case of the most general motion of a rigid body discussed in the following item.

account that point P is the instantaneous centre of zero velocity and therefore $\omega d = \omega \cdot PC = v_C$, where v_C is the velocity of the centre of mass, we obtain finally:

$$T_{\text{plane}} = \frac{1}{2} M v_C^2 + \frac{1}{2} J_C \omega^2. \tag{44}$$

Thus, in plane motion, the kinetic energy of a body is equal to the kinetic energy of translation of the centre of mass plus the kinetic energy of rotation relative to the centre of mass.

(4)* The Most General Motion of a Body. Taking the centre of mass C as the pole (Fig. 317), the most general motion of a body is a combination of a translation with the velocity $\boldsymbol{v}_C$ of the pole and

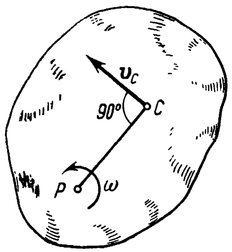

Fig. 316

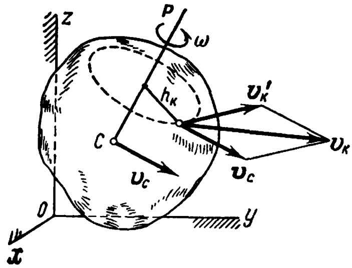

Fig. 317

a rotation about the instantaneous axis CP through the pole (see § 88). Then, as shown in the course of kinematics, the velocity $\boldsymbol{v}_k$ of any point of the body is equal to the geometrical sum of the velocity $\boldsymbol{v}_C$ of the pole and the velocity $\boldsymbol{v}_k'$ of the point in its rotation with the body about axis CP:

$$\boldsymbol{v}_k = \boldsymbol{v}_C + \boldsymbol{v}_k'.$$

In magnitude $v_k' = \omega h_k$, where h_k is the distance of the point from axis CP and ω is the *absolute* angular velocity of the body about that axis. It follows from this that*)

$$v_k^2 = \boldsymbol{v}_k^2 = (\boldsymbol{v}_C + \boldsymbol{v}_k')^2 = v_C^2 + v_k'^2 + 2\boldsymbol{v}_C \cdot \boldsymbol{v}_k'.$$

Substituting this expression into Eq. (41) and taking into account that $v_k' = \omega h_k$, we find:

$$T = \frac{1}{2}\left(\sum m_k\right) v_C^2 + \frac{1}{2}\left(\sum m_k h_k^2\right)\omega^2 + \boldsymbol{v}_C \cdot \sum m_k \boldsymbol{v}_k',$$

where the common multipliers have been taken outside of the parentheses.

---

*) From the definition of the scalar product of two vectors (see footnote on p. 300) it follows that $\boldsymbol{v}^2 = \boldsymbol{v} \cdot \boldsymbol{v} = vv \cos 0° = v^2$, i.e., the scalar square of a vector is equal to the square of its magnitude. This result has been employed here.

In this equation, the term in the first parentheses gives the mass M of the body and the term in the second parentheses gives the moment of inertia J_{CP} of the body with respect to the instantaneous axis CP. In the last term $\sum m_k v'_k = 0$ as it represents the linear momentum of the body in its rotation about axis CP, which passes through the centre of mass (see § 138).

Therefore, we finally have:

$$T = \frac{1}{2} Mv_C^2 + \frac{1}{2} J_{CP}\omega^2. \tag{45}$$

Thus, in the most general motion of a body, the kinetic energy is equal to the kinetic energy of translation of the centre of mass of the body plus the kinetic energy of rotation about an axis through the centre of mass.

If the pole is taken not in the centre of mass but in another point A such that axis AA' does not pass through the centre of mass, then for this axis $\sum m_k v'_k \neq 0$, and we cannot develop an equation of the form (45) (see Problem 136).

Problem 134. Find the kinetic energy of a uniform cylindrical wheel of mass M rolling without slipping if the velocity of its centre is v_C (Fig. 321*a*).

Solution. The wheel is in plane motion. By Eq. (44) or (45)

$$T = \frac{1}{2} Mv_C^2 + \frac{1}{2} J_C\omega^2.$$

As the body is a uniform cylinder, we have (see § 131) $J_C = 0.5MR^2$, where R is the radius of the wheel. On the other hand, since point B is the instantaneous centre of zero velocity of the wheel, $v_C = \omega \cdot BC = \omega R$, whence $\omega = v_C/R$.

Substituting these expressions into the equation, we find:

$$T = \frac{1}{2} Mv_C^2 + \frac{1}{4} MR^2 \frac{v_C^2}{R^2} = \frac{3}{4} Mv_C^2.$$

Problem 135. When body A in Fig. 318 translates with a velocity $\boldsymbol{u}$, body B of mass M moves in the slots of body A with a velocity $\boldsymbol{v}$. Determine the kinetic energy of body B. Angle α is known.

Solution. The absolute motion of body B is a translation with a velocity $\boldsymbol{v}_a = \boldsymbol{u} + \boldsymbol{v}$ (see § 93). Then

$$T = \frac{1}{2} Mv_a^2 = \frac{1}{2} M (u^2 + v^2 + 2uv \cos \alpha).$$

Note that in the most general case of relative motion, the total kinetic energy of a body does not equal the sum of the kinetic energies of its relative and transport motions. Thus, here

$$T_{rel} + T_{tr} = \frac{1}{2} Mv^2 + \frac{1}{2} Mu^2 \neq T.$$

Problem 136. A mechanism consists of a part which is translated with a velocity u and a rod AB of length l and mass M hinged at A (Fig. 319). The rod rotates about axis A with an angular velocity ω. Determine the kinetic energy of the rod if angle α is given.

Solution. The rod performs plane motion and by Eq. (44) or (45)

$$T=\frac{1}{2}Mv_C^2+\frac{1}{2}J_C\omega^2.$$

The velocity of point C is compounded of the velocities u and v_{rel}, where in magnitude $v_{rel}=\omega\frac{l}{2}$. Consequently (see Fig. 319), $v_C^2=u^2+v_{rel}^2+2uv_{rel}\cos\alpha$. The angular velocity of the rod

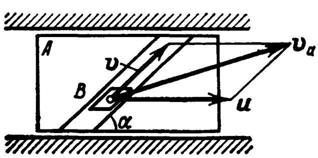

Fig. 318

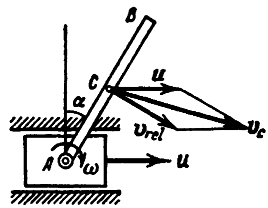

Fig. 319

about C is the same as about A, as ω does not depend on the location of the pole (see § 77). Furthermore, it was shown in Problem 122 (§ 132) that $J_C=Ml^2/12$.

Substituting all these expressions, we obtain:

$$T=\frac{1}{2}M\left(u^2+\omega^2\frac{l^2}{4}+u\omega l\cos\alpha\right)+\frac{1}{24}Ml^2\omega^2=$$
$$=\frac{1}{2}Mu^2+\frac{1}{6}Ml^2\omega^2+\frac{1}{2}Ml\omega u\cos\alpha.$$

A common mistake in this type of problem is to assume that

$$T=T_{\text{trans}}+T_{\text{rotation}}=\frac{1}{2}Mu^2+\frac{1}{2}J_A\omega^2=\frac{1}{2}Mu^2+\frac{1}{6}Ml^2\omega^2.$$

This is wrong because, as previously shown, the formula $T=T_{\text{trans}}+T_{\text{rotation}}$ is valid only when the axis of rotation passes through the centre of mass, which is not the case in this problem.

§ 148. Some Cases of Computation of Work

Work done by forces is computed from the equations developed in §§ 112 and 113. Let us consider several more cases.

(1) Work Done by Gravitational Forces Acting on a System. The work done by a gravitational force acting on a particle of weight

p_k will be $p_k\ (z_{k0} - z_{k1})$, where z_{k0} and z_{k1} are the coordinates of the initial and final positions of the particle (see § 113). Then the total work done by all the gravitational forces acting on a system will be, by Eqs. (74) in § 54:

$$A = \sum p_k z_{k0} - \sum p_k z_{k1} = P\,(z_{C0} - z_{C1}) = \pm Ph_C,$$

where P is the weight of the system and h_C is the vertical displacement of the centre of gravity (or centre of mass) of the system. Thus, the work done by the gravitational forces acting on a system is

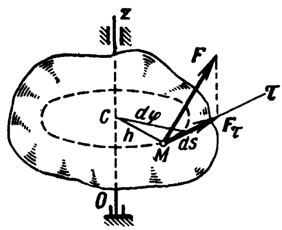

Fig. 320

computed as the work done by their resultant P in the displacement of the centre of gravity (or the centre of mass) of the system.

(2) Work Done by Forces Applied to a Rotating Body. The elemental work done by the force F applied to the body in Fig. 320 will be (see § 112)

$$dA = F_\tau\, ds = F_\tau h\, d\varphi,$$

since $ds = h\, d\varphi$, where $d\varphi$ is the angle of rotation of the body.

But it is evident that $F_\tau h = m_z\,(F)$. We shall call the quantity $M_z = m_z\,(F)$ the *turning moment*, or *torque*. Thus we obtain:

$$dA = M_z\, d\varphi, \tag{46}$$

i.e., the elemental work in this case is equal to the product of the torque and the elemental angle of rotation. Eq. (46) is valid when several forces are acting if it is assumed that $M_z = \sum m_z\,(F_k)$.

The work done in a turn through a finite angle φ_1 will be

$$A = \int_0^{\varphi_1} M_z\, d\varphi, \tag{47}$$

and, for a constant torque (M_z = const.),

$$A = M_z \varphi_1. \tag{47'}$$

If acting on a body is a force couple lying in a plane normal to Oz, then, evidently, M_z in Eqs. (46)-(47') will denote the moment of that couple.

Let us see how power is determined in this case (see § 112). From Eq. (46) we find:

$$W=\frac{dA}{dt}=\frac{M_z\,d\varphi}{dt}=M_z\omega.$$

Thus, the power developed by forces acting on a rotating body is equal to the product of the torque and the angular velocity of the body. For the same power, the torque increases as the angular velocity decreases.

(3) Work Done by Frictional Forces Acting on a Rolling Body. A wheel of radius R (Fig. 321) rolling without slipping on a plane (surface) is subjected to the action of a frictional force $\boldsymbol{F}_{\text{fr}}$, which prevents the slipping of the point of contact B on the surface. The elemental work done by this force is $dA = -F_{\text{fr}}\,ds_B$. But point B is the instantaneous centre of velocity (§ 81), and $v_B = 0$. As $ds_B = v_B\,dt$, $ds_B = 0$, and for every elemental displacement $dA = 0$.

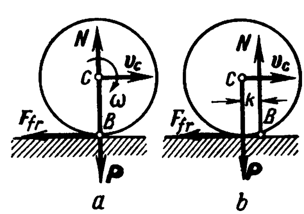

Fig. 321

Thus, in rolling without slipping, the work done by the frictional force preventing slipping is zero in any displacement of the body. For the same reason, the work done by the normal reaction $\boldsymbol{N}$ is also zero, assuming the body to be non-deformable and force $\boldsymbol{N}$ applied at point B, as shown in Fig. 321*a*.

The resistance to rolling due to deformation of the surfaces (Fig. 321*b*) creates a couple ($\boldsymbol{N}$, $\boldsymbol{P}$) with a moment $M = kN$, where k is the coefficient of rolling friction (see § 41). Then by Eq. (46) and taking into account that the angle of rotation of a rolling wheel is $d\varphi = \frac{ds_C}{R}$, we obtain:

$$dA_{\text{roll}} = -kN\,d\varphi = -\frac{k}{R}N\,ds_C, \tag{48}$$

where ds_C is the elemental displacement of the centre C of the wheel.

If N = const., then the total work done by the forces resisting rolling will be

$$A_{\text{roll}} = -kN\varphi_1 = -\frac{k}{R}Ns_C. \tag{48'}$$

As the quantity k/R is small, rolling friction can, in the first approximation, be neglected as compared with other resisting forces.

§ 149. Theorem of the Change in the Kinetic Energy of a System

The theorem proved in § 114 is valid for any point of a system. Therefore, if we take any particle of mass m_k and velocity v_k belonging to a system, we have for this particle

$$d\left(\frac{m_k v_k^2}{2}\right) = dA_k^e + dA_k^i,$$

where dA_k^e and dA_k^i are the elementary work done by the external and internal forces acting on the particle.

If we write similar equations for all the particles of a system and add them, we obtain:

$$d\left(\sum \frac{m_k v_k^2}{2}\right) = \sum dA_k^e + \sum dA_k^i,$$

or

$$dT = \sum dA_k^e + \sum dA_k^i. \qquad (49)$$

Equation (49) states the **theorem of the change in the kinetic energy of a system in differential form**. Integrating both parts of

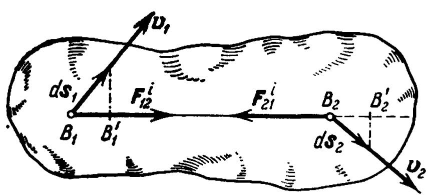

Fig. 322

the equation in the limits corresponding to the displacement of the system from some initial position where the kinetic energy is T_0 to a position where it is T_1, we obtain:

$$T_1 - T_0 = \sum A_k^e + \sum A_k^i. \qquad (50)$$

This equation states the **theorem of the change in kinetic energy in final form**: *The change in the kinetic energy of a system during any displacement is equal to the sum of the work done by all the external and internal forces acting on the system in that displacement.*

Unlike the previously proved theorems, in Eqs. (49) and (50) the internal forces are not ignored. For, if F_{12}^i and F_{21}^i are the forces of interaction between points B_1 and B_2 of a system (see Fig. 322), then $F_{12}^i + F_{21}^i = 0$, but at the same time point B_1 may be moving towards B_2 and point B_2 towards B_1. The work done by each force is positive, and the total work will not be zero. An example is the phenomenon of recoil (Problem 128, Fig. 307). The internal forces (the pressure of the gases) acting on the shell and the recoiling parts do a positive work. The sum of this work is not zero, and it changes the kinetic energy of the system from $T_0 = 0$ at the beginning of the shot to $T_1 = T_{\text{projectile}} + T_{\text{recoil}}$ at the end of the shot. Let us consider two important cases.

(1) Non-Deformable Systems. A non-deformable system is defined as one in which the distance between the points of application of the internal forces does not change during the motion of the system. Special cases of such systems are a rigid body and an inextensible string.

Let two points B_1 and B_2 of a non-deformable system (Fig. 322) be acting on each other with forces $\boldsymbol{F}^i_{12}$ and $\boldsymbol{F}^i_{21}(\boldsymbol{F}^i_{21} = -\boldsymbol{F}^i_{12}$ and let their velocities at some instant be $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$. Their displacements in a time interval dt will be $ds_1 = v_1 dt$ and $ds_2 = v_2\, dt$ directed along vectors $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$. But as line B_1B_2 is non-deformable, it follows from the laws of kinematics that the projections of vectors $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$, and consequently of the displacements ds_1 and ds_2, on the direction of B_1B_2 will be equal, i.e., $B_1B_1' = B_2B_2'$. Then the elemental work done by forces $\boldsymbol{F}^i_{12}$ and $\boldsymbol{F}^i_{21}$ will be equal in magnitude and opposite in sense, and their sum will be zero. This holds for all internal forces in any displacement of a system.

We conclude from this that the sum of the work done by all the internal forces of a non-deformable system is zero, and Eqs. (49) or (50) take the form

$$dT = \sum dA_k^e \text{ or } T_1 - T_0 = \sum A_k^e. \tag{51}$$

(2) Systems With Ideal Constraints. Consider a system with constraints that do not change with time. Dividing all the external and internal forces acting on the particles of the system into *active forces* and the *reactions of the constraints*, Eq. (49) can be written in the form:

$$dT = \sum dA_k^a + \sum dA_k^r,$$

where dA_k^a is the elementary work done by the external and internal forces acting on the k-th particle of the system, and dA_k^r is the elementary work done by the reactions of the external and internal constraints acting on that particle.

We see that the change in the kinetic energy of the system depends on the work done by both the acting forces and the reactions of the constraints. However, we can introduce the concept of "ideal" mechanical systems in which constraints do not affect the change in kinetic energy in the motion of the system. Such constraints should, evidently, satisfy the condition:

$$\sum dA_k^r = 0. \tag{52}$$

If for constraints that do not change with time the sum of the work done by all the reactions in an elementary displacement of a system is zero, such constraints are called *ideal*. Here are some known examples of ideal constraints.

In § 114 it was established that if a constraint is a fixed smooth surface (or curve), for which friction can be neglected, the work done by the reaction $\boldsymbol{N}$ in the motion of a body along that surface (curve) is zero. Then, in § 148 it was shown that, neglecting deformation, if a body rolls without slipping on a rough surface, the work done by the normal reaction $\boldsymbol{N}$ and the force of friction $\boldsymbol{F}$ (i.e., the tangential component of the reaction) is zero. Also, the work done by the reaction $\boldsymbol{R}$ of a hinge (see Fig. 10) is, neglecting friction, zero, as in any displacement of the system the point of application of force $\boldsymbol{R}$ is fixed. Finally, if the material particles B_1 and B_2 in Fig. 322 are assumed to be connected by a rigid rod B_1B_2, the forces $\boldsymbol{F}_{12}^i$ and $\boldsymbol{F}_{21}^i$ will be the reactions of the constraint; the work of each of these reactions in the displacement of the system is not zero, but their sum, as shown, is zero. Thus, all the mentioned constraints can, with the assumptions made, be regarded as ideal.

In the case of a mechanical system subject solely to ideal constraints that do not change with time we obviously have:

$$dT = \sum dA_k^a \quad \text{or} \quad T_1 - T_0 = \sum A_k^a. \tag{53}$$

Thus, the change in the kinetic energy of a system with ideal constraints that do not change with time is, in any displacement, equal to the sum of the work done in that displacement by the active external and internal forces.

All the foregoing theorems made it possible to exclude the internal forces from the equations of motion, but all the external forces, including the immediately unknown reactions of the external constraints, entered the equations. The theorem of the change in kinetic energy is useful because in the case of ideal constraints that do not change with time it makes it possible to exclude *all* the immediately unknown reactions of the constraints from the equations of motion.

§ 150. Solution of Problems

The theorem of the change in kinetic energy is useful when a moving system is non-deformable. In this case the theorem makes it possible to exclude from consideration all immediately unknown internal forces and, if there is no friction, also all the immediately unknown reactions of external constraints.

In the case of a deformable system, the theorem gives a solution only if the internal forces are known. If they are not known (Problems 124, 128, etc.), the energy theorem is insufficient to obtain a solution.

Eq. (50) is convenient in solving problems where the given and required quantities include: (1) the acting forces, (2) the displacement of the system, and (3) the velocities of the bodies (linear or angular) at the beginning and the end of the displacement. The acting forces must be constant or dependent only on the displacement (distance). We must bear in mind that the kinetic energy theorem can be used to develop the *differential equations of motion of a system* and, in particular, to find the accelerations of moving bodies. For this write Eq. (50), differentiate both sides with respect to time, and eliminate the velocity (see Problems 139 and 140). For arbitrary forces the equations are more conveniently written in the form (49), i.e., in differential form (see Problems 141 and 148).

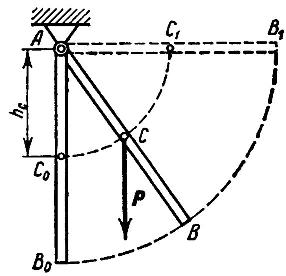

Fig. 323

Problem 137. A rod AB of length l is hinged, as shown, at point A (Fig. 323). Neglecting friction, determine the minimum velocity ω_0 that must be imparted to the rod so that it would swing into a horizontal configuration.

Solution. The given and required quantities include ω_0, $\omega_1 = 0$, and the displacement of the system as defined by angle B_0AB_1. Therefore, the problem is best solved by applying Eq. (51):

$$T_1 - T_0 = \sum A_k^e. \tag{a}$$

Denoting the mass of the rod by M, compute all the quantities in the equation. From Eqs. (6) and (43) we find:

$$T_0 = \frac{1}{2} J_A \omega_0^2 = \frac{1}{6} M l^2 \omega_0^2.$$

Since in the final configuration the velocity of the rod is zero, $T_1 = 0$. The superimposed constraint is ideal; thus, work is done only by force $P = Mg$, and $A^e = -Ph_C = -Mg\frac{l}{2}$. Substituting these values into equation (a), we obtain:

$$-\frac{1}{6} M l^2 \omega_0^2 = -Mg\frac{l}{2},$$

whence

$$\omega_0 = \sqrt{\frac{3g}{l}}.$$

Problem 138. Two pulleys A and B are connected by a belt (Fig. 324). When the motor is switched off, pulley A of radius R

has an angular velocity ω_0. The total weight of the pulleys is P and of the belt p. A brake shoe is applied to pulley A with a force Q to stop the rotation; the coefficient of friction of the shoe on the pulley is f. Neglecting friction in the axles and considering the pulleys to be homogeneous discs, determine how many revolutions pulley A will make before stopping.

Solution. We shall use Eq. (51) to determine the required number of revolutions N:

$$T_1 - T_0 = \sum A_k^e. \tag{a}$$

In computing kinetic energy, it should always be remembered that the kinetic energy of a system equals the sum of the kinetic

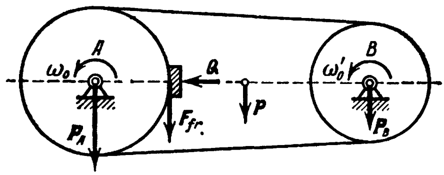

Fig. 324

energies of all the component bodies. From the conditions of the problem, $T_1 = 0$ and $T_0 = T_A + T_B + T_b$. Taking into account that the initial velocities of all the points of the belt $v_{b0} = \omega_0 R = \omega_0' r$, where ω_0' and r are the initial angular velocity and the radius of pulley B, we find from Eqs. (8) and (43):

$$T_A = \frac{1}{2}\left(\frac{P_A}{2g}R^2\right)\omega_0^2, \quad T_B = \frac{1}{2}\left(\frac{P_B}{2g}r^2\right)\omega_0'^2 = \frac{1}{4}\frac{P_B}{g}R^2\omega_0^2,$$

$$T_b = \frac{1}{2}\frac{p}{g}v_{b0}^2 = \frac{1}{2}\frac{p}{g}R^2\omega_0^2.$$

The last equation follows from the fact that all the points of the belt move with the same speed. Finally, as $P_A + P_B = P$, we obtain:

$$T_0 = \frac{P+2p}{4g}R^2\omega_0^2.$$

Now compute the work done by the forces. In this case, the work done by gravity is zero, as the centres of gravity of the pulleys and the belt do not change their position during the motion. The force of friction $F_{fr} = fQ$, and the work done by it is found from Eq. (47'):

$$A_{fr} = -(fQR)\cdot\varphi_1 = -fQR\ 2\pi N.$$

Substituting all the found values into equation (a), we obtain finally:

$$N = \frac{(P+2p)R\omega_0^2}{8\pi g f Q}.$$

Problem 139. A cart is drawn with a constant force $Q = 16$ kgf up an inclined plane making an angle $\alpha = 30°$ with the horizontal (Fig. 325). The platform of the cart weighs $P = 18$ kgf, and each of its uniform wheels weighs $p = 2$ kgf. Determine: (1) the linear velocity v_1 of the cart, when it has travelled a distance $l = 4$ m, if $v_0 = 0$; (2) the acceleration of the cart. The wheels roll without slipping. Neglect the rolling friction.

Solution. (1) Let us use Eq. (51) to determine v_1:

$$T_1 - T_0 = \sum A_k^e. \tag{a}$$

In our case $T_0 = 0$ and $T_1 = T_{\text{platform}} + 4T_{\text{wheel}}$. The cart is in translatory motion, and the kinetic energy of a uniform rolling wheel was calculated in Problem 134. Thus,

$$T_1 = \frac{1}{2}\frac{P}{g}v_1^2 + 4\left(\frac{3}{4}\frac{p}{g}v_1^2\right) = \frac{1}{2g}(P+6p)v_1^2.$$

Work is done by force Q and the force of gravity $P_1 = (P + 4p)$. The work done by the frictional forces preventing slippage and by

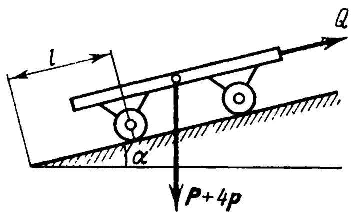

Fig. 325

the normal reactions is zero (§ 148). Making the necessary computations, we find:

$$A(Q) = Ql, \quad A(P_1) = -(P + 4p)h_C = -(P + 4p)l\sin\alpha.$$

Substituting these expressions into equation (a), we obtain:

$$\frac{1}{2g}(P + 6p)v_1^2 = [Q - (P + 4p)\sin\alpha]\,l, \tag{b}$$

whence

$$v_1 = \sqrt{\frac{2gl[Q-(P+4p)\sin\alpha]}{P+6p}} = 2.8 \text{ m/s}.$$

(2) To determine the acceleration w, let us consider the quantities v_1 and l in equation (b) as variables. Then, differentiating with respect to time, we have:

$$\frac{1}{g}(P + 6p)v\frac{dv}{dt} = [Q - (P + 4p)\sin\alpha]\frac{dl}{dt}.$$

But $\frac{dl}{dt}=v$ and $\frac{dv}{dt}=w$, and, eliminating v, we have:

$$w=\frac{Q-(P+4p)\sin\alpha}{P+6p}\,g=0.98\ \mathrm{m/s^2}.$$

Note the use of the theorem of the change in kinetic energy to determine the acceleration.

Problem 140. One end of a string passing over a pulley O (Fig. 326) is wound on a cylinder of radius R and weight P; attached to the other end is a load D of weight Q. If $v_{C0}=0$, determine the velocity v_C of the centre C of the cylinder after it has travelled a distance s and the acceleration w_C of the centre. The coefficient of rolling friction of the cylinder is k, the radius of gyration of the cylinder with respect to its axis is ρ. Neglect the masses of the string and the pulley.

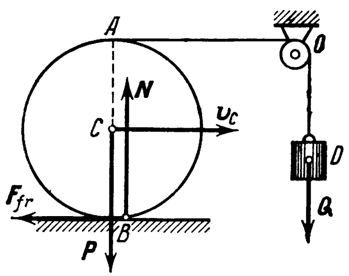

Fig. 326

Solution. (1) We use Eq. (51) to determine the velocity v_C:

$$T-T_0=\sum A_k^e. \tag{a}$$

In our case $T_0=0$ and $T=T_{\text{cyl}}+T_D$. From Eqs. (4), (42), and (44),

$$T_D=\frac{1}{2}\,\frac{Q}{g}\,v_D^2,$$

$$T_{\text{cyl}}=\frac{1}{2}\,\frac{P}{g}\,v_C^2+\frac{1}{2}\left(\frac{P}{g}\rho^2\right)\omega^2.$$

As point B is the instantaneous centre of zero velocity, $\omega=\frac{v_C}{R}$ and $v_D=v_A=2v_C$. Consequently,

$$T=\frac{1}{2g}\left[4Q+P\left(1+\frac{\rho^2}{R^2}\right)\right]v_C^2.$$

The forces doing work are Q and the couple (N, P). As $v_D=2v_C$, the displacement of load D is $h=2s$ and $A(Q)=Q\cdot 2s$. The work done by the forces opposing the rolling can be found from Eq. (48′), as $N=P=\text{const}$. Then,

$$\sum A_k^e=2Qs-\frac{k}{R}Ps.$$

Substituting the found expressions into equation (a), we obtain:

$$\frac{1}{2g}\left[4Q+P\left(1+\frac{\rho^2}{R^2}\right)\right]v_C^2=\left(2Q-\frac{k}{R}P\right)s, \tag{b}$$

whence

$$v_C=\sqrt{\frac{2g\,(2QR-kP)\,Rs}{4QR^2+P\,(R^2+\rho^2)}}.$$

(2) As in the preceding problem, to determine ω_C differentiate both sides of equation (b) with respect to time. Taking into account

that $\frac{ds}{dt} = v_C$, we finally obtain:

$$w_C = \frac{(2QR - kP)R}{4QR^2 + P(R^2 + \rho^2)} g.$$

Problem 141. Referring to Fig. 327, a spiral spring is attached to gear *1* of radius r and weight P and to crank OC of length l and weight Q on which the gear is mounted. Gear *1* rolls on a fixed gear *2* of radius $R = l - r$. The moment of the spring $M_{sp} = c\alpha$, where α is the angle of rotation of gear *1* with respect to the crank. Neglecting friction, determine the period of vibration of the crank if it is disturbed from its position of equilibrium. The mechanism works in the horizontal plane.

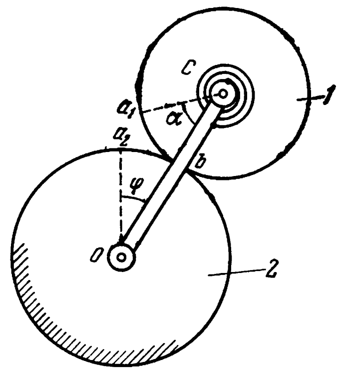

Fig. 327

Solution. We shall define the position of the crank by the angle φ measured from its equilibrium position. To exclude the unknown reaction of axis C from the computation, consider gear *1* and the crank as a single system and develop the differential equation of its motion from Eq. (49).

First compute the kinetic energy T of the system in terms of the angular velocity ω_{cr} of the crank (as we are developing the law of motion of the crank). We have:

$$T = T_{cr} + T_{gear} = \frac{1}{2} J_{O\,cr}\omega_{cr}^2 + \frac{1}{2}\frac{P}{g} v_C^2 + \frac{1}{2} J_{C\,gear}\omega_{gear}^2. \qquad \text{(a)}$$

Considering the crank as a homogeneous rod and the gear as a uniform disc, and taking into account that the point of contact is the instantaneous centre of zero velocity of gear *1*, we have:

$$J_{O\,cr} = \frac{1}{3}\frac{Q}{g} l^2, \quad J_{C\,gear} = \frac{1}{2}\frac{P}{g} r^2,$$

$$v_C = \omega_{cr} l. \qquad \omega_{gear} = \frac{v_C}{r} = \frac{l}{r}\omega_{cr}.$$

Note again that Eq. (44), which is used to compute T_{gear}, contains the *absolute* angular velocity of the gear, not its relative velocity of rotation with respect to the crank. Substituting all the determined quantities into equation (a), we finally obtain:

$$T = \frac{1}{12g}(2Q + 9P) l^2 \omega_{cr}^2. \qquad \text{(b)}$$

Now let us compute the elemental work. In this case no external forces do any work, therefore $dA^e = 0$. The elemental work done by the elastic force of the spring (an internal force) in turning the gear through an angle α about the crank is $dA^i = -M_{sp}\,d\alpha = -c\alpha\,d\alpha$ (the minus sign indicates that the moment is directed opposite the direction through which the gear is turned). As we are seeking the law of motion of the crank, let us express angle α in terms of φ. As $a_2b = a_1b$, we have $R\varphi = r\alpha$, or $(l - r)\,\varphi = r\alpha$, whence

$$\alpha = \frac{l-r}{r}\varphi, \quad dA^i = -c\frac{(l-r)^2}{r^2}\varphi\,d\varphi.$$

Writing now the equation $dT = dA^i$, we have:

$$\frac{1}{6g}(2Q+9P)\,l^2\omega_{cr}\cdot d\omega_{cr} = -c\frac{(l-r)^2}{r^2}\varphi\,d\varphi.$$

Dividing by dt and taking into account that $\frac{d\varphi}{dt} = \omega_{cr}$ and $\frac{d\omega_{cr}}{dt} = \frac{d^2\varphi}{dt^2}$, we finally obtain the differential equation of motion of the system in the form

$$\frac{d^2\varphi}{dt^2} + k^2\varphi = 0,$$

where

$$k^2 = \frac{6gc\,(l-r)^2}{(2Q+9P)\,l^2r^2}.$$

The equation is a differential equation of harmonic motion (§ 123). When moved from its equilibrium position the crank will perform simple harmonic motion the period of which will be

$$\tau = \frac{2\pi}{k} = 2\pi\frac{lr}{l-r}\sqrt{\frac{2Q+9P}{6gc}}.$$

This problem shows the great opportunities for investigating the motion of a system provided by the theorem of the change in kinetic energy.

§ 151. Conservative Force Field and Force Function

The problems examined in the preceding section (and in § 115) were solved with the help of the theorem of the change in kinetic energy because in each case the work done by the acting forces could be computed without knowing the law of motion. The important thing is to determine the type of the forces involved.

The work done in a displacement M_1M_2 by a force F applied at a point M of a body is computed according to Eq. (38′) of § 112:

$$A_{(M_1M_2)} = \int_{(M_1)}^{(M_2)} dA = \int_{(M_1)}^{(M_2)} (F_x\,dx + F_y\,dy + F_z\,dz). \qquad (54)$$

As pointed out in § 112, the integral on the right can be evaluated without knowledge of the law of motion involved (i.e., of the dependence of x, y, z on time) only if the force depends solely on the location of the point, i.e., on its x, y, z coordinates. Such forces are said to form a force field, or field of force. *A force field is defined as a region of space in which any article experiences a force of certain magnitude and direction.* Examples are planetary or stellar gravitational fields. As any force can be defined by its projections on a set of coordinate axes, a force field can be described by the equations:

$$F_x = \Phi_1(x, y, z), \; F_y = \Phi_2(x, y, z), \; F_z = \Phi_3(x, y, z). \quad (55)$$

But in the most general case, to compute the work done by such forces, in Eq. (54) it is necessary to go over to one variable in the integrand; for example, one must know the dependencies $y = f_1(x)$ and $z = f_2(x)$, which give the spatial equation of the curve that is the path of particle M. Consequently, in the most general case the work done by the forces constituting a force field depends on the type of path of the point of application of the relevant force.

However, if the integrand in Eq. (54), which represents the elementary work done by force $\boldsymbol{F}$, is the full differential of a function $U(x, y, z)$, i.e., if

$$dA = dU(x, y, z), \text{ or } F_x\,dx + F_y\,dy + F_z\,dz = dU(x, y, z), \quad (56)$$

the work $A_{(M_1M_2)}$ can be computed without knowing the path of point M.

The function U of the coordinates x, y, z, the differential of which equals the elementary work, is called a *force function*. A force field for which there is a force function is called a *conservative force field*, and the forces acting in that field are called *conservative forces*. We shall regard a force function as a unique function of coordinates.

Substituting the expression for dA from Eq. (56) into Eq. (54), we obtain:

$$A_{(M_1M_2)} = \int\limits_{(M_1)}^{(M_2)} dU(x, y, z) = U_2 - U_1, \quad (57)$$

where $U_1 = U(x_1, y_1, z_1)$ and $U_2 = U(x_2, y_2, z_2)$ are the values of the force functions at points M_1 and M_2 of the field, respectively. Consequently, the work done by a conservative force acting on a moving particle equals the difference between the values of the force function at the terminal and initial points of the displacement and does not depend on the particle's path. In a displacement along a closed path $U_2 = U_1$, and the work done by a conservative force is zero.

The basic property of a conservative force field is that the work done by its forces acting on a moving material particle depends only on the particle's initial and final positions and does not depend on its path followed or the law of motion.

When the work done by a force depends on the path or law of motion of the point at which it is applied, the force is said to be *non-conservative*, or *dissipative*. Examples are friction and the resistance of a medium.

If the relationship (56) is found to apply, the force function can be determined from the equation

$$U = \int dA + C, \text{ or } U = \int (F_x\,dx + F_y\,dy + F_z\,dz) + C, \quad (58)$$

where C is a constant having any value [it is apparent from Eq. (57) that work does not depend on C]. However, it is conventionally assumed that at some point O, called the "zero point", $U_O = 0$, and C is determined on that basis.

Examples of conservative forces examined before are gravity, elastic and gravitational forces (§ 113). Let us show that for the fields of these forces the force functions really do exist and determine the expressions for the forces. With reference to Eqs. (39), (40), and (41′) of § 113, as the integrals from which they were obtained include the elementary works done by the respective forces, we have:

(1) For *gravity*, if axis z is directed vertically up, $dA = -P\,dz$, whence, assuming $U = 0$ at $z = 0$ (the zero point is at the origin of the coordinate system),

$$U = -Pz. \quad (59)$$

(2) For an *elastic force* acting along axis x, $dA = -cx\,dx$, whence, assuming $U = 0$ at $x = 0$,

$$U = -\frac{1}{2}cx^2. \quad (59')$$

(3)* For a *gravitational force*, $dU = kmd\left(\frac{1}{r}\right) = mgR^2d\left(\frac{1}{r}\right)$, whence, assuming $U = 0$ at infinity,

$$U = mgR^2\frac{1}{r}, \quad (59'')$$

where $r = \sqrt{x^2 + y^2 + z^2}$.

Introducing these values of U, Eq. (57) yields the same expressions for the work done by the respective forces as Eqs. (39), (40) and (41′) in § 113.

Let us show that knowing the force function one can determine the force acting at any point of a field. From Eq. (56), calculating the differential with respect to the function $U(x, y, z)$, we have:

$$F_x\,dx + F_y\,dy + F_z\,dz = \frac{\partial U}{\partial x}dx + \frac{\partial U}{\partial y}dy + \frac{\partial U}{\partial z}dz,$$

whence, equating the coefficients of dx, dy, dz in both sides of the equations, we obtain:

$$F_x = \frac{\partial U}{\partial x}, \quad F_y = \frac{\partial U}{\partial y}, \quad F_z = \frac{\partial U}{\partial z}. \tag{60}$$

Consequently, in a conservative force field the projections of a force equal the partial derivatives of the force function with respect to the corresponding coordinates. Vector $\boldsymbol{F}$, the projections of which are given by equations of the form (60), is called the *gradient* of the scalar function U (x, y, z). Thus, $\boldsymbol{F}$ = grad U.

From Eqs. (60) we obtain:

$$\frac{\partial F_x}{\partial y} = \frac{\partial^2 U}{\partial y\,\partial x}, \quad \frac{\partial F_y}{\partial x} = \frac{\partial^2 U}{\partial x\,\partial y}, \quad \text{etc.}$$

It follows that if for a given field there exists a force function, the projections of the force satisfy the relationships:

$$\frac{\partial F_x}{\partial y} = \frac{\partial F_y}{\partial x}, \quad \frac{\partial F_y}{\partial z} = \frac{\partial F_z}{\partial y}, \quad \frac{\partial F_z}{\partial x} = \frac{\partial F_x}{\partial z}. \tag{61}$$

The reverse can also be proved, i.e., if Eqs. (61) are valid, for a field it has a force function U. Consequently, conditions (61) are the *necessary and sufficient conditions* for a force field to be conservative.

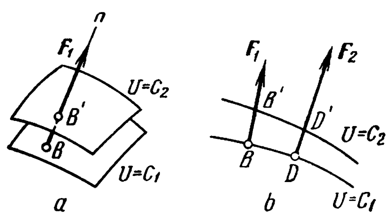

Fig. 328

Thus, if a force field is given by Eqs. (55), Eqs. (61) can be used to determine whether it is conservative or not. If the field is conservative, Eq. (58) yields its force function, and Eq. (51) the work done by the forces. Conversely, if the force function is known, Eqs. (60) can be used to find the force field determined by that function.

Assuming U $(x, y, z) = C$, where C is a constant, we have an equation of a spatial surface in which U has the same value C at every point. Such surfaces are called *equipotential*. If, as assumed, a force function is a unique function of the coordinates, it follows that equipotential surfaces cannot intersect, and only one equipotential surface passes through any point of the field. It follows from Eq. (57) that, in any displacement M_1M_2 along an equipotential surface, $U_1 = U_2 = C$, and the work done by the forces is zero. But as the forces are not zero we conclude that at any point of a conservative force field the force is normal to the equipotential surface through that point.

In Fig. 328 are shown two equipotential surfaces, U $(x, y, z) = C_1$ and U $(x, y, z) = C_2$, and their intersection with a plane through a normal Bn. If the force is in the direction shown in the drawing, the work done in the displacement BB' is positive. But, by Eq. (57), that work equals $C_2 - C_1$. Consequently, $C_2 > C_1$, i.e., the forces of a conservative field are pointed in the direction of the increase in the force function. Furthermore, the work done by force F_1 in the displacement BB' and by F_2 in the displacement DD' is the same, as in both cases $C_2 - C_1$ is the same. But as $DD' < BB'$, apparently $F_2 > F_1$. Consequently, the forces of a conservative field are greater where the equipotential surfaces are more dense. These properties make it possible to present a clear picture of the distribution of forces in a conservative force field by means of equipotential surfaces. Furthermore, as Eq. (57) indicates, the work done by a conservative force depends, in the final analysis, only on the equipotential surfaces between which the displacement takes place.

Here are a few examples of what has been said.

(1) For a homogeneous field of gravity (see Fig. 254), it follows from Eq. (59) that U = const. when z = const. Hence, the equipotential surfaces are horizontal planes. The force of gravity P is directed normal to them in the direction of increasing U and is constant at all points of the field.

(2) For a field of gravitational forces, according to Eq. (59″), U = const. when r = const. Hence, the equipotential surfaces are concentric spheres whose centres coincide with that of the attracting centre. The force at every point of a gravitational field is normal to the respective sphere in the direction of increasing U (or decreasing r), i.e., pointed towards the centre of the sphere.

In the case of a **system of material particles** moving in a conservative force field, it is possible to determine the force function $U_k(x_k, y_k, z_k)$ for every point with coordinates x_k, y_k, z_k, thus enabling the computation of the elementary work done by the force acting on the particle concerned. Hence the function of the coordinates of all particles of the system

$$U(x_1, y_1, z_1, \ldots, x_n, y_n, z_n) = \sum U_k(x_k, y_k, z_k)$$

is the force function of the mechanical system.

As $dU = \sum dU_k$, according to Eq. (56), which is valid for all points of the system,

$$dU = \sum dA_k, \tag{62}$$

i.e., the differential of the force function of a system equals the sum elementary works done by all the forces acting on the system.

§ 152. Potential Energy

For conservative forces we can introduce the concept of *potential energy* as a measure of the capacity of a particle for doing work by virtue of its position in the force field. In order to compare different "capacities for doing work", we must agree on the choice of a zero point O, in which we assume the capacity to do work to be zero (the choice of the zero point, as of any initial point or origin, is arbitrary). *The potential energy of a particle in any configuration M is defined as the scalar quantity V equal to the work done on the particle by the forces of a field in the passage from configuration M to the zero configuration*:

$$V = A_{(MO)}.$$

It follows from the definition that potential energy is dependent on the coordinates of the particle M, i.e., $V = V(x, y, z)$.

Assuming that the zero points of the functions $V(x, y, z)$ and $U(x, y, z)$ coincide, we have $U_O = 0$ and, by Eq. (57), $A_{(MO)} = U_O - U = -U$, where U is the force function at point M of the field; whence,

$$V(x, y, z) = -U(x, y, z),$$

i.e., the potential energy at any point of a force field is equal to the magnitude of the force function at that point taken with the opposite sign.

It is thus apparent that in investigating the properties of a conservative force field we can replace the force function with potential energy. In particular, in computing the work done by a conservative force we can use instead of Eq. (57) the formula

$$A_{(M_1M_2)} = V_1 - V_2. \tag{63}$$

Thus, the work done by a conservative force is equal to the difference between a moving particle's potential energy in its initial and final positions.

The expressions for the potential energy of the conservative forces examined before can be determined from Eqs. (59)-(59″), taking into account that $V = -U$. Thus, for gravity $V = Pz$, etc.

§ 153. The Law of Conservation of Mechanical Energy

Let us assume that all the external and internal forces acting on a system are conservative forces. Then, for any particle belonging to the system, the work done by the applied forces is

$$A_k = V_{k0} - V_{k1},$$

and for all the external and internal forces

$$\sum A_k = \sum V_{k0} - \sum V_{k1} = V_0 - V_1,$$

where $V = \sum V_k$ is the potential energy of the whole system.

Substituting this expression for work into Eq. (50), we obtain:

$$T_1 - T_0 = V_0 - V_1 \text{ or } T_1 + V_1 = T_0 + V_0 = \text{const.}$$

Thus, in the motion of a system subjected to the action of only conservative forces, the sum of the kinetic and potential energies of the system remains constant for any configuration. This is the *law of conservation of mechanical energy*, which is a particular case of the general physical law of conservation of energy. The quantity $T + V$ is called the *total mechanical energy of the system.*

If the acting forces include dissipative forces, such as friction, the total mechanical energy of the system will decrease during its motion due to transformation into other forms of energy, e.g., thermal energy.

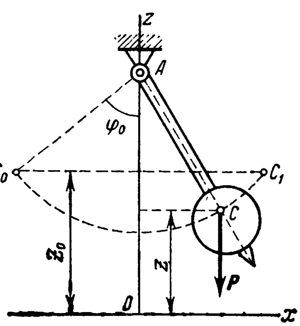

Fig. 329

The whole meaning of the foregoing law becomes apparent when it is considered in connection with the general physical law of conservation of energy. However, in solving purely mechanical problems, the theorem of the change in the kinetic energy of a system can always be immediately applied.

Example. Consider a pendulum (Fig. 329) displaced through an angle φ_0 from the vertical and released from rest. Then, in its initial position, $V_0 = Pz_0$ and $T_0 = 0$, where P is the weight of the pendulum and z is the coordinate of its centre of gravity. Consequently, neglecting all resisting forces in any configuration we shall have $V + T = V_0$, or

$$Pz + \frac{1}{2} J_A \omega^2 = Pz_0.$$

Thus, the centre of gravity of the pendulum cannot be higher than position z_0. When the pendulum swings down, its potential energy decreases and its kinetic energy increases; during an upward swing the potential energy increases and the kinetic energy decreases.

It follows from the previous equation that

$$\omega^2 = \frac{2P}{J_A}(z_0 - z).$$

Thus, the instantaneous angular velocity of a pendulum depends only on the position occupied by its centre of gravity and is always the same for any given configuration. Such relations occur only during motion under the action of conservative forces. Due to friction in the axis and the resistance of the air (non-conservative forces), the above relationships do not actually hold; the total mechanical energy of a pendulum decreases with time, and its vibrations are damped.

Chapter 28
Applications of the General Theorems to Rigid-Body Dynamics

§ 154. Rotation of a Rigid Body

Let us consider the application of the general theorems of dynamics to some problems on the motion of absolutely rigid bodies. Since the investigation of translatory motion of a rigid body is reduced to particle dynamics, we shall begin with rotational motion.

Let there be a system of forces F_1^e, F_2^e, . . ., F_n^e acting on a rigid body with a fixed axis of rotation z (Fig. 330). Also acting on the body are the reactions R_A and R_B of the bearings. To exclude these immediately unknown forces from the equations of motion, we make use of the theorem of moments with respect to axis z (§ 144). As the moments of forces R_A and R_B with respect to the axis are zero, we have

$$\frac{dK_z}{dt} = M_z^e,$$

where

$$M_z^e = \sum m_z(F_k^e).$$

We shall call the quantity M_z^e the *turning moment*, or *torque*.

Substituting the expression $K_z = J_z\omega$ into the foregoing equation, we obtain:

$$J_z \frac{d\omega}{dt} = M_z^e \quad \text{or} \quad J_z \frac{d^2\varphi}{dt^2} = M_z^e. \qquad (64)$$

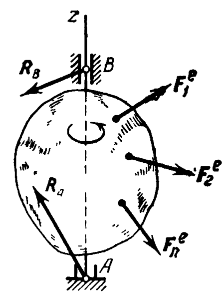

Fig. 330

Eq. (64) is the differential equation of the rotational motion of a rigid body. It follows from the equation that the product of the moment of inertia of a body with respect to its axis of rotation and its angular acceleration is equal to the turning moment:

$$J_z\varepsilon = M_z^e. \qquad (64')$$

Equation (64′) shows that, for a given torque M_z^e, the greater the moment of inertia of a body, the less the angular velocity, and vice versa. Thus, we see that in rotational motion the moment of inertia of a body actually plays the same role as mass in translational motion, i.e., it is the measure of a body's inertia in rotational motion (see § 131). Eq. (64) can be applied: (1) to develop the equation of rotation of a body $\varphi = f(t)$ or to find its angular velocity ω if the applied torque is known, (2) to determine the torque M_z^e if the law of motion, i.e., $\varphi = f(t)$, is known. In solving the first problem it should be borne in mind that in the general case M_z^e may be a variable dependent on t, φ, and $\omega = \dot{\varphi}$.

Instead of Eq. (64), rotational motion may be investigated with the help of the theorem of the change in kinetic energy: $T_1 - T_0 = A^e$, where T and A^e are found from Eqs. (43) and (47).

Note the following special cases:

(1) If $M_z^e = 0$, $\omega = \text{const}$, i.e., the rotation is uniform;

(2) If $M_z^e = \text{const.}$, $\varepsilon = \text{const.}$, i.e., the rotation is uniformly variable.

Eq. (64) is analogous in form to the differential equation of rectilinear motion of a particle (§ 104); therefore, the methods of integration are also analogous (see Problem 143).

Eq. (64) is most conveniently used in problem solutions when a system consists only of a single rotating body. If a system has, besides a rotating body, other moving bodies (as, for instance, in Problems 133, 138, and others), the equation of motion is more conveniently developed with the help of the general theorems or the methods described in §§ 168, 173 and 178.

In solving problems of the type 133 it should be remembered that acting on the drum is not the force Q but the tension in the string F, which is not equal to Q, and Eq. (64) takes the form $J_O\varepsilon = Fr$. To solve it, F must be found additionally from the equation of motion of the load A, which makes the computations longer.

Problem 142. A wheel of radius R and mass M rotates on its axis O with an angular velocity ω_0 (Fig. 331). A brake shoe is applied to the wheel at some instant with a force Q. The coefficient of friction of the shoe on the wheel is f. Neglecting friction in the axle and the mass of the spokes, determine in how many seconds the wheel will stop.

Solution. Writing Eq. (64) and considering the moment positive in the direction of the rotation, we have:

$$J_O \frac{d\omega}{dt} = -fQr, \tag{a}$$

as the force of friction $F = fQ$. From this, integrating, we obtain

$$J_O\omega = -fQrt + C_1.$$

According to the initial conditions, at $t = 0$ we have $\omega = \omega_0$; consequently, $C_1 = J_O\omega_0$, and finally

$$\omega = \omega_0 - \frac{fQr}{J_O} t. \tag{b}$$

At $t = t_1$, when the wheel stops, $\omega = 0$. Substituting this expression and taking into account that for the rim (a ring) $J_O = Mr^2$, we obtain:

$$t_1 = \frac{J_O\omega_0}{fQr} = \frac{Mr\omega_0}{fQ}.$$

If we want to determine the number of revolutions of the wheel until it stops, it is more convenient to apply the theorem of the change in kinetic energy without integrating equation (b) again.

Problem 143. A vertical cylindrical rotor whose moment of inertia with respect to its axis is J_z (Fig. 332) is made to revolve by an

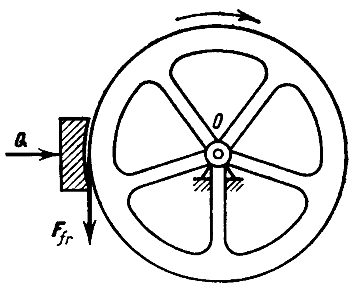

Fig. 331

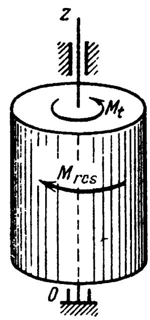

Fig. 332

applied torque M_t. Determine how the angular velocity ω of the rotor will change during the motion if $\omega_0 = 0$ and the moment of the resisting force of the air is proportional to ω, i.e., $M_{res} = \mu\omega$.

Solution. The differential equation (64) of the rotation of the rotor has the form (assuming the moments in the direction of rotation to be positive)

$$J_z \frac{d\omega}{dt} = M_t - \mu\omega.$$

Separating the variables and assuming $\frac{\mu}{J_z} = n$, we evaluate the respective definite integrals for both sides of the equation; we obtain:

$$\int_0^\omega \frac{-\mu\, d\omega}{M_t - \mu\omega} = -n \int_0^t dt.$$

Hence,

$$\ln \frac{M_t - \mu\omega}{M_t} = -nt, \text{ or } \frac{M_t - \mu\omega}{M_t} = e^{-nt}.$$

And finally we obtain:

$$\omega = \frac{M_t}{\mu}(1 - e^{-nt}).$$

The angular velocity of the rotor increases with time and tends towards the limiting value $\omega_{\text{lim}} = \frac{M_t}{\mu}$.

§ 155. The Compound Pendulum

Any rigid body free to oscillate about a fixed horizontal axis under the action of gravity is called a compound, or physical, pendulum. Let Fig. 333*a* denote a cross section of a pendulum normal to the axis of support through the centre of mass C of the pendulum, and let us introduce the notation P for the weight of the pendulum, a for the distance OC from the centre of mass to the axis of suspension, and J_O for the moment of inertia of the pendulum with respect to the axis of suspension. We specify the position of the pendulum by the angle φ that OC makes with the vertical.

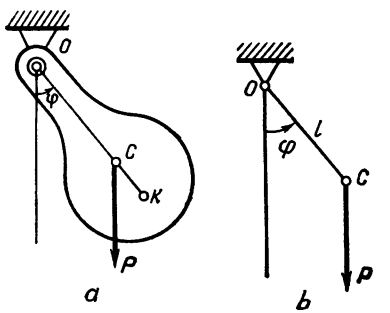

Fig. 333

To develop the equation of motion of the pendulum, we shall employ the differential equation of rotational motion (64). In our case $M_z = M_O = -Pa \sin \varphi$ (the "minus" being taken to indicate that the direction of the moment is opposite to the positive direction of angle φ), and Eq. (64) takes the form

$$J_O \frac{d^2\varphi}{dt^2} = -Pa \sin \varphi.$$

Dividing through by J_O and introducing the notation

$$\frac{Pa}{J_O} = k^2,$$

we obtain the differential equation of motion of the pendulum in the form

$$\frac{d^2\varphi}{dt^2} + k^2 \sin \varphi = 0.$$

This differential equation cannot be integrated with respect to ordinary functions. Considering only *small oscillations* and assuming approximately $\sin \varphi \approx \varphi$ (which can be done when angle φ is smaller than one radian), we have

$$\frac{d^2\varphi}{dt^2}+k^2\varphi=0.$$

This differential equation has the same form as the equation of free rectilinear oscillations of a particle and, by analogy with Eq. (63) of § 123, its general solution has the form:

$$\varphi = C_1 \sin kt + C_2 \cos kt.$$

Assuming that at the initial moment $t=0$ the pendulum is deflected through a small angle $\varphi=\varphi_0$ and released with no initial velocity ($\omega_0=0$), we obtain for the integration constants the values $C_1=0$ and $C_2=\varphi_0$. Then the law of motion for small oscillations is, given the stated initial conditions:

$$\varphi = \varphi_0 \cos kt.$$

Consequently, small oscillations of a compound pendulum are harmonic. The period of oscillation is given by the formula (substituting the value of k):

$$T_{\text{compound}} \approx \frac{2\pi}{k} = 2\pi\sqrt{\frac{J_O}{Pa}}. \qquad (65)$$

Thus, for small oscillations, the period does not depend on the initial angle of deflection φ_0. The result is approximate, for integrating the differential equation of oscillations written earlier, and not assuming angle φ to be small (i.e., not assuming $\sin\varphi \approx \varphi$), we see that T_{compound} depends on φ_0. Approximately the dependency has the form

$$T_{\text{compound}} \approx 2\pi\sqrt{\frac{J_O}{Pa}}\left(1+\frac{\varphi_0^2}{16}\right).$$

It follows, for example, that at $\varphi_0=0.4$ radian (about 23°), Eq. (65) gives the period of oscillation to an accuracy of 1%.

These results are also valid in the case of the so-called *simple pendulum*, i.e., a weight of small size (which can be assumed a material particle) suspended by an inextensible string of length l and negligible mass as compared with the mass of the pendulum (Fig. 333*b*). As a simple pendulum is, by definition, a system consisting of one material particle, obviously

$$J_O = ml^2 = \frac{P}{g}l^2, \quad a = OC = l.$$

Substituting these values into Eq. (65), we find that the period of small oscillations of a simple pendulum is given by the equation:

$$T_{\text{simple}} = 2\pi\sqrt{\frac{l}{g}}. \qquad (65')$$

Comparing Eqs. (65) and (65′), we see that for a length

$$l_1 = \frac{J_O g}{Pa} = \frac{J_O}{Ma} \tag{66}$$

the period of oscillation of a simple pendulum is equal to that of an equivalent compound pendulum.

The length l_1 of such a simple pendulum whose period of oscillation is equal to that of a given compound pendulum is called the *equivalent length of the compound pendulum*. The point K which lies at a distance $OK = l_1$ from the axis of support is called the *centre of oscillation*, or the *centre of percussion*, of the coupound pendulum (see Fig. 333).

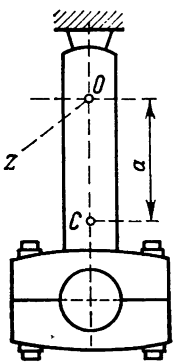

Fig. 334

Noting that, from the parallel axis theorem, $J_O = J_C + Ma^2$, we can transform Eq. (66) to the form

$$l_1 = a + \frac{J_C}{Ma}. \tag{66′}$$

It follows from this that OK is always greater than $OC = a$, i.e., that the centre of percussion of a compound pendulum is always located below its centre of mass.

We see from Eq. (66′) that $KC = J_C/Ma$. Therefore, if the axis of suspension passes through point K, the equivalent length l_2 of the resulting pendulum will, by Eq. (66′), be

$$l_2 = KC + \frac{J_C}{M \cdot KC} = \frac{J_C}{Ma} + a = l_1.$$

Consequently, points K and O are interchangeable, i.e., if the axis of suspension passes through K, the centre of percussion will be at O (as $l_2 = l_1$), and the period of oscillation will not change. This property is employed in the so-called inverted pendulum which is used to determine the acceleration of gravity.

Determination of Moments of Inertia by Experiment. One of the methods of determining the moment of inertia of bodies by experiment is based on the application of Eq. (65) for the period of small oscillations.

Let it be required to determine the moment of inertia with respect to an axis Oz of a connecting rod of weight P (Fig. 334). For this the body is suspended so that the axis Oz is horizontal, and the period of small oscillations T is determined by direct observation. Then the distance $OC = a$ is found by the method of weighing (see § 56, Fig. 131). Substituting the obtained values into Eq. (65), we have:

$$J_{Oz} = \frac{PaT^2}{4\pi^2}.$$

If it is required to determine the moment of inertia of a body with respect to an axis Ox through its centre of gravity, the body may be suspended by two strings so that the axis Ox is horizontal (Fig. 335), and the moment of inertia J_{AB} is determined with respect to axis AB (the value of a is immediately known). Then the required moment of inertia is computed by the parallel axis theorem:

$$J_{Ox}=J_{AB}-\frac{P}{g}a^2.$$

§ 156. Plane Motion of a Rigid Body

The position of a body performing plane motion is specified at any instant by the position of any pole and the angle of rotation of the body about that pole (§ 77). Dynamical problems are much more simple solved if the centre of mass C of a body is taken as the

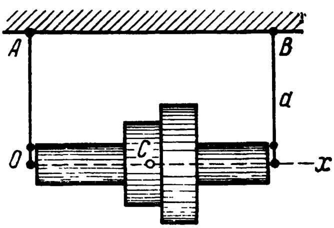

Fig. 335

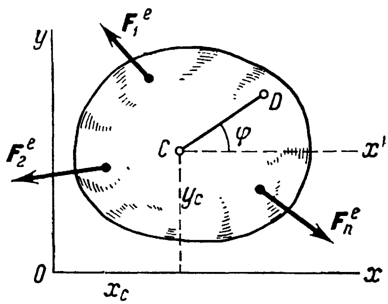

Fig. 336

pole (Fig. 336) and the position of the body is defined by coordinates x_C, y_C, and angle φ (in Fig. 336 the body is depicted as intersected by a plane parallel to the plane of motion and passing through point C).

Let there be acting on the body a coplanar system of external forces $\boldsymbol{F}_1^e, \boldsymbol{F}_2^e, \ldots, \boldsymbol{F}_n^e$. The equation of motion of point C can be found from the theorem of the motion of centre of mass:

$$M\boldsymbol{w}_C=\sum \boldsymbol{F}_k^e, \tag{67}$$

and the rotation about C is given by Eq. (64), since the theorem from which it was derived is also valid for the motion of a system about the centre of mass. Finally, after projecting both sides of Eq. (67) on the coordinate axes, we obtain:

$$Mw_{Cx}=\sum F_{kx}^e,\quad Mw_{Cy}=\sum F_{ky}^e,\quad J_C\varepsilon=\sum m_C(\boldsymbol{F}_k^e), \tag{68}$$

or

$$M\frac{d^2x_C}{dt^2}=\sum F^e_{kx},\quad M\frac{d^2y_C}{dt^2}=\sum F^e_{ky},\quad J_C\frac{d^2\varphi}{dt^2}=\sum m_C(F^e_k). \quad (68')$$

Eqs. (68) are the *differential equations of plane motion of a rigid body*. With their help we can develop the equation of motion of a body if the forces are given or we can determine the principal vector and principal moment of the acting forces if the law of motion is known.

For the case of constrained motion, when the path of the centre of mass is known, the equation of motion of point C is more conveniently expressed in terms of the projections on the tangent τ and principal normal n to the path. Then, instead of the equations (68), we obtain:

$$M\frac{dv_C}{dt}=\sum F^e_{k\tau},\quad M\frac{v_C^2}{\rho_C}=\sum F^e_{kn},\quad J_C\frac{d^2\varphi}{dt^2}=\sum m_C(F^e_k), \quad (69)$$

where ρ_C is the radius of curvature of the path of the centre of mass.

Note that in constrained motion the right side of Eqs. (68) or (69) will additionally include the unknown reactions of the constraints. They will have to be determined by deriving additional equations describing the conditions of motion imposed on the body by the constraints (see Problem 144 and others). The equations of constrained motion will often be more conveniently derived with the help of the theorem of the change in kinetic energy, which can be used in place of one of the equations (68) or (69).

Problem 144. A uniform circular cylinder rolls down an inclined plane without slipping (Fig. 337). Neglecting rolling friction, determine the acceleration of the centre of the cylinder and the limiting friction of impending slip.

Solution. Let us introduce the following notations: α for the angle of inclination of the surface, P for the weight of the cylinder, R for its radius, and F for the limiting friction of impending slip; let us also direct axis x along the inclined plane and axis y perpendicular to it.

As the centre of mass of the cylinder does not move parallel to axis y, $w_{Cy}=0$, and by the first of Eqs. (68), the sum of the projections of all the forces on axis y is also zero. Thus,

$$N=P\cos\alpha.$$

In writing the last two of the equations (68), take into account that $w_{Cx}=w_C$. Neglecting rolling friction and taking the direction of rotation of the cylinder as the positive direction of the moment of force, we find:

$$Mw_C=P\sin\alpha-F,\quad J_C\varepsilon=FR. \quad \text{(a)}$$

Equations (a) contain three unknown quantities, w_C, ε, and P (we cannot consider $F = fN$ here, because this is valid only when the point of contact *slides* on the surface; when there is no sliding it is possible for $F \leqslant fN$; see § 37). We obtain an additional relationship between the unknown quantities if we take into account that in pure rolling $v_C = \omega R$, whence, differentiating, we obtain $w_C = \varepsilon R$.

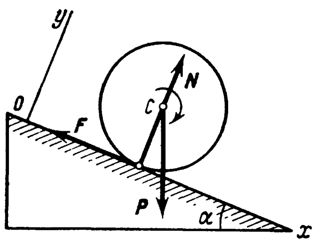

Fig. 337

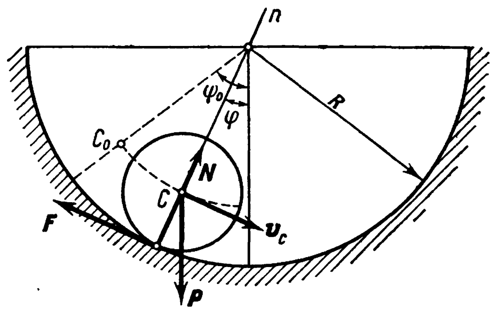

Fig. 338

Then, remembering that for a uniform cylinder $J_C = 0.5MR^2$, the second of equations (a) takes the form

$$\frac{1}{2} M w_C = F. \tag{b}$$

Substituting this expression of F into the first of equations (a), we obtain:

$$w_C = \frac{2}{3} g \sin \alpha. \tag{c}$$

Now from (b) we find:

$$F = \frac{1}{3} P \sin \alpha. \tag{d}$$

This is the frictional force that must act on the rolling cylinder if it is not to slip. It was pointed out before that $F \leqslant fN$. Consequently, pure rolling takes place when

$$\frac{1}{3} P \sin \alpha \leqslant fP \cos \alpha,$$

or

$$f \geqslant \frac{1}{3} \tan \alpha.$$

If the coefficient of friction is less than this, force F cannot attain the value given by equation (d), and the cylinder will slip. In this case v_C and ω are not related by the equality $v_C = \omega R$ (the point of contact is not the instantaneous centre of zero velocity), but now F has a limiting value, i.e., $F = fN = fP \cos \alpha$, and equations (a) take the form

$$Mw_C = P(\sin\alpha - f\cos\alpha);\ \frac{1}{2}MR^2\varepsilon = fPR\cos\alpha,$$

whence

$$w_C = g(\sin\alpha - f\cos\alpha); \qquad \varepsilon = \frac{2gf}{R}\cos\alpha. \tag{e}$$

In this case the centre of the cylinder moves with an acceleration w_C, while the cylinder itself rotates with an angular acceleration ε, the values of which are determined from equations (e).

Problem 145. Solve the previous problem taking into account the resistance to rolling, assuming the coefficient of rolling friction to be k.

Solution. In order to give an example of another method of computation, let us find w_C with the help of the theorem of the change in kinetic energy, i.e., the equation

$$dT = dA^e. \tag{a}$$

In our case (see Problem 134, § 147),

$$T = \frac{3}{4}Mv_C^2.$$

Only the force P and the resisting moment perform any work. The work done by force F is zero (see § 148). Then, taking into account Eq. (48), we obtain (see Fig. 337, but now with force N shifted by the value k in the direction of the motion, i.e., located as shown in Fig. 99):

$$dA^e = P\sin\alpha\cdot ds_C - \frac{k}{R}N\,ds_C = P\left(\sin\alpha - \frac{k}{R}\cos\alpha\right)ds_C.$$

Substituting the determined quantities into equations (a) and dividing by dt, we have:

$$\frac{3}{2}\frac{P}{g}v_C\frac{dv_C}{dt} = P\left(\sin\alpha - \frac{k}{R}\cos\alpha\right)\frac{ds_C}{dt}.$$

The last multiplier is equal to v_C, and we finally obtain:

$$w_C = \frac{2}{3}g\left(\sin\alpha - \frac{k}{R}\cos\alpha\right).$$

At $k = 0$ this formula gives the result of the previous problem. The frictional force can now be found from the equation $Mw_C = P\sin\alpha - F$, which does not change its form.

Problem 146. A uniform cylinder of weight P and radius r starts rolling from rest without slipping from a point on a cylindrical surface of radius R defined by angle φ_0 (Fig. 338). Determine: (1) the pressure of the cylinder on the surface for any angle φ and (2) the law of motion of the cylinder when angle φ_0 is small. Neglect rolling friction

Solution. (1) Acting on the cylinder in any position is force $\boldsymbol{P}$, the normal reaction $\boldsymbol{N}$, and the frictional force $\boldsymbol{F}$, without which rolling is impossible. The path of the centre C is known: a circle of radius $R - r$. To determine N we make use of the second of equations (69). Directing the normal Cn inwards to the path, we obtain:

$$M\frac{v_C^2}{R-r} = N - P\cos\varphi. \tag{a}$$

The quantity v_C in this equation can be found from the theorem of the change in kinetic energy (compare with § 119):

$$T - T_0 = \sum A_k^e. \tag{b}$$

In our case $T_0 = 0$ and $T = \frac{3}{4}Mv_C^2$ (see Problem 134). Only force P does any work, consequently,

$$\sum A_k^e = Ph = P(R-r)(\cos\varphi - \cos\varphi_0),$$

and equation (b) takes the form

$$\frac{3}{4}Mv_C^2 = P(R-r)(\cos\varphi - \cos\varphi_0). \tag{c}$$

Computing from here Mv_C^2 and substituting into equation (a), we obtain finally:

$$N = \frac{P}{3}(7\cos\varphi - 4\cos\varphi_0).$$

If, for example, $\varphi_0 = 60°$ and $\varphi = 0°$, then $N = \frac{3}{5}P$.

(2) To determine the law of motion of point C, differentiate equation (c) with respect to time. We obtain:

$$\frac{3}{2}\frac{P}{g}v_C\frac{dv_C}{dt} = -P(R-r)\sin\varphi\frac{d\varphi}{dt}.$$

In our case angle φ decreases when the cylinder moves, and $\frac{d\varphi}{dt} < 0$. Then,

$$v_C = (R-r)\left|\frac{d\varphi}{dt}\right| = -(R-r)\frac{d\varphi}{dt}; \quad \frac{dv_C}{dt} = -(R-r)\frac{d^2\varphi}{dt^2}.$$

Substituting these expressions into the previous equation, we obtain finally:

$$\frac{d^2\varphi}{dt^2} + \frac{2}{3}\frac{g}{R-r}\sin\varphi = 0.$$

If angle φ_0 is small, then, as $\varphi \leqslant \varphi_0$, we can assume that $\sin\varphi \approx \varphi$, and the equation takes the form

$$\frac{d^2\varphi}{dt^2} + k^2\varphi = 0,$$

where

$$k^2 = \frac{2}{3}\frac{g}{R-r}.$$

Consequently (see § 123), the centre of the cylinder performs simple harmonic motion, its period being

$$\tau = \frac{2\pi}{k} = 2\pi \sqrt{\frac{3(R-r)}{2g}}.$$

Problem 147. A body of weight P (Fig. 339) rests at B on a piezoelectric sensor of an instrument for measuring pressure, and at A it is attached to a string AD. When the system is in equilibrium, AC is horizontal and the pressure at B is Q_0. Determine the moment of inertia J_C of the body with respect to an axis through the centre of mass C, if at the instant when the string is severed the pressure at B becomes Q_1. The distance l is known.

Solution. (1) In the equilibrium position $Q_0 l = P(l - a)$, whence

$$a = \frac{P - Q_0}{P} l.$$

(2) When the string is severed, the body begins plane motion. Its displacement in the initial time increment can be neglected. Then Eqs. (68), which are valid only for this initial time interval, will take the form

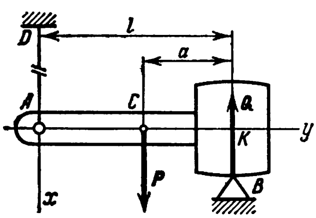

Fig. 339

$$Mw_{Cx} = P - Q_1, \quad w_{Cy} = 0, \quad J_C\varepsilon = Q_1 a. \tag{a}$$

As $w_{Cy} = 0$, point C starts moving vertically down and point B slides horizontally (assuming the friction in the support to be very small). Erecting perpendiculars to the directions of these displacements, we find that the instantaneous centre of zero velocity is at point K. Consequently, $v_C = a\omega$. Assuming $a =$ const. for the elementary time interval, we obtain, after differentiating, $w_C = a\varepsilon$. Then the first of equations (a) gives

$$\frac{P}{g} a\varepsilon = P - Q_1.$$

Determining ε from here, we obtain finally:

$$J_C = \frac{Q_1 a}{\varepsilon} = \frac{P}{g} \frac{Q_1}{P - Q_1} a^2.$$

This result can be used for the experimental determination of moment of inertia.

Problem 148. The weight of an automobile together with its wheels is P, the weight of each wheel is p, and their radii are r (Fig. 340). Acting on the rear (driving) wheels is a turning moment M_t. The car starts from rest and is subjected to the resistance of the air, which is proportional to the square of the translatory velocity: $R = \mu v^2$. The frictional moment acting on the axle of each wheel is M_{fr}. Neglecting rolling friction, determine (1) the maximum velocity of the car and (2) the sliding friction acting on the driving and driven wheels during motion.

Solution. (1) To determine the maximum velocity, write the equation of motion from Eq. (49):

$$dT = \sum dA_k^e + \sum dA_k^i. \qquad \text{(a)}$$

The kinetic energy of the car is equal to the energy of the body plus the energy of the wheels. Taking into account that P is the

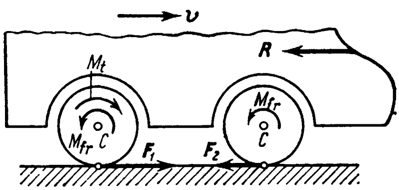

Fig. 340

weight of the whole car and $v_C = \omega r$, and denoting the radius of gyration of each wheel by the symbol ρ, we obtain:

$$T = \frac{1}{2}\frac{P}{g}v_C^2 + 4\left(\frac{1}{2}J_C\omega^2\right) = \frac{1}{2g}\left(P + 4p\frac{\rho^2}{r^2}\right)v_C^2.$$

Of all the external forces, only the resistance of the air does work, as we have neglected rolling resistance, and in this case the work done by the frictional forces F_1 and F_2 of the wheels on the road is zero (see § 148). Therefore,

$$\sum dA_k^e = -R\,ds_C = -\mu v_C^2\,ds_C.$$

The work done by the internal forces (the torque and the friction in the axles) is

$$\sum dA_k^i = (M_t - 4M_{fr})\,d\varphi = (M_t - 4M_{fr})\frac{ds_C}{r}.$$

Substituting all these expressions into equation (a) and dividing through by dt, we obtain:

$$\frac{1}{g}\left(P + 4p\frac{\rho^2}{r^2}\right)v_C\frac{dv_C}{dt} = \frac{1}{r}(M_t - 4M_{fr} - \mu r v_C^2)\frac{ds_C}{dt},$$

from which, cancelling out $v_C = \frac{ds_C}{dt}$, we find:

$$\left(P + 4p\frac{\rho^2}{r^2}\right)w_C = \frac{g}{r}(M_t - 4M_{fr} - \mu r v_C^2). \qquad \text{(b)}$$

When the velocity reaches its limiting value, the acceleration w_C becomes zero. Therefore, $v_C^{\lim}$ can be found from the equation

$$M_t - 4M_{fr} - \mu r v_C^2 = 0,$$

whence

$$v_C^{\lim} = \sqrt{\frac{M_t - 4M_{fr}}{\mu r}}.$$

This result could have been obtained immediately by equating the total work done by all the forces to zero. The purpose of the above computations was to show how to develop the equation of motion (b).

(2) To determine the frictional forces acting on each wheel, we deduce the equations of the rotation of the wheels about their axes. For the driving wheels, taking into account that the frictional force F_1 acting on each of them is directed forward (see § 136, Fig. 301), we obtain:

$$2\frac{p}{g}\rho^2\varepsilon = M_t - 2M_{fr} - 2F_1 r.$$

Since in rolling $w_C = \varepsilon r$, we obtain finally:

$$F_1 = \frac{0.5M_t - M_{fr}}{r} - \frac{\rho^2}{r^2}\frac{p}{g}w_C. \qquad \text{(c)}$$

The frictional force F_2 acting on each of the driven wheels is directed backwards. Therefore, for the driven wheels we have

$$\frac{p}{g}\rho^2\varepsilon = F_2 r - M_{fr},$$

whence

$$F_2 = \frac{M_{fr}}{r} + \frac{\rho^2}{r^2}\frac{p}{g}w_C. \qquad \text{(d)}$$

We see from equation (b) that when the velocity increases the acceleration w_C decreases, and tends to zero when $v_C \to v_C^{\lim}$.

Thus, the frictional forces acting on the driving wheels increase somewhat during the acceleration and reach a maximum value when the motion becomes uniform ($w_C = 0$). If we substitute the value of w_C from equation (b), we shall readily notice that the last member in equation (c) is much smaller than the first, since $P \gg p$. Therefore, for all cases of practical interest the value of F_1 changes only slightly.

The frictional forces acting on the driven wheels will be greatest at the beginning of the motion, then decreasing till they reach their smallest value M_{fr}/r when the motion is uniform ($w_C = 0$).

If the coefficient of friction of the wheels on the road is too small for the frictional force to reach the values of F_1 or F_2, the respective wheels will slip. As M_t is much larger than M_{fr}, the driving wheels are primarily in danger of slipping.

When the motor is switched off, all the wheels become driven, and initially they will all be subjected to the frictional force $F = M_{fr}/r$. The action of brake shoes is equivalent to an increase in M_{fr} acting on the axles, and consequently in the frictional force acting on each wheel, and the automobile slows down quicker (see § 136).

§ 157*. Approximate Theory of Gyroscopic Action

A gyroscope is a rigid body rotating about an axis whose direction in space may change with time. In future we shall deal only with symmetrical gyroscopes, i.e., possessing an axis of material symmetry about which it rotates. In gyroscopic instruments, the gyroscope is usually supported in a system of concentric rings or gimbals (Fig. 341) in such a way that, no matter how it is turned, its centre of gravity remains motionless.

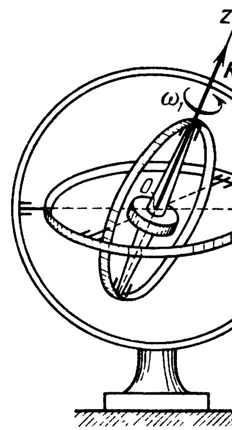

Fig. 341

Gyroscopes used in engineering have a high angular velocity ω_1 of spin about the axis of symmetry (axis of spin). This makes it possible to neglect in the first approximation the additional rotations of the gyroscope due to the motion of its axis, and to develop an approximate theory of gyroscopic action.

In § 143 it was established that when a body rotates about a fixed axis Oz, which is the body's axis of symmetry, vector $\boldsymbol{K}_O$ is directed along the axis of spin and is computed according to Eq. (32). The basic assumption in the elementary theory of gyroscopic action is that even if the axis does move slowly, at any instant the principal angular momentum vector $\boldsymbol{K}_O$ of a gyroscope with respect to its fixed point remains directed along the axis of spin in the same direction as vector $\boldsymbol{\omega}_1$ and is equal to $J_z\omega_1$:

$$K_O = K_z = J_z\omega_1, \tag{70}$$

where J_z is the moment of inertia of the gyroscope with respect to its axis of symmetry. The faster the spin of the gyroscope, the more valid this assumption. Proceeding from this assumption, let us establish the main properties of the gyroscope.

(1) Free Gyroscope. A gyroscope mounted so that its centre of gravity is fixed and its axis can turn in any way about that centre (see Fig. 341) is called a *free gyroscope*. For if, neglecting friction in the axles of the gimbals, we have $\sum m_O(F_k^e = 0$*) and $K_O =$ const., i.e., the total angular momentum is constant both in magnitude and direction (see § 145). But as vector K_O is directed continually along the axis of spin, it follows that the axis of a free gyroscope remains constantly aligned in space with respect to an inertial (stellar) frame of reference. This is one of the important properties of gyroscopic action which is employed in all gyroscopic instruments.

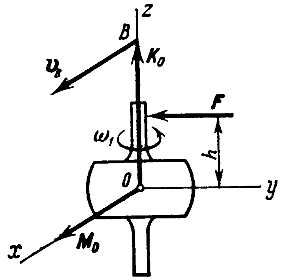

Fig. 342

By virtue of this property, the axis of a free gyroscope rotates relative to the earth in the opposite direction of the latter's rotation. A free gyroscope can thus be used to prove the fact of the earth's rotation round its axis. Such an experiment was first carried out by Foucault in 1852.

(2) Action of a Force Applied to the Axis of a Gyroscope. Let a force F whose moment with respect to the centre O is $M_O = Fh$ start acting on the axis of a rapidly spinning gyroscope (Fig. 342). Then, by the principle of moments (§ 144):

$$\frac{dK_O}{dt} = M_O \quad \text{or} \quad \frac{d(\overline{OB})}{dt} = M_O,$$

where B is a point on the axis coincident with the tip of vector K_O. From this, taking into account that the derivative of vector $\overline{OB}$ with respect to time gives the velocity v_B of point B, we obtain:

$$v_B = M_O. \tag{71}$$

Eq. (71) indicates that the velocity of the tip of the vector of the total angular momentum of a body with respect to a centre O is equal in magnitude and direction to the principal moment of the external forces with respect to that centre (*Resal's theorem*).

Thus, point B, and with it the axis of the gyroscope, will move in the direction of vector M_O. We find, therefore, that if a force is made to act on the axis of a rapidly spinning gyroscope, the axis starts moving not in the direction of the acting force but in the

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*) The forces F_k^e include the force of gravity and the reactions of the bearings in which the axles are mounted. When the gyroscope is motionless, the equation $\sum m_O(F_k^e) = 0$ enters as one of the equilibrium conditions. When the gyroscope spins about its axis of symmetry, the reactions have the same value as at rest (this is proved in § 169); consequently, the forces acting on a spinning gyroscope also satisfy the equation $\sum m_O(F_k^e) = 0$.

direction of the vector of the moment of the force with respect to the fixed point O of the gyroscope, i.e., perpendicular to the force (how the direction of vector $\boldsymbol{M}_O$ is determined was explained in § 42).

Note that if the gyroscope were made to spin in the opposite direction (see Fig. 342), vector $\boldsymbol{\omega}_1$ and with it vector $\boldsymbol{K}_O$ applied at point O would be directed downward, and the lower end of the gyroscope would move in the direction of $\boldsymbol{M}_O$, i.e., towards us, and the upper end pointing into the paper.

Another important result follows from Eq. (71). When the action of the force ends, $\boldsymbol{M}_O$, and v_B as well, vanish and the gyroscope axis stops. Thus, a gyroscope does not retain the motion imparted to it by the force. If the force is of short duration (an impact), the gyroscope axis will practically not change its orientation. This is the cause of the *stability* of the axis of a rapidly spinning gyroscope.

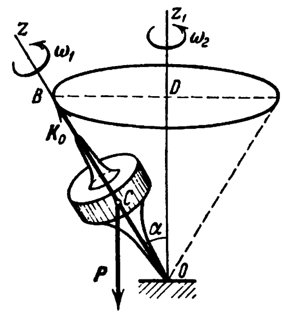

Fig. 343

(3) Regular Precession of a Heavy Gyroscope. Let us consider a gyroscope whose fixed point O is not coincident with its centre of gravity C (Fig. 343). In this case continuously acting on the axis of spin will be a force $\boldsymbol{P}$ which, as just shown, will deflect the axis Oz not downwards (not in the direction of angle α increasing) but in the direction of $\boldsymbol{m}_O(\boldsymbol{P})$, i.e., normal to plane Ozz_1. As a result, the axis of spin will turn about the vertical axis Oz_1, describing a conical surface. This motion of the axis of a gyroscope is called *precession*.

Let us find the angular velocity ω_2 of the precession. From Eq. (71), we should have $v_B = M_O$. Introducing the notation $OC = a$, we find that in this case $M_O = Pa \sin \alpha$. On the other hand, $v_B = \omega_2 \cdot BD = \omega_2 \cdot OB \sin \alpha = \omega_2 \cdot K_O \sin \alpha$, or, taking into account Eq. (70),

$$v_B = J_z \omega_1 \omega_2 \sin \alpha.$$

Consequently, the equation $v_B = M_O$ yields $J_z \omega_1 \omega_2 \sin \alpha = Pa \sin \alpha$, whence

$$\omega_2 = \frac{Pa}{J_z \omega_1}. \tag{72}$$

As ω_1 is large, the angular velocity of precession is small. As ω_1 decreases, ω_2 increases, a phenomenon which is familiar to anyone who has ever seen a spinning top.

The earth's axis also performs motion of precession, due to the fact that the earth is not an ideal sphere and also to the inclination of its axis, thanks to which the resultant of the attractive forces of the sun and the moon does not pass through the earth's centre of mass, producing some moments with respect to the centre. The rate of precession of the earth's axis (the time of one complete revolution) is approximately 26 000 years.

(4) **The Gyroscopic Effect.** Consider a rapidly spinning gyroscope supported by bearings A and A' in a gimbal ring which in turn can rotate with an angular velocity ω_2 ($\omega_2 \ll \omega_1$) about axis DD' (Fig. 344). Since under such conditions the gyroscope axis precesses, point B (the tip of vector $\boldsymbol{K}_O$), as in the previous case, will have an angular velocity $v_B = J_z\omega_1\omega_2 \sin\alpha$. From Eq. (71) we conclude that acting on the axis is a moment of magnitude

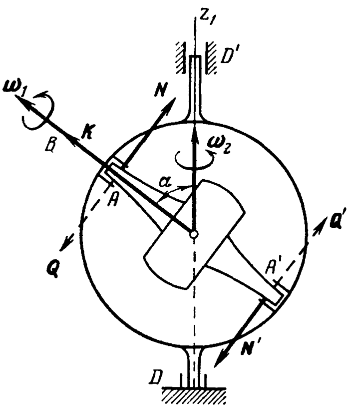

Fig. 344

$$M_O = v_B = J_z\omega_1\omega_2 \sin\alpha.$$

This moment is, evidently, created by forces $\boldsymbol{Q}$, $\boldsymbol{Q}'$ with which bearings A and A' act on the axis. As the centre of mass of the gyroscope is fixed it follows from the theorem of the motion of the centre of mass that the sum of the forces is zero; thus, these forces form a couple whose moment $\boldsymbol{M}_O$ must be directed the same as the velocity $\boldsymbol{v}_B$, i.e., upward (out of the paper in Fig. 344).

But then the axis of the gyroscope will also press on bearings A and A' with forces $\boldsymbol{N}$, $\boldsymbol{N}'$ equal in magnitude to forces $\boldsymbol{Q}$, $\boldsymbol{Q}'$, but oppositely directed.

The force couple ($\boldsymbol{N}$, $\boldsymbol{N}'$) is called a *gyroscopic couple*, and its moment, the *gyroscopic moment*. Since in magnitude $M_{gyr} = M_O$, then

$$M_{gyr} = J_2\omega_1\omega_2 \sin\alpha. \tag{73}$$

We deduce from this **Zhukovsky's Rule:** *If a forced precession is imparted to a rapidly spinning gyroscope, a couple of moment M_{gyr} will start to act on the bearings which will tend to move the axis of spin along the shortest path to set it parallel to the axis of precession, so that the directions of vectors ω_1 and ω_2 would coincide.*

Besides bringing forces to act on the bearings, the gyroscopic effect may cause the body to which the bearings are attached to move too if the motion is not opposed by constraints.

Consider the following example. If the rotor of a ship's turbine rotates with an angular velocity ω_1 (Fig. 345) and the ship turns with an angular velocity ω_2, gyroscopic forces $\boldsymbol{N}_1$ and $\boldsymbol{N}_2$ will be acting on bearings A and B, directed as shown in the diagram*). If $AB = l$ and the moment of inertia of the rotor is J_z, then, by Eq. (73),

$$M_{\text{gyr}} = Nl = J_z\omega_1\omega_2 \quad \text{and} \quad N = \frac{J_z\omega_1\omega_2}{l}.$$

These forces may reach a value of several tons and they must be taken into account in calculating the bearings. Through the bearings the gyroscopic forces are transmitted to the hull, and if the vessel is very light they can cause the stern or bow to "dip" during a turn. A similar phenomenon takes place when a propeller aircraft banks in a horizontal plane.

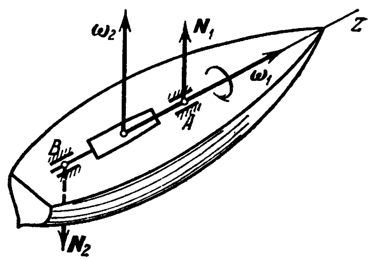

Fig. 345

The gyroscopic phenomena discussed in this section are used for gyroscopic stabilisers, navigation and other special instruments.

An example of a direct action gyroscopic stabiliser is the ship anti-rolling gyroscope. This is a heavy gyroscope (Fig. 346*a*) whose axis of spin AA_1 is mounted in a frame the axis of rotation DD_1 of which is attached to the ship's hull. When the ship starts to roll

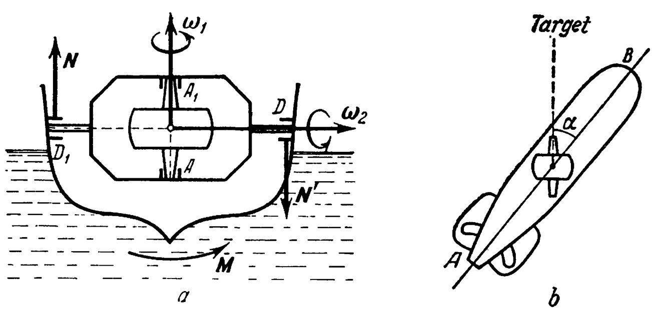

Fig. 346

and a moment M starts to act on it, a specially controlled motor begins to turn the frame with a certain angular velocity ω_2 (see diagram). As a result, a gyroscopic couple (N, N') of moment $J\omega_1\omega_2$ is brought to act on bearings D and D_1, which reduces the roll. When the moment M changes its direction, the motor will reverse the rotation of the frame and the couple (N, N') will also change its sense.

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*) Gyroscopic forces also appear in the bearings due to rolling and pitching of the ship. Their directions, of course, are different.

Another gyroscopic stabiliser (not a direct-action one, though) is a device which controls the motion of a torpedo in the horizontal plane (Fig. 346*b*). The stabilising element is a free gyroscope (see Fig. 341), whose axis is coincident with the axis of the torpedo, which is directed to the target at the launching moment. If the torpedo departs from its set course by an angle α, the gyroscope axis will continue, by virtue of the property of a free gyroscope, to point towards the target and will, therefore, turn through the same angle with respect to the axis of the torpedo. This turn actuates a governor which controls the steering mechanism. The rudder will turn correspondingly and the torpedo will return to its course. A similar idea is incorporated in automatic pilots, which sense any deflection of an aircraft from its set course and actuate the necessary steering assemblies.

To other gyroscopic navigation instruments belong the gyro-compass, the gyro-horizon, the turn-indicator, and others. They are of various design, but all of them are based on the properties of the gyroscope discussed here.

§ 158*. Motion of a Rigid Body About a Fixed Point and Motion of a Free Rigid Body

To write the differential equations of motion of a body that has a fixed point it is necessary to find the expressions for the total angular momentum $\boldsymbol{K}_O$ and for the kinetic energy T of the body.

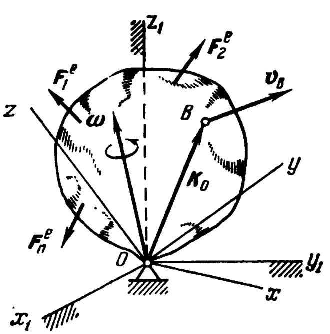

Fig. 347

(1) Angular Momentum of a Body Moving About a Fixed Point. Vector $\boldsymbol{K}_O$ can be determined by obtaining its projections on any set of three coordinate axes $Oxyz$. For the equations to take the simplest form, direct the coordinate axes, as in Fig. 347 below, along the *principal axes of inertia* of the body through point O (§ 133), which are rigidly connected with the body.

Commencing the computation with K_x, by analogy with Eq. (43′) in § 116 [or with Eqs. (52) in § 43], we have

$$m_x(m_k v_k) = m_k(y_k v_{kz} - z_k v_{ky}).$$

But by Euler's equations [§ 87, Eqs. (67′)],

$$v_{ky} = \omega_z x_k - \omega_x z_k, \quad v_{kz} = \omega_x y_k - \omega_y x_k,$$

where ω_x, ω_y, ω_z are the projections of the instantaneous angular velocity of the body on xyz axes, and x_k, y_k, z_k are the coordinates of the body's points.

Substitute these values of v_{ky} and v_{kz} into the preceding equation, noting that there is no need to compute the members with the products of the coordinates as xyz axes are the principle axes of inertia, and all the centrifugal moments of inertia with respect to them are zero, i.e., $\sum m_k x_k y_k = \sum m_k y_k z_k = 0$. Consequently, taking the common multiplier ω_x outside the parentheses, we obtain:

$$K_x = \sum m_x (m_k v_k) = [\sum m_k (y_k^2 + z_k^2)] \omega_x,$$

where the quantity in brackets is, according to Eqs. (3), the principal moment of inertia of the body with respect to axis Ox. With similar expressions for K_y and K_z, we finally obtain:

$$K_x = J_x \omega_x, \quad K_y = J_y \omega_y, \quad K_z = J_z \omega_z. \tag{74}$$

Eqs. (74) give the expressions for the projections of vector $\boldsymbol{K}_O$ on the body's *principal axes of inertia* through O.

It can readily be observed that, if $Oxyz$ are not the principal axes, Eqs. (74) take a much more complex form:

$$\left.\begin{aligned} K_x &= J_x\omega_x - J_{xy}\omega_y - J_{xz}\omega_z, \\ K_y &= -J_{xy}\omega_x + J_y\omega_y - J_{yz}\omega_z, \\ K_z &= -J_{xz}\omega_x - J_{yz}\omega_y + J_z\omega_z. \end{aligned}\right\} \tag{74'}$$

(2) Kinetic Energy of a Body Moving About a Fixed Point. As the motion of a body having a fixed point is compounded of a series of elementary rotations about the instantaneous axes of rotation OP through that point, its kinetic energy can be computed according to the formula $T = 0.5 J_{OP}\omega^2$, where ω is the instantaneous angular velocity of the body. The formula, however, is inconvenient as axis OP continuously changes its orientation and, consequently, the value of J_{OP} changes too. Let us find another formula for computing T, introducing instead of $\boldsymbol{\omega}$ its projections on the *principal axes of inertia of the body* through point O ($Oxyz$ in Fig. 347 below). By definition,

$$T = \frac{1}{2} \sum m_k v_k^2 = \frac{1}{2} \sum m_k (v_{kx}^2 + v_{ky}^2 + v_{kz}^2).$$

From Euler's equations [Eqs. (67′) in § 87], we have:

$$v_{kx} = \omega_y z_k - \omega_z y_k, \quad v_{ky} = \omega_z x_k - \omega_x z_k, \quad v_{kz} = \omega_x y_k - \omega_y x_k.$$

Substituting these values into the expression for T, and taking into account that, as $Oxyz$ are the principal axes and $\sum m_k x_k y_k = \sum m_k y_k z_k = \sum m_k z_k x_k = 0$, the members containing the product of the coordinates can be discarded, and after taking the common multiplier outside the parentheses we obtain:

$$T=\frac{1}{2}\left[\sum m_k(y_k^2+z_k^2)\omega_x^2+\sum m_k(z_k^2+x_k^2)\omega_y^2+\sum m_k(x_k^2+y_k^2)\omega_z^2\right].$$

From Eqs. (3), the sums in the right-hand side equals the respective axial moments of inertia. Consequently, the kinetic energy of a body moving about a fixed point O can be computed according to the formula

$$T=\frac{1}{2}(J_x\omega_x^2+J_y\omega_y^2+J_z\omega_z^2), \tag{75}$$

where ω_x, ω_y, ω_z are the projections of the instantaneous angular velocity of the body on the principal axes of inertia through O.

It will be readily observed that if $Oxyz$ are not the principal axes of inertia, Eq. (75) takes a more complex form:

$$T=\frac{1}{2}(J_x\omega_x^2+J_y\omega_y^2+J_z\omega_z^2-2J_{xy}\omega_x\omega_y-2J_{yz}\omega_y\omega_z-2J_{zx}\omega_z\omega_x). \tag{75'}$$

(3) The Dynamic Euler Equations. Let a system of given forces $\boldsymbol{F}_1^e$, $\boldsymbol{F}_2^e$, . . ., $\boldsymbol{F}_n^e$ be acting on a rigid body having a fixed point O (Fig. 347). Also acting on the body is the reaction $\boldsymbol{R}_O$ of the constraint (not shown in the drawing). To exclude this unknown force from the equations of moment we apply the principle of moments with respect to the centre O (§ 144), representing it in the form (71), i.e., Resal's theorem. Then, as $\boldsymbol{m}_O(\boldsymbol{R})=0$, we have:

$$\boldsymbol{v}_B=\boldsymbol{M}_O^e,$$

where $\boldsymbol{M}_O^e=\sum\boldsymbol{m}_O(\boldsymbol{F}_k^e)$ and $\boldsymbol{v}_B$ is the velocity of point B at the tip of vector $\boldsymbol{K}_O$ with respect to an inertial frame of reference $Ox_1y_1z_1$.

The motion can also be investigated with respect to that inertial frame, but to obtain the equations of motion in the simplest form we shall project the last equation on the principal axes of inertia of the body through point O, using them as a set of coordinate axes $Oxyz$ rigidly connected with the body. Then the expressions for the projections of vector $\boldsymbol{K}_O$ take the simple form given by Eqs. (74), and the moments of inertia J_x, J_y, J_z entering them are constants.

To compute the projections of the absolute velocity $\boldsymbol{v}_B$ on the moving axes, let us represent $\boldsymbol{v}_B$ as a sum of the relative (with respect to axes $Oxyz$) and transport velocities. Then Eq. (71) (Resal's theorem) takes the form

$$\boldsymbol{v}_B^{\text{rel}}+\boldsymbol{v}_B^{\text{tr}}=\boldsymbol{M}_O^e. \tag{76}$$

First we determine the equations of motion for the projection on axis x. From Eq. (76) we have:

$$v_{Bx}^{\text{rel}}+v_{Bx}^{\text{tr}}=M_x^e. \tag{76'}$$

The projection of the relative velocity on the moving axes is computed as though the axes were fixed, i.e., according to Eqs. (15) in § 64. Then, taking into account that the x coordinate of B equals the projection of vector $\overline{OB}$ or $\boldsymbol{K}_O$ on axis x, we have:

$$v_{Bx}^{\text{rel}}=\frac{d(\overline{OB})_x}{dt}=\frac{dK_x}{dt}=J_x\frac{d\omega_x}{dt},$$

since, according to Eqs. (74), $K_x = J_x\omega_x$, where $\boldsymbol{\omega}$ is the instantaneous angular velocity of the body and $J_x = \text{const}$.

Furthermore, $\boldsymbol{v}_B^{\text{tr}}$ is the velocity of point B, which coincides at the given instant with the tip of vector $\boldsymbol{K}_O$ and is rigidly connected with the axes $Oxyz$ (i.e., with the body). Consequently, by Eq. (67) of § 87, $\boldsymbol{v}_B^{\text{tr}} = \boldsymbol{\omega} \times \overline{OB} = \boldsymbol{\omega} \times \boldsymbol{K}_O$. But from vector algebra, if $\boldsymbol{c} = \boldsymbol{a} \times \boldsymbol{b}$, then $c_x = a_y b_z - a_z b_y$ [compare equations (49′) and (50) in § 42]. Consequently, taking into account Eqs. (74), we have:

$$v_{Bx}^{\text{tr}}=\omega_y K_z-\omega_z K_y=(J_z-J_y)\,\omega_y\omega_z.$$

Substituting the values for v_{Bx}^{rel} and v_{Bx}^{tr} into the left-hand member of Eq. (76′), we obtain $J_x\frac{d\omega_x}{dt}+(J_z-J_y)\,\omega_y\omega_z=M_x^e$. Similar expressions are obtained for the projections of Eq. (76) on the y and z axes. Finally, we obtain the following *differential equations of motion of a rigid body about a fixed point* in terms of the projections on the principal axes of inertia of the body through that point:

$$\left.\begin{aligned}J_x\frac{d\omega_x}{dt}+(J_z-J_y)\,\omega_y\omega_z&=M_x^e,\\ J_y\frac{d\omega_y}{dt}+(J_x-J_z)\,\omega_z\omega_x&=M_y^e,\\ J_z\frac{d\omega_z}{dt}+(J_y-J_x)\,\omega_x\omega_y&=M_z^e.\end{aligned}\right\}\qquad(77)$$

Equations (77) are called the *dynamic Euler equations.* If a body's position is given by the Euler angles φ, ψ, θ (§ 86), the principal problem of dynamics is: knowing M_x^e, M_y^e, M_z^e, determine the law of motion of the body, i.e., $\varphi = \varphi(t)$, $\psi = \psi(t)$, $\theta = \theta(t)$. To solve the problem, supplement Eqs. (77) with the kinematic Euler equations [see § 97, Eqs. (98)] giving the relationship between ω_x, ω_y, ω_z and angles φ, ψ, θ. The dynamic and kinematic Euler equations constitute a set of six nonlinear differential equations of the first order; integrating them is a complex mathematical problem. In § 157 we set forth the approximate theory of gyroscopic motion. Gyroscopic motion is precisely described by equations (77). Various approximate mathematical methods are usually employed in integrating these equations to solve specific problems.

One of Eqs. (77) can, if necessary, be replaced by the theorem of the change in kinetic energy. Formula (75) can also be used to write equations by the method described in § 177 (see Problem 176).

(4) Example. For a simple example of application of the equations obtained above consider the motion of a free gyroscope whose centre of gravity is fixed and on which no other force but gravity is acting [see § 157, item (1)]. In that case, $\boldsymbol{M}_O^e = 0$, and the principle of moments (§ 144) yields

$$\frac{d\boldsymbol{K}_O}{dt} = 0, \quad \text{or} \quad \boldsymbol{K}_O = \text{const.} \tag{a}$$

Thus, the direction of vector $\boldsymbol{K}_O$ is constant in an inertial frame of reference. Accordingly, to simplify the subsequent computations, direct the fixed axis Oz_1 along vector $\boldsymbol{K}_O$ as in Fig. 348. The other two axes, not shown in the drawing, can be directed arbitrarily. Draw the moving axes connected with the gyroscope so that axis Oz is along the gyroscope's axis of symmetry. Then $J_x = J_y$ and, as in our case $M_z^e = 0$, the last of Eqs. (77) yields $d\omega_z/dt = 0$, whence

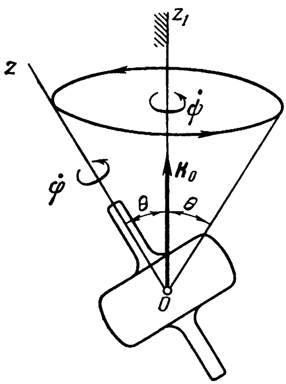

Fig. 348

$$\omega_z = \text{const.} \tag{b}$$

It follows, therefore, from Eqs. (74) that $K_z = J_z \omega_z = \text{const.}$ But at the same time, as can be seen in Fig. 348, $K_z = K_O \cos\theta$, where $\theta = \angle z_1 O z$ is the angle of nutation (see Fig. 200). As by equation (a) $K_O = \text{const.}$, we conclude that $\cos\theta = \text{const.}$, or

$$\theta = \text{const.} = \theta_0, \tag{c}$$

where θ_0 is the initial value of the angle of nutation.

Now multiply both sides of the first of Eqs. (77) by ω_x and of the second by ω_y, and add the respective members taking into account that in our case $M_x^e = M_y^e = 0$ and $J_x = J_y$. We thus obtain:

$$J_x \left(\omega_x \frac{d\omega_x}{dt} + \omega_y \frac{d\omega_y}{dt} \right) = 0.$$

Whence, integrating and dividing both sides by a constant multiplier, we obtain:

$$\omega_x^2 + \omega_y^2 = \text{const.}$$

Substituting for ω_x and ω_y their values from the kinematic Euler equations [§ 97, Eqs. (98)] and taking into account that $\theta = \text{const.}$

and $\dot{\theta} = 0$, we obtain:

$$\omega_x = \dot{\psi} \sin \theta \sin \varphi, \quad \omega_y = \dot{\psi} \sin \theta \cos \varphi,$$

whence

$$\omega_x^2 + \omega_y^2 = \dot{\psi}^2 \sin^2 \theta.$$

But, as proved, the left-hand member of the equation and $\sin \theta$ are constant. Consequently,

$$\dot{\psi} = \text{const.} = \dot{\psi}_0 \tag{d}$$

Finally, the last of the kinematic Euler equations yields $\omega_z = \dot{\varphi} + \dot{\psi} \cos \theta$. Here, as we found, ω_z, $\dot{\psi}$, and $\cos \theta$ are constants. Consequently,

$$\dot{\varphi} = \text{const.} = \dot{\varphi}_0 \tag{e}$$

Thus, whatever the initial conditions, the gyroscope revolves about its axis of symmetry with a constant angular velocity $\dot{\varphi}_0$; the axis itself rotates about the fixed axis Oz_1 with a constant angular velocity $\dot{\psi}_0$, describing a conic surface with a constant angle $2\theta_0$ at the apex (see Fig. 348). Such gyroscopic motion is called *regular precession.*

(5) Motion of a Free Rigid Body. We know that the motion of a free rigid body is compounded of a translation together with a pole, assumed in solving problems of dynamics to be at the body's centre of mass C, and of a rotation about the centre of mass as about a fixed point (§ 88). If external forces $\boldsymbol{F}_1^e, \boldsymbol{F}_2^e, \ldots, \boldsymbol{F}_n^e$ are acting on a body, the motion of the pole C is described by the theorem of the motion of the centre of mass, $M\boldsymbol{w}_C = \sum \boldsymbol{F}_k^e$, where M is the mass of the body. In terms of the projections on the fixed axes $Ox_1y_1z_1$ this equation yields

$$M\frac{d^2x_{1C}}{dt^2} = \sum F_{kx_1}^e, \quad M\frac{d^2y_{1C}}{dt^2} = \sum F_{ky_1}^e, \quad M\frac{d^2z_{1C}}{dt^2} = \sum F_{kz_1}^e \tag{78}$$

where x_{1C}, y_{1C}, and z_{1C} are the coordinates of the centre of mass.

But for motion about the centre of mass the principle of moments expressed by Eq. (40), yields, in terms of the projections on the principal central axes of inertia of the body, three equations coinciding in form with Eqs. (77). Thus, the set of differential equations (77), (78) describes the motion of a free rigid body (artillery projectile, aircraft missile, etc.).

Chapter 29

Applications of the General Theorems to the Theory of Impact

§ 159. The Fundamental Equation of the Theory of Impact

In the motion under the action of conventional forces treated hitherto, the velocities of the points of a body change continuously, i.e., to any infinitesimal time interval there corresponds an infinitesimal velocity increment. For, if we represent the linear impulse of any force $\boldsymbol{F}_k$ in a time interval τ in the form $\boldsymbol{F}_k^{\mathrm{av}}\tau$, where $\boldsymbol{F}_k^{\mathrm{av}}$ is the average value of this force in the time interval τ, the theorem of the change in the momentum of a particle acted upon by forces $\boldsymbol{F}_k$ gives

$$m(\boldsymbol{v}_1-\boldsymbol{v}_0)=\sum \boldsymbol{F}_k^{\mathrm{av}}\tau.$$

We see from this that when the time τ is infinitesimal (tends to zero), for conventional forces the velocity increment $\Delta \boldsymbol{v} = \boldsymbol{v}_1 - \boldsymbol{v}_0$ will also be infinitesimal (will tend to zero).

If, however, the acting forces include very large forces (of the order $1/\tau$), the velocity increment in the small time interval τ will become a finite quantity.

The phenomenon in which the velocities of the points of a body suffer a finite change in a very small time interval τ *is called impact.* We shall call the forces developed during impact *impulsive forces*, or *forces of impact* (denoted $\boldsymbol{F}_{\mathrm{imp}}$) and the very small time interval τ during which collision takes place, the *impact time*.

As forces of impact are very large and change within considerable limits in the impact time, the theory of impact considers not the forces themselves as a measure of the interaction of colliding bodies but their impulses. The impulse of a force of impact

$$\boldsymbol{S}_{\mathrm{imp}}=\int_0^{\tau}\boldsymbol{F}_{\mathrm{imp}}\,dt=\boldsymbol{F}_{\mathrm{imp}}^{\mathrm{av}}\tau$$

is a finite quantity. The impulses of other forces in the time interval τ will be very small and for all practical purposes can be neglected.

Denoting the velocity of a particle at the beginning of impact—the velocity of approach—by the symbol $\boldsymbol{v}$, and the velocity at the end of impact—the velocity of separation—by $\boldsymbol{u}$, Eq. (75) takes the form *)

$$m(\boldsymbol{u}-\boldsymbol{v})=\sum \boldsymbol{S}_k. \tag{79}$$

This equation states the **theorem of the change in the momentum of a particle during impact**: *The change in the linear momentum of a particle during impact is equal to the sum of the impulses of the forces of impact acting on the particle.* Eq. (79) is *the fundamental equation of the theory of impact* and it plays the same role in the theory of impact as the fundamental law of dynamics $m\boldsymbol{w}=\boldsymbol{F}$ does in studying motion under the action of non-impulsive forces.

Finally, the displacement of a particle during impact is equal to $v^{av}\tau$, a very small quantity which in practice can be neglected.

From the above discussion we conclude:

(1) the action of non-impulsive forces (the force of gravity, for example) in the time of impact can be neglected;

(2) the displacement of the points of a body in the time of impact can be neglected and the body regarded as motionless;

(3) the change in the velocities of the points of a body during impact is defined by the fundamental equation of the theory of impact (76).

§ 160. General Theorems of the Theory of Impact

The following theorems are used in investigating impact in a system of particles or bodies.

(1) The Theorem of the Change in the Linear Momentum of a System in Impact. Eq. (22) obtained § 139 has the same form in the case of impact, but as in impact the impulses of ordinary forces are neglected, only the impulses of the forces of impact remain in the right-hand side of the equation. Consequently, in impact

$$\boldsymbol{Q}_1-\boldsymbol{Q}_0=\sum \boldsymbol{S}_k^e, \tag{80}$$

i.e., *the change in the momentum of a system during impact is equal to the total impulse of all the external forces of impact acting on the system.*

For the projections on any coordinate axis x, Eq. (80) gives:

$$Q_{1x}-Q_{0x}=\sum S_{kx}^e. \tag{80'}$$

If the geometrical sum of the impulses of the external forces of impact is zero, then, as can be seen from Eq. (80), the linear mo-

*) We shall denote the impulse of a force of impact simply by $\boldsymbol{S}$, since other impulses are not considered in the theory of impact.

mentum of the system will not change during impact. Consequently, the internal impulses cannot change the linear momentum of the system as a whole.

(2) The Theorem of the Change in the Total Angular Momentum of a System (the Principle of Moments) During Impact. In the case of impact the principle of moments takes a somewhat different form than that obtained in § 144. That is because during the time of impact the points of a body undergo no displacement. Consider a system of n material particles. Denote the resultant impulse of all the external forces of impact acting on a particle of mass m_k by the symbol $\boldsymbol{S}_k^e$, and the resultant impulse of all the internal forces of impact acting on that particle by the symbol $\boldsymbol{S}_k^i$. Then, from Eq. (79), $m_k(\boldsymbol{u}_k - \boldsymbol{v}_k) = \boldsymbol{S}_k^e + \boldsymbol{S}_k^i$, or

$$m_k \boldsymbol{u}_k = m_k \boldsymbol{v}_k + \boldsymbol{S}_k^e + \boldsymbol{S}_k^i.$$

The vectors in this equation are applied to a particle which remains motionless during the impact. Hence, taking the moments of all the vectors with respect to any centre O, we obtain by Varignon's theorem, which is valid for any vector quantities:

$$\boldsymbol{m}_O(m_k \boldsymbol{u}_k) = \boldsymbol{m}_O(m_k \boldsymbol{v}_k) + \boldsymbol{m}_O(\boldsymbol{S}_k^e) + \boldsymbol{m}_O(\boldsymbol{S}_k^i).$$

Writing such equations for all the particles of a system and adding them, we obtain:

$$\sum \boldsymbol{m}_O(m_k \boldsymbol{u}_k) - \sum \boldsymbol{m}_O(m_k \boldsymbol{v}_k) = \sum \boldsymbol{m}_O(\boldsymbol{S}_k^e) + \sum \boldsymbol{m}_O(\boldsymbol{S}_k^i).$$

The summations in the left side represent the total angular momentum of the system with respect to O at the end and the beginning of the impact, which we shall denote by $\boldsymbol{K}_1$ and $\boldsymbol{K}_0$, respectively. From the property of internal forces, the sum of the moments of the internal impulses in the right side of the equation is zero. Thus, we finally have:

$$\boldsymbol{K}_1 - \boldsymbol{K}_0 = \sum \boldsymbol{m}_O(\boldsymbol{S}_k^e), \tag{81}$$

i.e., *the change during impact of the total angular momentum of a system with respect to any centre is equal to the sum of the moment a of all the external impulses acting on the system taken with respect to the same centre.*

For the projections on any axis x, Eq. (81) gives

$$K_{1x} - K_{0x} = \sum m_x(\boldsymbol{S}_k^e). \tag{81'}$$

It follows from these equations that if the sum of the moments of all external impulses with respect to any centre (or axis) is zero, then the total angular momentum of the system with respect to the centre (or axis) will not change during the impact. Consequently, internal impulses cannot change the total angular momentum of a system.

The theorem of the change in kinetic energy is not used to solve the fundamental problem of dynamics in the theory of impact, as all the particles of a body are considered motionless during impact, and instead of the forces of impact their impulses are considered. Therefore, it is not possible to calculate directly (from the force and displacement) the work done by impulsive forces. Later on we shall consider only the question of determining the loss in the kinetic energy of bodies during impact (§ 164).

§ 161. Coefficient of Restitution

The magnitude of the impulse when two bodies collide depends not only on their masses and velocities before impact, but also on their elastic or plastic properties, which are characterised by the *coefficient of restitution.*

Consider a sphere falling onto a fixed horizontal plate (Fig. 349). There are two stages in the direct impact that will take place between them. During the first stage the velocities of the particles of the sphere, which at the beginning of the impact were v (assuming the sphere to be in translational motion) drop to zero. The sphere deforms (assuming the plate to be rigid) and its total initial kinetic energy $\frac{1}{2}Mv^2$ turns into the internal potential energy of deformation. During the second stage of the impact the internal elastic forces of the sphere work to restore its shape and the internal potential energy turns into the kinetic energy of motion of the particles of the sphere. At the end of the impact the velocities of the particles will be u and the kinetic energy of the sphere $\frac{1}{2}Mu^2$. Actually, however, the mechanical energy of the sphere is not restored completely, as part of it is spent on giving the sphere some residual deformation and on heating it. Therefore, u is always less than v.

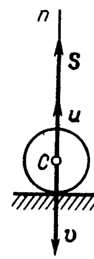

Fig. 349

The coefficient of restitution k for direct impact of a body against a fixed obstacle is equal to the ratio of the magnitudes of the velocity of separation to the velocity of approach:

$$k=\frac{u}{v}. \tag{82}$$

The coefficient of restitution for different bodies is determined experimentally. Experiments show that for changes of the velocity v within small limits the value of k depends only on the material of the colliding bodies.

The limiting values of k are $k = 1$ for *perfectly elastic impact* in which the mechanical energy of a body is completely restored after impact, and $k = 0$ for *perfectly inelastic impact* when the impact ends with the end of the period of deformation and the mechanical energy of a body is lost completely in deformation and heating.

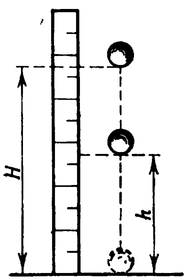

Fig. 350

The coefficient of restitution is determined experimentally by investigating the free fall of a sphere from a known height H to a plate. The height of the rebound h is then measured on a vertical rule (Fig. 350). From Galileo's formula,

$$v = \sqrt{2gH} \text{ and } u = \sqrt{2gh},$$

whence

$$k = \frac{u}{v} = \sqrt{\frac{h}{H}}.$$

Coefficients of restitution for some bodies with a velocity of approach in the order of 3 m/s are given in the table below.

Wood on wood	$k = 1/2$
Steel on steel	$k = 5/9$
Ivory on ivory	$k = 8/9$
Glass on glass	$k = 15/16$

§ 162. Impact of a Body Against a Fixed Obstacle

Consider a body (sphere) of mass M hitting a fixed plate The impulsive force acting on the body will be the reaction of the plate. Let us denote by $\boldsymbol{S}$ the impulse of this force during the impact. If the normal to the surface of the body at its point of contact with the plate passes through the centre of mass of the body (for a sphere this is always the case) the impact is called *central* impact. If the velocity of approach $\boldsymbol{v}$ of the centre of mass of the body is directed along the normal n to the plate, the impact is called *direct* impact; otherwise it is oblique impact.

(1) Direct Impact. Writing Eq. (80′) in terms of the projections on the normal n (see Fig. 349), and taking into account that $\boldsymbol{Q}_0 = M\boldsymbol{v}$ and $\boldsymbol{Q}_1 = M\boldsymbol{u}$, we obtain:

$$M(u_n - v_n) = S_n.$$

But in direct impact $u_n = u$, $v_n = -v$, and $S_n = S$. Consequently,

$$M(u + v) = S.$$

The second equation necessary to solve problems is given by Eq. (82): $u = kv$.

Knowing M, v, and k, we can obtain from these equations the unknown quantities u and S. We have:

$$S = M(k+1)v.$$

We see that the greater the coefficient of restitution k, the greater the impulse in impact is. This dependence between S and k was pointed out in § 161.

In order to determine the average value of the impulsive force (the reaction) we must also know the time of impact τ, which can be found experimentally.

Example. When a steel sphere of weight $p = 1$ kgf falls from a heigth $H = 3$ m on a steel plate ($k = 5/9$), we have $v = \sqrt{2gH} \approx$ ≈ 7.7 m/s, and $u = kv \approx 4.3$ m/s. The impulse of impact will be

$$S = \frac{p}{g} v(1+k) \approx 1.2 \text{ kgf-s}.$$

If the time of impact $\tau = 0.0005$ s, the average value of the impulse reaction will be

$$N_{\text{imp}}^{\text{av}} = \frac{S}{\tau} = 2\,400 \text{ kgf}.$$

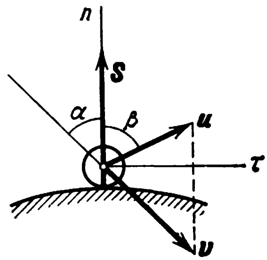

Fig. 351

(2) Oblique impact. Let the velocity of approach $\boldsymbol{v}$ of the centre of mass of a body make an angle α with the normal to a plate, and the velocity of separation $\boldsymbol{u}$ an angle β (Fig. 351). Then Eq. (80) in terms of the projections on the tangent τ and the normal $\boldsymbol{n}$ gives

$$M(u_\tau - v_\tau) = 0, \quad M(u_n - v_n) = S.$$

The coefficient of restitution in this case is equal to the ratio of the magnitudes of $|u_n|$ and $|v_n|$, as impact acts only along the normal to the surface (we neglect friction). Then, taking the signs into account, we obtain $u_n = -kv_n$, and finally we have:

$$u_\tau = v_\tau, \quad u_n = -kv_n, \quad S = M|v_n|(1+k).$$

These equations enable us to find the magnitude and direction of the velocity of separation and the impulse of the forces of impact if the quantities M, v, α, and k are known. In particular, noting that $v_\tau = |v_n| \tan\alpha$ and $u_\tau = |u_n| \tan\beta$, from the first equation we obtain $|u_n| \tan\beta = |v_n| \tan\alpha$, whence

$$k = \frac{|u_n|}{|v_n|} = \frac{\tan\alpha}{\tan\beta}.$$

Thus, in oblique impact the coefficient of restitution equals the ratio of the tangent of the angle of approach to the tangent of the angle of rebound. As $k < 1$, $\alpha < \beta$, i.e., the angle of rebound is always smaller than the angle of approach.

§ 163. Direct Central Impact of Two Bodies (Impact of Spheres)

Impact of two colliding bodies is said to be *direct* and *central* when the common normal to the surfaces of the bodies through their point of contact passes through their centres of mass and when the approach velocities of the centres of mass are directed along this common normal. An example of direct central impact is the collision of two homogeneous balls whose centres moved collinearly before impact.

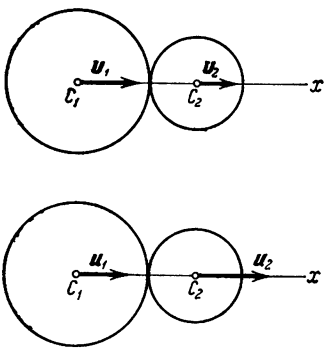

Fig. 352

Let the masses of the colliding bodies be M_1 and M_2, their approach velocities be v_1 and v_2, and their velocities of separation be u_1 and u_2. Draw through their centres of mass C_1 and C_2 a coordinate axis C_1x, always directed from C_1 to C_2 (Fig. 352). For a collision to take place, we must have $v_{1x} > v_{2x}$ (otherwise the first body will never catch up with the second); furthermore we shall have $u_{1x} \leqslant u_{2x}$, since a body hitting another body can never overtake it.

Assuming that M_1, M_2, v_{1x}, v_{2x}, and k are known, let us find u_{1x} and u_{2x}. For this let us apply the theorem of the change in linear momentum to the colliding bodies. If we consider them as a single system, the forces of impact will be internal, and $\sum S_k^e = 0$. As a result, Eq. (80′) gives $Q_{1x} = Q_{0x}$, or

$$M_1u_{1x} + M_2u_{2x} = M_1v_{1x} + M_2v_{2x}. \tag{83}$$

The second equation is obtained from the expression for the coefficient of restitution. When two bodies collide, the impulses of the forces of impact depend only on the relative velocity of the bodies, i.e., on the difference $v_{1x} - v_{2x}$. Therefore, for two colliding bodies, taking into account that always $v_{1x} > v_{2x}$, and $u_{1x} \leqslant u_{2x}$, we obtain:

$$k = \left| \frac{u_{1x} - u_{2x}}{v_{1x} - v_{2x}} \right| = -\frac{u_{1x} - u_{2x}}{v_{1x} - v_{2x}}, \tag{84}$$

or

$$u_{1x} - u_{2x} = -k(v_{1x} - v_{2x}). \tag{84′}$$

The set of equations (83), (84) makes it possible to solve the problem. The impulse of the forces of impact can be found by writing

Eq. (80′) for one of the bodies, the first, for instance. Then

$$S_{1x}=M_1(u_{1x}-v_{1x}),\quad S_{2x}=-S_{1x}. \tag{85}$$

Let us consider two limiting cases.

(a) **Perfectly inelastic impact** ($k=0$). In this case, from Eqs. (84) and (83) we find

$$u_{1x}=u_{2x}=\frac{M_1v_{1x}+M_2v_{2x}}{M_1+M_2}. \tag{86}$$

After impact both bodies move with the same velocity. The impulse of the forces of impact acting on the bodies is

$$S_{2x}=-S_{1x}=\frac{M_1M_2}{M_1+M_2}(v_{1x}-v_{2x}).$$

(b) **Perfectly elastic impact** ($k=1$). In this case, from Eqs. (83) and (84) we obtain

$$\left.\begin{aligned} u_{1x}&=v_{1x}-\frac{2M_2}{M_1+M_2}(v_{1x}-v_{2x}),\\ u_{2x}&=v_{2x}+\frac{2M_1}{M_1+M_2}(v_{1x}-v_{2x}). \end{aligned}\right\} \tag{87}$$

The impulse of the forces of impact is

$$S_{2x}=-S_{1x}=\frac{2M_1M_2}{M_1+M_2}(v_{x1}-v_{2x}).$$

We see that for perfectly elastic impact the impulse is double the impulse in perfectly inelastic impact.

In the special case when $M_1=M_2$ we obtain from Eqs. (87) $u_{1x}=v_{2x}$, $u_{2x}=v_{1x}$. Thus, in perfectly elastic impact, two bodies of equal mass exchange their velocities.

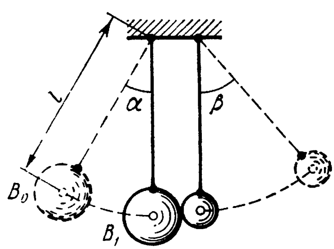

Fig. 353

Problem 149. Two balls of masses M_1 and M_2 are suspended as shown in Fig. 353. The first ball is pulled back through an angle α and let free from rest. After impact, the second ball swings away through an angle β. Find the coefficient of restitution for the balls.

Solution. The conditions of the problem make it possible to find the velocity of approach v_1 of the centre of the first ball and the velocity of separation u_2 of the centre of the second ball. From the theorem of the change in kinetic energy in a displacement B_0B_1, we find that for the first ball

$$\frac{1}{2}M_1v_1^2=P_1h=M_1gl(1-\cos\alpha),$$

where l is the distance of the centre of the ball from the point of suspension. Hence,

$$v_1 = 2\sqrt{gl}\sin\frac{\alpha}{2}.$$

Similarly, we find that

$$u_2 = 2\sqrt{gl}\sin\frac{\beta}{2}.$$

Since in our case $v_2 = 0$, Eqs. (83) and (84) give:

$$M_1u_{1x} + M_2u_{2x} = M_1v_{1x}, \quad u_{2x} - u_{1x} = kv_{1x}.$$

Eliminating u_{1x} and noting that $v_{1x} = v_1$ and $u_{2x} = u_2$, we obtain:

$$M_1v_1(1 + k) = (M_1 + M_2)u_2,$$

and finally we have:

$$k = \frac{(M_1 + M_2)u_2}{M_1v_1} - 1 = \frac{(M_1 + M_2)\sin\frac{1}{2}\beta}{M_1\sin\frac{1}{2}\alpha} - 1.$$

§ 164. Loss of Kinetic Energy in Perfectly Inelastic Impact. Carnot's Theorem

It follows from the reasoning in § 161 that in inelastic impact the colliding bodies lose kinetic energy. This loss is greater in the case of perfectly inelastic impact. Let us calculate the amount of kinetic energy lost by a system of two bodies in perfectly inelastic impact.

Considering the colliding bodies to be in translational motion and denoting their common velocity after impact by $\boldsymbol{u}$, we obtain for the kinetic energy of the system at the beginning and the end of impact:

$$T_0 = \frac{1}{2}(M_1v_{1x}^2 + M_2v_{2x}^2), \quad T_1 = \frac{1}{2}(M_1 + M_2)u_x^2. \tag{88}$$

The kinetic energy lost in impact will be $T_0 - T_1$. Let us write this difference in the form

$$T_0 - T_1 = T_0 - 2T_1 + T_1. \tag{89}$$

As, from Eq. (86),

$$(M_1 + M_2)u_x = M_1v_{1x} + M_2v_{2x},$$

then

$$2T_1 - (M_1 + M_2)u_x^2 = (M_1v_{1x} + M_2v_{2x})u_x. \tag{90}$$

Substituting into the right side of Eq. (89) the expressions for T_0 and T_1 from formulas (88), and for $2T_1$ the right side of Eq. (90), we obtain:

$$T_0 - T_1 = \frac{1}{2}(M_1 v_{1x}^2 + M_2 v_{2x}^2 - 2M_1 v_{1x} u_x - - 2M_2 v_{2x} u_x + M_1 u_x^2 + M_2 u_x^2),$$

or

$$T_0 - T_1 = \frac{1}{2} M_1 (v_{1x} - u_x)^2 + \frac{1}{2} M_2 (v_{2x} - u_x)^2. \qquad (91)$$

The differences $(v_{1x} - u_x)$ and $(v_{2x} - u_x)$ give the velocity lost by each body as a result of the impact. From Eq. (91) follows **Carnot's***) **theorem**: *The kinetic energy lost by a system of bodies in a perfectly inelastic impact is equal to the kinetic energy the system would have had if its bodies moved with the lost velocities.*

If an impact is not perfectly inelastic ($k \neq 0$), we can similarly establish that the kinetic energy lost by a system of two bodies is given by the equation

$$T_0 - T_1 = \frac{1-k}{1+k}\left[\frac{1}{2} M_1 (v_{1x} - u_{1x})^2 + \frac{1}{2} M_2 (v_{2x} - u_{2x})^2\right]. \qquad (91')$$

Let us consider the special case of collision with a body initially at rest. In this case $v_2 = 0$ and

$$T_0 = \frac{1}{2} M_1 v_1^2, \quad u = \frac{M_1 v_1}{M_1 + M_2}.$$

Then

$$T_1 = \frac{1}{2}(M_1 + M_2) u^2 = \frac{1}{2} \frac{M_1^2 v_1^2}{M_1 + M_2} = \frac{M_1}{M_1 + M_2} \frac{M_1 v_1^2}{2},$$

or

$$T_1 = \frac{M_1}{M_1 + M_2} T_0. \qquad (92)$$

Eq. (92) gives the energy left in a system after impact. Let us consider two interesting extremes.

(a) The mass of the hitting body is much greater than the mass of the hit body ($M_1 \gg M_2$). In this case we may consider $M_1 + M_2 \approx M_1$, and Eq. (92) gives $T_1 \approx T_0$. Thus, even if impact is perfectly inelastic, there is practically no loss of kinetic energy and the system moves after impact with practically the same kinetic energy it had at the beginning of the impact.

In practice this is of importance, for example, in hammering nails, driving piles, etc. Clearly, the mass of the hammer should be much greater than the mass of the nail (Fig. 354*a*).

*) Lazare Carnot (1753-1823), an outstanding French mathematician and a prominent public figure in the French Revolution.

(b) The mass of hit body is much greater than the mass of the hitting body ($M_2 \gg M_1$). In this case we may consider $\frac{M_1}{M_1+M_2} \approx \approx 0$, and Eq. (92) gives $T_1 \approx 0$, i.e., practically all the kinetic

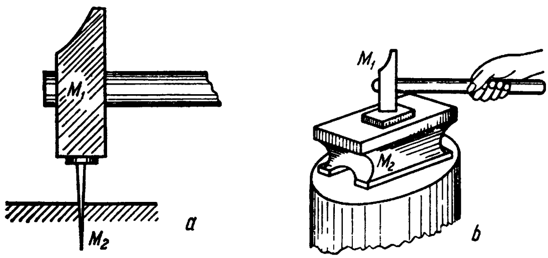

Fig. 354

energy is lost on deformation and at the end of impact the bodies may be considered motionless.

In practice this is of importance in forging, riveting, etc. In these cases, therefore, the combined mass of forging and anvil (or rivet and support) should be much greater than the mass of the hammer (Fig. 354*b*).

§ 165*. Impact with a Rotating Body

Consider a body rotating about an axis z (Fig. 355). Let an impact impulse $\boldsymbol{S}$ be applied at any instant to the body. Then, from Eq. (81′), we have:

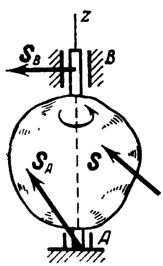

Fig. 355

$$K_{1z} - K_{0z} = m_z(\boldsymbol{S}),$$

as the moments with respect to axis z of the impulse reactions $\boldsymbol{S}_A$ and $\boldsymbol{S}_B$ in the bearings are zero. If at the beginning of impact the body had an angular velocity ω_0, and at the end of impact its angular velocity became ω_1, then $K_{0z} = J_z\omega_0$ and $K_{1z} = = J_z\omega_1$, and we obtain:

$$J_z(\omega_1 - \omega_0) = m_z(\boldsymbol{S}) \qquad (93)$$

or

$$\omega_1 = \omega_0 + \frac{m_z(\boldsymbol{S})}{J_z}. \qquad (93')$$

Eq. (93) gives the change in the angular velocity of a body due to impact. It follows from the equation that *the change in the angular velocity of a body during impact is equal to the ratio of the moment of the impulse of impact to the moment of inertia of the body, both moments taken with respect to the axis of rotation.*

Impulsive Reactions. Let us determine the impulsive reactions of bearings A and B in Fig. 356 on impact. Draw axes $Axyz$ so that the body's centre of mass C lies in plane Ayz (Fig. 356*a*) and denote the required impulsive reactions in terms of their components along the axes. Let $AB = b$ and the distance of point C from axis Az be a. Write equations (80′) in terms of the projections on all three axes and equations (81′) in terms of the projections on Ax and Ay [the equation for the projection on Az was already used in obtaining Eq. (93)]. As during the time of impact the body undergoes no displacement, vectors $\boldsymbol{v}_C$ and $\boldsymbol{u}_C$ are parallel to axis Ax; consequently,

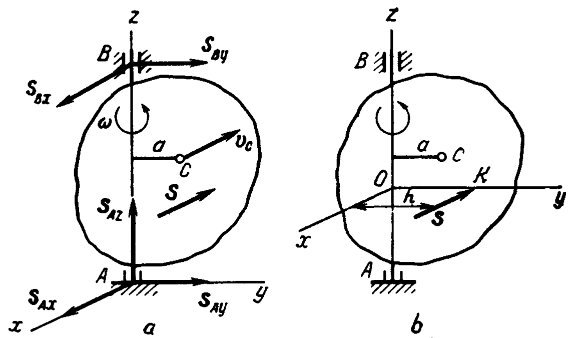

Fig. 356

$Q_{0x} = -Mv_C = -Ma\omega_0$, $Q_{1x} = -Ma\omega_1$, and $Q_y = Q_z = 0$. Using Eqs. (34) from § 143 in writing equations (81′), we obtain:

$$\left.\begin{aligned} &-Ma(\omega_1-\omega_0)=S_{Ax}+S_{Bx}+S_x,\\ &0=S_{Ay}+S_{By}+S_y, \quad 0=S_{Az}+S_z,\\ &-J_{xz}(\omega_1-\omega_0)=-S_{By}b+m_x(\boldsymbol{S}),\\ &-J_{yz}(\omega_1-\omega_0)=S_{Bx}b+m_y(\boldsymbol{S}). \end{aligned}\right\} \tag{94}$$

Equations (94) are used to determine the unknown impulsive reactions S_{Ax}, S_{Ay}, S_{Az}, S_{Bx}, S_{By}. The difference $\omega_1 - \omega_0$ is found from Eq. (93).

Centre of Impact. Impulsive reactions in impact are highly undesirable, as they result in the increased wear and tear, if not destruction, of structural elements (bearings, shafts, etc.). Let us see whether impact with a body mounted on a shaft is possible without the appearance of impulsive reactions in bearings A and B. For this let us determine the conditions that satisfy Eqs. (94), assuming $\boldsymbol{S}_A = \boldsymbol{S}_B = 0$. In that case the second and third of Eqs. (94) take the form $\boldsymbol{S}_y = 0$ and $S_z = 0$. To satisfy those equations the impulse $\boldsymbol{S}$ must be directed perpendicular to plane Ayz, i.e., under the conditions assumed, to the plane through the axis of rotation and centre of mass. Consider the direction of $\mathbf{S}$ to be the one in Fig. 356*b*. As at $\mathbf{S}_A = \mathbf{S}_B = 0$ the form of the system (94) does not depend on the location of the origin of the coordinate system on axis Az, to simplify the subsequent equations draw plane Oxy so that the impulse $\mathbf{S}$ lies in it (Fig. 356*b*). Then $m_x(\mathbf{S}) = m_y(\mathbf{S}) = 0$, and the last two of Eqs. (94) yield $J_{xz} = J_{yz} = 0$. This means (see § 133) that plane Oxy, in which the impulse $\mathbf{S}$ lies, must pass through a point O with respect to which axis z is the body's principal axis of inertia; in particular, as shown in § 133, conditions $J_{xz} = J_{yz} = 0$ are satisfied if plane Oxy is the body's plane of symmetry.

Referring, finally, to the first of Eqs. (94), as $\mathbf{S}_A = \mathbf{S}_B = 0$ and $S_x = -S$ (see Fig. 356*b*), it takes the form $Ma(\omega_1 - \omega_0) = S$. Simultaneously, as in our case $m_z(\mathbf{S}) = Sh$, Eq. (93) yields $J_z(\omega_1 - \omega_0) = Sh$. Cancelling out the difference, we obtain:

$$h = \frac{J_z}{Ma}. \tag{95}$$

Eq. (95) defines the distance h from the axis at which the impulse of impact must be applied.

Thus, in order to prevent the generation of impulsive reactions in the points of support of an axis z fixed in a body, the following conditions are necessary:

(1) the impact impulse must lie in a plane Oxy perpendicular to axis z through a point O of the body with respect to which axis z is the principal axis of inertia (in particular, Oxy may be the body's plane of symmetry);

(2) the impact must be directed perpendicular to the plane through the axis of rotation z and the centre of mass C of the body;

(3) the impulse of impact must be applied at a distance $h = J_z/Ma$ from the axis (on the same side of the axis as the centre of mass).

The point K through which the impulse will pass without causing impulsive reactions in the points of support of the axis is called the *centre of impact*.

Note that, from Eq. (95), the centre of impact coincides with the centre of percussion of a compound pendulum. Therefore, as was shown in § 155, $h > a$, i.e., the distance from the axis to the centre of impact is greater than to the centre of mass. If the axis of rotation passes through the centre of mass, then $a = 0$ and $h = \infty$. In this case the centre of impact is at infinity, and any impact will be transmitted to the axis.

Applications of the above results are illustrated by the following examples.

(1) In designing a pivoted gunlock hammer (see Problem 150) or a pendulum impact-testing machine the axis of rotation should

be so chosen that the point of impact should coincide with the centre of impact with respect to the axis.

(2) In working with a hammer, you should hold the handle so that the hitting end would be the centre of impact with respect to your hand. Failure to do this results in a characteristic "sting".

(3) In order not to "sting" your hand when you hit something with a stick (Fig. 357) the blow should be dealt with the point which is the centre of impact with respect to your hand. If we consider the stick to be a homogeneous rod of length l and the axis of rotation to be at the end in your hand, then $a = l/2$, $J_z = Ml^2/3$, and $h = J_z/Ma = 2l/3$.

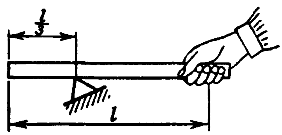

Fig. 357

Thus, in the case shown in Fig. 357, the blow should be dealt with a point of the stick two-thirds of the total distance away from the hand, or one-third of the distance from the opposite end.

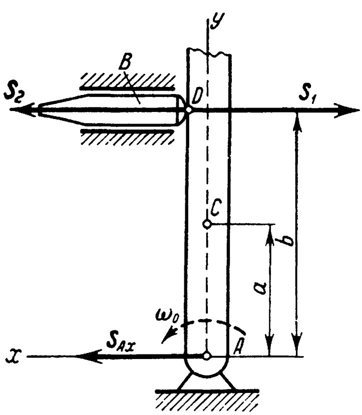

Fig. 358

Problem 150. At the beginning of impact with the firing pin B of a gun-lock, a pivoted hammer AD (Fig. 358) has an angular velocity ω_0. Determine the acceleration of the firing pin at the end of impact and the impulsive pressure on axis A, if the masses M and m of the hammer and firing pin, the moment of inertia J_A of the hammer with respect to axis A, and the distances a and b are all known. Point C is the centre of mass of the hammer.

Solution. Let us denote the impulses acting on the hammer and the firing pin during impact by S_1 and S_2. Then, taking into account that $S_1 = S_2 = S$ and $v_B = 0$, we obtain for the hammer [by Eq. (93)] and for the firing pin [by Eq. (80')]:

$$J_A(\omega_1 - \omega_0) = -Sb, \quad mu_B = S. \tag{a}$$

The moment Sb is taken with a minus sign because it is directed opposite to the rotation of the hammer. Moreover, since for point D of the hammer $v_D = \omega_0 b$ and $u_D = \omega_1 b$ (v_D is the velocity of approach, and u_D the velocity of separation), equation (84), which gives the coefficient of restitution for direct impact, gives

$$u_D - u_B = -k(v_D - v_B),$$

or

$$\omega_1 b - u_B = -k\omega_0 b.$$

Substituting into this equation the expression of ω_1 and S from equation (a), we obtain the velocity of the firing pin at the end of impact:

$$u_B = \frac{J_A b\,(1+k)}{J_A + mb^2}\,\omega_0.$$

To determine the impulsive reaction S_A of the axis on the hammer, we write Eq. (80) in terms of the projections on the axes Ax and Ay. Taking into account that

$$Q_{0x} = Mv_{Cx} = M\omega_0 a, \quad Q_{1x} = Mu_{Cx} = M\omega_1 a,$$

we obtain:

$$Ma\,(\omega_1 - \omega_0) = -S + S_{Ax}, \quad S_{Ay} = 0. \qquad \text{(b)}$$

But from equations (a)

$$S = mu_B, \quad \omega_1 - \omega_0 = -\frac{mb}{J_A}\,u_B.$$

Substituting these values into equation (b) and replacing u_B by its expression, we obtain finally:

$$S_{Ax} = \frac{J_A - Mab}{J_A + mb^2}\,mb\,(1+k)\,\omega_0.$$

We see from this that at $b = \frac{J_A}{Ma}$, i.e., when point D is the centre of impact, $S_A = 0$. When $Mab < J_A$, we have $S_{Ax} > 0$, i.e., the impulsive pressure on the hammer is directed to the left and the pressure on the axis to the right. At $Mab > J_A$, the pressure on the axis is directed to the left.

Chapter 30
D'Alembert's Principle. Forces Acting on the Axis of a Rotating Body

§ 166. D'Alembert's Principle

All the methods of solving the problems of dynamics examined up till now were based on equations derived either directly from Newton's laws or from the general theorems, which are corollaries of those laws. However, the equations of motion or equilibrium conditions of a mechanical system can also be obtained on the basis of other general propositions called the *principles of mechanics*. We shall see that in many cases application of those principles offers better methods of problem solutions. In this chapter we shall examine one of the general principles of mechanics known as *D'Alembert's principle*.

Let there be a system of n material particles. Selecting any particle of mass m_k, assume it to be acted upon by external and internal forces $\boldsymbol{F}_k^{\text{ext}}$ and $\boldsymbol{F}_k^{\text{int}}$ *) (which include both active forces and the reactions of constraints), which impart it an acceleration $\boldsymbol{w}_k$ with respect to an inertial reference frame.

Let us introduce the quantity

$$\boldsymbol{F}_k^{\text{i}} = -m_k \boldsymbol{w}_k, \tag{96}$$

with the dimension of force. The vector quantity equal in magnitude to the product of the particle's mass and acceleration and directed in the opposite sense of the acceleration is called the *force of inertia* of that particle (sometimes the *D'Alembert inertia force*).

Motion of a particle, we then find, satisfies the following **D'Alembert's principle for a material particle:** *If, at any moment of time, to the effective forces* $\boldsymbol{F}_k^{\text{ext}}$ and $\boldsymbol{F}_k^{\text{int}}$ *acting on the particle is added the inertia force* $\boldsymbol{F}_k^{\text{i}}$, *the resultant force system will be in equilibrium*, i.e.,

$$\boldsymbol{F}_k^{\text{ext}} + \boldsymbol{F}_k^{\text{int}} + \boldsymbol{F}_k^{\text{i}} = 0. \tag{97}$$

*) In this chapter we introduce the notation $\boldsymbol{F}^{\text{int}}$ (and $\boldsymbol{F}^{\text{ext}}$) for internal and external forces to avoid confusion between internal forces and inertia forces, for which the notation $\boldsymbol{F}^{\text{i}}$ is retained.

It will be readily observed that D'Alembert's principle is equivalent to Newton's second law, and vice versa. For Newton's second law gives for this particle $m_k\boldsymbol{w}_k = \boldsymbol{F}_k^{\text{ext}} + \boldsymbol{F}_k^{\text{int}}$. Transferring $m_k\boldsymbol{w}_k$ to the right-hand side of the equation, and taking into account the notation (96), we arrive at Eq. (97). Conversely, by transferring $\boldsymbol{F}_k^{\text{i}}$ to the other side of Eq. (97), and taking into account (96), we obtain the formula expressing Newton's second law.

Reasoning similarly for all the particles of the system, we arrive at the following result, which expresses **D'Alembert's principle for a system**: *If, at any moment of time, to the effective external and internal forces acting on every particle of a system are added the respective inertia forces, the resultant force system will be in equilibrium, and the equations of statics will apply to it.*

Mathematically D'Alembert's principle is expressed by a set of n simultaneous vector equations of the form (97) which, apparently, are equivalent to the differential equations of motion of a system (13) obtained in § 134. Consequently, like Eqs. (13), D'Alembert's principle can be used to derive all the general theorems of dynamics.

The value of D'Alembert's principle is that, when directly applied to problems of dynamics, the equations of motion of a system can be written in the form of the well-known equations of equilibrium; this makes for uniformity in the approach to problem solutions and usually greatly simplifies the computations. Furthermore, when used in conjunction with the principle of virtual displacement, which will be examined in the following chapter, D'Alembert's principle yields a new general method of solution of problems of dynamics (§ 173).

In applying D'Alembert's principle it should be remembered that, like the fundamental law of dynamics, it refers to motion considered with respect to an inertial frame of reference. That means that acting on the particles of the mechanical system whose motion is being investigated are only the external and internal forces $\boldsymbol{F}_k^{\text{ext}}$ and $\boldsymbol{F}_k^{\text{int}}$ that appear as a consequence of the interactions of the particles of the system among themselves and with bodies not belonging to the system; it is under the action of those forces that the particles of the system are moving with their respective accelerations $\boldsymbol{w}_k$. The inertia forces mentioned in D'Alembert's principle do not act on the moving particles [otherwise, by Eqs. (97), the points would be at rest or in uniform motion in which case, as is apparent from Eq. (96), there would be no inertia forces]. The introduction of inertia forces is but a device making it possible to examine the equations of dynamics by the simpler methods of statics.

In § 120 we investigated inertia forces introduced in examining motion in *noninertial reference frames*; the meaning there was different, namely, their addition to the forces of interaction of a moving body with other bodies makes it possible to preserve the same form for the equations of motion with respect to a noninertial reference frame as with respect to an inertial one. As they are called the *transport* and *Coriolis forces of inertia*, this precludes any confusion with the *inertia forces* of D'Alembert's principle.

We know from statics that the geometrical sum of balanced forces and the sum of their moments with respect to any centre O are zero; we know, further, from the principle of solidification (§ 3) that this holds not only for forces acting on a rigid body but for any deformable system. Thus, according to D'Alembert's principle, we must have:

$$\left.\begin{aligned}\sum (\boldsymbol{F}_k^{\text{ext}} + \boldsymbol{F}_k^{\text{int}} + \boldsymbol{F}_k^{\text{i}}) = 0;\\ \sum [\boldsymbol{m}_O(\boldsymbol{F}_k^{\text{ext}}) + \boldsymbol{m}_O(\boldsymbol{F}_k^{\text{int}}) + \boldsymbol{m}_O(\boldsymbol{F}_k^{\text{i}})] = 0.\end{aligned}\right\} \tag{98}$$

Let us introduce the following notation:

$$\boldsymbol{R}^{\text{i}} = \sum \boldsymbol{F}_k^{\text{i}}; \quad \boldsymbol{M}_O^{\text{i}} = \sum \boldsymbol{m}_O(\boldsymbol{F}_k^{\text{i}}). \tag{99}$$

The quantities $\boldsymbol{R}^{\text{i}}$ and $\boldsymbol{M}_O^{\text{i}}$ are respectively the *principal vector of the inertia forces* and their *principal moment with respect to a centre O*. Taking into account that the sum of the internal forces and the sum of their moments are each zero (§ 129), we obtain:

$$\sum \boldsymbol{F}_k^{\text{ext}} + \boldsymbol{R}^{\text{i}} = 0; \quad \sum \boldsymbol{m}_O(\boldsymbol{F}_k^{\text{ext}}) + \boldsymbol{M}_O^{\text{i}} = 0. \tag{100}$$

The use of Eqs. (100), which follow from D'Alembert's principle, simplifies the process of problem solution because the equations do not contain the internal forces. Actually, Eqs. (100) are equivalent to the equations expressing the theorems of the change in the momentum and the total angular momentum of a system, differing from them only in form.

Eqs. (100) are especially convenient in investigating the motion of a rigid body or a system of rigid bodies. For the complete investigation of any deformable system these equations, however, are insufficient*).

For the projections on a set of coordinate axes, Eqs. (100) give equations analogous to the corresponding equations of statics (see §§ 24 and 49). To use these equations for solving problems we must know the principal vector and the principal moment of the inertia forces.

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*) This follows from the reasoning in § 3 when the principle of solidification was examined. See also the remarks on the general theorems of dynamics in in § 134.

§ 167. The Principal Vector and the Principal Moment of the Inertia Forces of a Rigid Body

It follows from Eqs. (99) (see § 47) that a system of inertia forces applied to a rigid body can be replaced by a single force equal to $\boldsymbol{R}^{i}$ and applied at the centre O, and a couple of moment $\boldsymbol{M}_O^{i}$. The principal vector of a system, it will be recalled, does not depend on the centre of reduction and can be computed at once. As $\boldsymbol{F}_k^{i} = -m_k \boldsymbol{w}_k$, taking into account Eq. (15) (§ 135), we will have:*)

$$\boldsymbol{R}^{i} = -\sum m_k \boldsymbol{w}_k = -\mathcal{M} \boldsymbol{w}_C. \qquad (101)$$

Thus, *the principal vector of the inertia forces of a moving body is equal to the product of the mass of the body and the acceleration of its centre of mass, and is opposite in direction to the acceleration.*

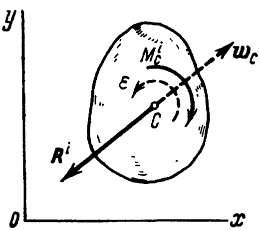

Fig. 359

If we resolve the acceleration $\boldsymbol{w}_C$ into its tangential and normal components, then vector $\boldsymbol{R}^{i}$ will resolve into components

$$\boldsymbol{R}_\tau^{i} = -\mathcal{M} \boldsymbol{w}_{C\tau} \quad \text{and} \quad \boldsymbol{R}_n^{i} = -\mathcal{M} \boldsymbol{w}_{Cn}. \qquad (101')$$

Let us determine the principal moment of the inertia forces for particular types of motion.

(1) Translational Motion. In this case a body has no rotation about its centre of mass C, from which we conclude that $\sum m_C (\boldsymbol{F}_k^{ext}) = 0$, and Eq. (100) gives $M_C = 0$.

Thus, *in translational motion, the inertia forces of a rigid body can be reduced to a single resultant* $\boldsymbol{R}^{i}$ *through the centre of mass of the body.*

(2) Plane Motion. Let a body have a plane of symmetry, and let it be moving parallel to the plane. By virtue of symmetry, the principal vector and the resultant couple of inertia forces lie, together with the centre of mass C, in that plane.

Therefore, placing the centre of reduction in point C, we obtain from Eq. (100) $M_C^{i} = -\sum m_C (\boldsymbol{F}_k^{ext})$. On the other hand (see § 156), from the last of Eqs. (68), $\sum m_C (\boldsymbol{F}_k^{ext}) = J_C \varepsilon$. We conclude from this that

$$M_C^{i} = -J_C \varepsilon. \qquad (102)$$

Thus, in such motion *a system of inertia forces can be reduced to a resultant force* $\boldsymbol{R}^{i}$ [Eq. (101)] *applied at the centre of mass C* (Fig. 359) *and a couple in the plane of symmetry of the body whose moment is*

*) To avoid confusion between the symbols for mass and moment, in this chapter we shall denote mass by the symbol $\mathcal{M}$ (see footnote on p. 367)

given by Eq. (102). The minus sign shows that the moment M_C^i is in the opposite direction of the angular acceleration of the body.

(3) Rotation About an Axis Through the Centre of Mass. Let a body have a plane of symmetry, and let the axis of rotation Cz be normal to the plane through the centre of mass. This case will thus be a particular case of the previous motion. But here $w_C = 0$, and consequently, $\boldsymbol{R}^i = 0$.

Thus, in this case *a system of inertia forces can be reduced to a couple in the plane of symmetry of the body of moment*

$$M_z^i = -J_z\varepsilon. \tag{102'}$$

In applying Eqs. (101) and (102) to problem solutions, the magnitudes of the respective quantities are computed and the directions are shown in a diagram.

§ 168. Solution of Problems

D'Alembert's principle supplies a uniform method for developing the equations of motion of any constrained system*). It provides simple, graphic solutions of problems where we have to determine the reactions of constraints when the motion of a system is known. In these solutions all the immediately unknown internal forces are excluded from consideration. When we have to determine the reactions of internal constraints, the system is divided into parts so that the required forces would be external.

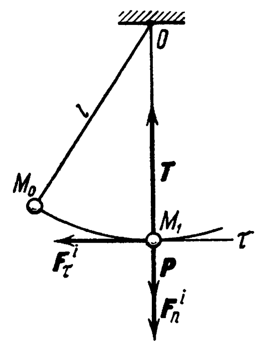

Fig. 360

D'Alembert's principle is also conveniently used to develop the differential equations of motion and, in particular, to determine the acceleration of moving bodies.

In the case of motion of a single constrained material particle application of D'Alembert's principle yields equations similar to those examined in §§ 118 and 119 (see Problem 151).

Problem 151. Solve Problem 112 (see § 119) using D'Alembert's principle.

Solution. Let us depict the load in the position for which the tension in the string has to be found, i.e., in position M_1 (Fig. 360). Acting on the load is its weight $\boldsymbol{P}$ and the reaction of the string $\boldsymbol{T}$.

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*) D'Alembert's principle is especially effective when used in combination with the principle of virtual work (see § 173).

To these forces we add the normal and tangential components of the inertia force, F_n^i and F_τ^i. The full system of forces will be in equilibrium due to D'Alembert's principle. Setting the projection of the sum of these forces on normal M_1O equal to zero, we obtain:

$$T - P - F_n^i = 0.$$

As $F_n^i = mw_n = mv_1^2/l$, where v_1 is the velocity of the load in position M_1,

$$T = P + F_n^i = P + \frac{mv_1^2}{l}.$$

Thus, we have come to the same result as in Problem 112. Using the theorem of the change in the kinetic energy, we can find v_1 in the same way as we did this in Problem 112.

Setting the projection of the sum of the forces on the tangent equal to zero, we obtain $F_\tau^i = 0$. This result follows from the fact that at

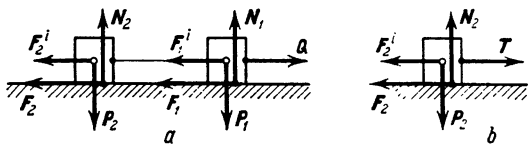


Fig. 361

point M_1 the derivative dv/dt equals zero since the velocity there is maximal.

Problem 152. Two weights P_1 and P_2 are connected by a thread and move along a horizontal plane under the action of a force Q applied to the first weight (Fig. 361*a*). The coefficient of friction of the weights on the plane is f. Determine the accelerations of the weights and the tension in the thread.

Solution. Denote all the external forces acting on the system and add to them the inertia forces of the weights. As both weights are translated with the same acceleration w, in magnitude

$$F_1^i = \frac{P_1}{g} w \quad \text{and} \quad F_2^i = \frac{P_2}{g} w.$$

The forces are directed as shown. The frictional forces are

$$F_1 = fP_1, \quad F_2 = fP_2.$$

According to D'Alembert's principle, the force system must be in equilibrium. Writing the equilibrium equation in terms of the projections on axis Ox, we find:

$$Q - f(P_1 + P_2) - \frac{1}{g}(P_1 + P_2) w = 0,$$

whence

$$w = \left(\frac{Q}{P_1 + P_2} - f\right) g.$$

Evidently, the weights will move if $f < \frac{Q}{P_1 + P_2}$.

In our force system the required tension in the thread is an internal force. To determine it we divide the system and apply D'Alembert's principle to one of the weights, say the second (Fig. 361*b*). Acting on it is force $\boldsymbol{P}_2$, the normal reaction $\boldsymbol{N}_2$, the frictional force $\boldsymbol{F}_2$, and the tension $\boldsymbol{T}$ in the thread. Add to them the inertia force $\boldsymbol{F}_2^{\text{i}}$ and write the equilibrium equation in terms of the projections on axis Ox. We have:

$$T - fP_2 - \frac{P_2}{g} w = 0.$$

Substituting the earlier found value of w, we obtain finally:

$$T = \frac{QP_2}{P_1 + P_2}.$$

Note that the tension in the thread does not depend on the friction and, given the same total weight of the system, decreases with the reduction of the second (rear) weight. That is why, for example, in making up a goods train it is better to place the heavier vans closer to the locomotive.

Assigning specific values to the quantities of this problem, let $Q = 20$ kgf, $P_1 = 40$ kgf, and $P_2 = 10$ kgf, then motion is possible when $f < 0.4$ and the tension in the thread is 4 kgf. If the weights are reversed, the tension in the thread will be 16 kgf.

Problem 153. Solve Problem 133 (§ 146) with the help of D'Alembert's principle and also determine the tension in the thread.

Solution. (1) Considering the drum and the load as a single system, we add to the bodies of the system inertia forces (Fig. 362). Load A is in translatory motion, and for it $R^{\text{i}} = \frac{Q}{g} w_A = \frac{Q}{g} r\varepsilon$. The inertia forces of the drum can be reduced to a couple with a moment M_O^{i} equal in magnitude to $J_O\varepsilon = \frac{P}{g}\rho^2\varepsilon$ and directed opposite the rotation (see § 167). Writing now for all the forces the equilibrium conditions in the form $\sum m_O(\boldsymbol{F}_k) = 0$, we obtain:

$$|M_O^{\text{i}}| + R^{\text{i}} r - Qr = 0,$$

or

$$\frac{P}{g}\rho^2\varepsilon + \frac{Q}{g} r^2\varepsilon - Qr = 0,$$

from which we find

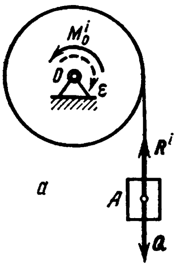
$$\varepsilon = \frac{Qgr}{P\rho^2 + Qr^2}.$$

(2) Considering now load A separately and adding to the active forces $\boldsymbol{Q}$ and $\boldsymbol{T}$ the inertia force $\boldsymbol{R}^i$, we obtain from the equilibrium conditions the tension in the thread

$$T=Q-R^i=Q\left(1-\frac{r\varepsilon}{g}\right)=\frac{PQ\rho^2}{P\rho^2+Qr^2}.$$

Problem 154. Determine the forces acting on a uniformly spinning flywheel of mass $\mathcal{M}$, assuming its mass to be distributed along the rim. The radius of the flywheel is r and its angular velocity is ω.

Solution. The required force is an internal one. In order to determine it, cut the rim into two and apply D'Alembert's principle to one portion (Fig. 363). We denote the action of the separated half by two forces $\boldsymbol{F}'$ both equal in magnitude to the required force $\boldsymbol{F}$. For



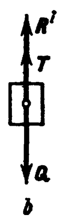


Fig. 362

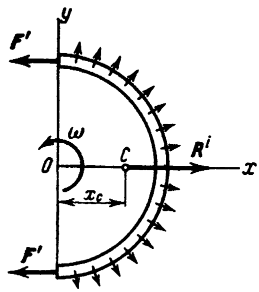


Fig. 363

each element of the rim, the inertia force (a centrifugal force) is directed along the radius. These concurrent forces intersecting at O have a resultant equal to the principal vector $\boldsymbol{R}^i$ of the inertia forces directed, by virtue of symmetry, along axis Ox. By Eq. (101), $R^i = 0.5\mathcal{M}w_C = 0.5\mathcal{M}x_C\omega^2$, where x_C is the coordinate of the mass centre of the semicircular arc, which is equal to $2r/\pi$ (see § 57). Therefore,

$$R^i=\frac{\mathcal{M}r\omega^2}{\pi}.$$

The equilibrium conditions give $2F = R^i$, and finally

$$F=\frac{\mathcal{M}r\omega^2}{2\pi}.$$

This formula can be used to determine the limiting angular velocity beyond which a flywheel made of a specific material may be torn apart.

Problem 155. A homogeneous rod AB of length l and weight P is hinged at A to a vertical shaft rotating with an angular velocity ω (Fig. 364). Determine the tension T in the horizontal thread securing the rod at an angle α to the shaft.

Solution. Applying D'Alembert's principle, we add to the external forces $\boldsymbol{P}$, $\boldsymbol{T}$, $\boldsymbol{X}_A$, and $\boldsymbol{Y}_A$, acting on the rod, the inertia forces. For each element of the rod of mass Δm the centrifugal inertia force is $\Delta m\omega^2 x$, where x is the distance of the element from the axis of rotation Ay. The resultant of these parallel forces distributed according to a linear law (see § 28) passes through the centre of gravity of triangle ABE, i.e., at a distance $h = 2/3 l \cos\alpha$ from axis Ax. As this resultant is equal to the principal vector of the inertia forces*⁾, by Eq. (101) we have:

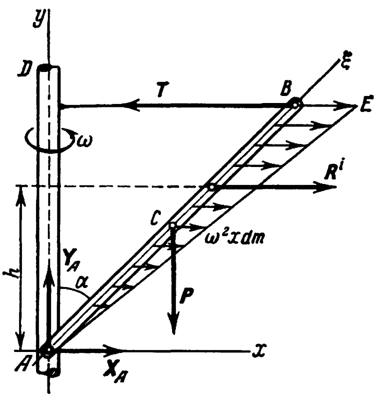


Fig. 364

$$R^i = \mathcal{M} w_C = \mathcal{M}\omega^2 x_C = \frac{P}{g}\,\omega^2 \frac{l}{2}\sin\alpha$$

(here x_C is the coordinate of the centre of gravity of the rod).

Writing now the statics equation $\sum m_A(\boldsymbol{F}_k) = 0$, we obtain:

$$Tl\cos\alpha - R^i h - P\frac{l}{2}\sin\alpha = 0.$$

Substituting the values of R^i and h into this equation, we obtain finally:

$$T = P\left(\frac{l\omega^2}{3g}\sin\alpha + \frac{1}{2}\tan\alpha\right).$$

Alternate Solution. The problem can be solved without applying the results of § 28 by directly computing the sum of the moments of the inertia forces with respect to centre A by integration. Draw along rod AB an axis $A\xi$. Acting on each element $d\xi$ of the rod, whose coordinate is ξ, there is an inertia force equal to $\omega^2 x dm$. Its moment with respect to A is $-y\omega^2 x dm$. Then the equation of moments gives

$$\sum m_A(\boldsymbol{F}_k) \equiv Tl\cos\alpha - P\frac{l}{2}\sin\alpha - \int_0^l \omega^2 yx\,dm = 0. \qquad \text{(a)}$$

Expressing all the quantities under the integration sign in terms of ξ, we obtain:

$$x = \xi\sin\alpha,\quad y = \xi\cos\alpha,\quad dm = \frac{\mathcal{M}}{l}d\xi,$$

*⁾ We know from statics that the resultant of any force system, if there is one, is equal to the principal vector of the forces. Therefore, the resultant of the inertia forces, when there is one, is equal to $\boldsymbol{R}^i$, though in non-translational motion it does not necessarily pass through the centre of mass, as is the case in this problem.

from which we have

$$\int_0^l \omega^2 yx\,dm = \frac{\mathcal{M}}{l}\,\omega^2 \sin\alpha\cos\alpha \int_0^l \xi^2\,d\xi = \frac{1}{3}\,\frac{P}{g}\,l^2\omega^2 \sin\alpha\cos\alpha.$$

Substituting this expression into equation (a), we obtain for T the same expression as in the first solution.

Problem 156. A homogeneous rod bent at a right angle as shown in Fig. 365 rotates in a horizontal plane about its end A with an angular velocity ω and an angular acceleration ε. The respective distances $AB = a$ and $BC = b$; the mass of unit length of the rod is ρ_1. Determine the stresses at a cross section of the rod at point D at a distance h from end B.

Solution. The required forces are internal. To determine them, cut the rod in two and examine the motion of section DC, the length of which is $b - h$. The action of the removed portion ABD is replaced by a force applied at the centre D of the cross section, which force we shall represent by its components P and Q, and a couple with a moment M_D (see § 27, Fig. 72). The quantities P, Q, and M_D will specify the required stresses in section D of the rod, i.e., the forces with which portions ABD and DC act on one another.

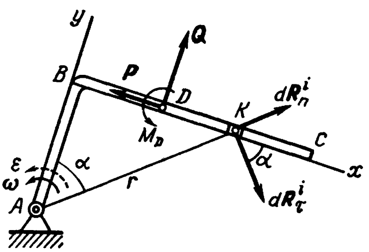


Fig. 365

To compute these quantities, we use D'Alembert's principle.

Resolve the inertia forces of the rod's particles into tangential and normal components and denote their principal vectors as $\boldsymbol{R}_\tau^i$ and $\boldsymbol{R}_n^i$, respectively. Then, drawing axes Bxy as shown in the drawing and writing the respective equilibrium equations for the effective forces and the inertia forces, we have:

$$\left.\begin{aligned} -P + R_{\tau x}^i + R_{nx}^i = 0, \quad Q + R_{\tau y}^i + R_{ny}^i = 0, \\ M_D + M_{D1}^i + M_{D2}^i = 0, \end{aligned}\right\} \qquad \text{(a)}$$

where M_{D1}^i is the sum of the moments of the tangential inertia forces with respect to the centre D, and M_{D2}^i is the sum of the moments of the normal forces.

Now compute the quantities in Eq. (a). For element K of length dx at a distance x from B ($BK = x$) the tangential and normal components of the inertia forces are, in magnitude, $dR_\tau^i = \varepsilon r\,dm$ and $dR_n^i = \omega^2 r\,dm$, where $r = AK$ and $dm = \rho_1\,dx$. Then, taking into account that $r\cos\alpha = a$, and $r\sin\alpha = x$, we obtain:

$$dR_{\tau x}^i + (\rho_1 \varepsilon r\,dx)\cos\alpha - \rho_1 \varepsilon a\,dx, \quad dR_{\tau y}^i = -(\rho_1 \varepsilon r\,dx)\sin\alpha = -\rho_1 \varepsilon x\,dx$$

Similarly, we find that

$$dR^{i}_{nx}=\rho_1\omega^2 x\,dx,\quad dR^{i}_{ny}=\rho_1\omega^2 a\,dx.$$

Hence,

$$R^{i}_{\tau x}=\rho_1\varepsilon a\int_h^b dx=\rho_1\varepsilon a\,(b-h),\ \ R^{i}_{\tau y}=-\rho_1\varepsilon\int_h^b x\,dx=-\frac{1}{2}\rho_1\varepsilon\,(b^2-h^2);$$

$$R^{i}_{nx}=\frac{1}{2}\rho_1\omega^2\,(b^2-h^2),\ \ R^{i}_{ny}=\rho_1\omega^2 a\,(b-h).$$

Now, noting that the moments of the components of the inertia forces along axis Bx with respect to D are zero, we obtain (see Fig. 365):

$$M^{i}_{D1}=-\int_h^b (x-h)\,|dR^{i}_{\tau y}|=-\rho_1\varepsilon\int_h^b (x-h)\,x\,dx=$$

$$=-\frac{1}{6}\rho_1\varepsilon\,(h^3+2b^3-3b^2h),$$

$$M^{i}_{D2}=\int_h^b (x-h)\,dR^{i}_{ny}=\rho_1\omega^2 a\int_h^b (x-h)\,dx=\frac{1}{2}\rho_1\omega^2 a\,(b-h)^2.$$

Substituting all the computed quantities into equation (a), we find that acting at section D are a tensile force P, lateral force Q and bending moment M_D; in magnitude:

$$P=\rho_1\left[a\,(b-h)\,\varepsilon+\frac{1}{2}(b^2-h^2)\,\omega^2\right],$$

$$Q=\rho_1\left[\frac{1}{2}(b^2-h^2)\,\varepsilon-a\,(b-h)\,\omega^2\right],$$

$$M_D=\rho_1\left[\frac{1}{6}(2b^3+h^3-3b^2h)\,\varepsilon-\frac{1}{2}a\,(b-h)^2\,\omega^2\right].$$

The quantity ρ_1 equals the ratio of the mass of the rod to its length. For decelerated motion in the same direction substitute $-\varepsilon$ instead of ε. At $a=0$ we have the case of rod BC revolving about end B.

§ 169*. Dynamic Reactions on the Axis of a Rotating Body. Dynamic Balancing of Masses

Consider a rigid body rotating uniformly with an angular velocity ω about an axle mounted in bearings A and B (Fig. 366). Let us find the dynamic pressures X_A, Y_A, Z_A, X_B, Y_B exerted by the bearings on the axle, i.e. the reactions that appear when the body rotates. Let a set of given forces $\boldsymbol{P}_1^{\text{ext}}, \boldsymbol{P}_2^{\text{ext}}, \ldots, \boldsymbol{P}_n^{\text{ext}}$ be acting on the body, and denote the projections of their principal vector on the coordinate axes $Axyz$ rotating together with the body by the symbols R_x^{ext}, R_y^{ext}, R_z^{ext} ($R_x^{\text{ext}} = \sum P_{kx}^{\text{ext}}$, etc.), and their principal moments with respect to the axes by the symbols M_x^{ext}, M_y^{ext}, M_z^{ext} ($M_x^{\text{ext}} = \sum m_x(\boldsymbol{P}_k^{\text{ext}})$, etc.). As the body is rotating uniformly, $M_z^{\text{ext}} = 0$. To determine the required reactions, apply D'Alembert's principle and, adding the respective inertia forces to the forces acting on all the points of the body, write Eqs. (100) in terms of the projections on axes $Axyz$. In our case these equations (or the corresponding equations of § 49) take the form (denoting $AB = b$):

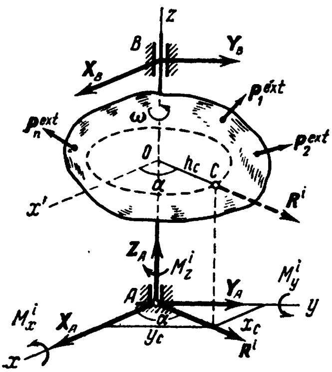


Fig. 366

$$\left.\begin{aligned} X_A + X_B + R_x^{\text{ext}} + R_x^{\text{i}} &= 0, \\ Y_A + Y_B + R_y^{\text{ext}} + R_y^{\text{i}} &= 0, \\ Z_A + R_z^{\text{ext}} + R_z^{\text{i}} &= 0, \\ -Y_B b + M_x^{\text{ext}} + M_x^{\text{i}} &= 0, \\ X_B b + M_y^{\text{ext}} + M_y^{\text{i}} &= 0. \end{aligned}\right\} \quad (103)$$

The last equation $M_z^{\text{ext}} + M_z^{\text{i}} = 0$ is an identity (as $\varepsilon = 0$), and we omit it.

The principal vector of the inertia forces $\boldsymbol{R}^{\text{i}} = -\mathcal{M}\boldsymbol{w}_C$. At $\omega = \text{const.}$, the centre of mass C has only a normal acceleration $w_{Cn} = \omega^2 h_C$, where h_C is the distance of point C from the axis of rotation. Consequently, vector $\boldsymbol{R}^{\text{i}}$ is directed along OC*). Computing the projections of $\boldsymbol{R}^{\text{i}}$ on the coordinate axes and taking into account that $h_C \cos\alpha = x_C$ and $h_C \sin\alpha = y_C$, where x_C and y_C are the coordinates of the centre of mass, we obtain:

$$\boldsymbol{R}_x^{\text{i}} = \mathcal{M}\omega^2 h_C \cos\alpha = \mathcal{M}\omega^2 x_C,$$

$$\boldsymbol{R}_y^{\text{i}} = \mathcal{M}\omega^2 h_C \sin\alpha = \mathcal{M}\omega^2 y_C, \quad \boldsymbol{R}_z^{\text{i}} = 0.$$

In order to determine M_x^{i} and M_y^{i}, consider any particle of the body of mass m_k located at a distance h_k from the axle. For this particle, at $\omega = \text{const.}$, the inertia force has only a centrifugal component $F_n^{\text{i}} = m_k\omega^2 h_k$, the projections of which, like the projections of $\boldsymbol{R}^{\text{i}}$, are

$$F_{kx}^{\text{i}} = m_k\omega^2 x_k, \quad F_{ky}^{\text{i}} = m_k\omega^2 y_k, \quad F_{kz}^{\text{i}} = 0.$$

*) The dashed line in the diagram indicates only the direction of $\boldsymbol{R}^{\text{i}}$. But if we reduce the inertia forces to a centre, say A, then they will be replaced by a force $\boldsymbol{R}^{\text{i}}$ applied at A and a couple with a moment, which for point A is made up of $\boldsymbol{M}_x^{\text{i}}$ and $\boldsymbol{M}_y^{\text{i}}$.

Then [see § 43, Eqs. (52)]

$$m_x(F_k^i) = -F_{ky}^i z_k = -m_k\omega^2 y_k z_k, \quad m_y(F_k^i) = F_{kx}^i z_k = m_k\omega^2 x_k z_k.$$

Writing similar expressions for all the particles of the system, adding them, and taking the common multiplier ω^2 outside of the parentheses, we obtain:

$$M_x^i = -\left(\sum m_k y_k z_k\right)\omega^2 = -J_{yz}\omega^2, \quad M_y^i = \left(\sum m_k x_k z_k\right)\omega^2 = J_{xz}\omega^2, \quad (104)$$

where J_{xz} and J_{yz} are the respective products of inertia*). Substituting all the obtained quantities into Eqs. (103), we obtain:

$$\left.\begin{aligned} X_A + X_B = -R_x^{\text{ext}} - \mathcal{M}x_C\omega^2, \quad Y_A + Y_B = -R_y^{\text{ext}} - \mathcal{M}y_C\omega^2, \\ Z_A = -R_z^{\text{ext}}, \quad X_B b = -M_y^{\text{ext}} - J_{xz}\omega^2, \quad Y_B b = M_x^{\text{ext}} - J_{yz}\omega^2. \end{aligned}\right\} \quad (105)$$

Equations (105) specify the *dynamic reactions* acting on the axle of a uniformly rotating rigid body, when axis Oz is taken coincident with the axle.

Let us call *static reactions* those values of the reactions that Eqs. (105) give when $\omega = 0$ (the reactions will be static in the sense of acting on axle AB at rest when the projections of the applied forces on axes $Axyz$ are constant). As Eqs. (105) show, dynamic reactions can in general be substantially greater than static, and this depends not only on the value of ω but also on the quantities x_C, y_C, J_{xz}, J_{yz}, which characterise the distribution of mass with respect to the axis of rotation z.

However, it is apparent from Eqs. (105) that if

$$x_C = 0, \quad y_C = 0, \quad (106)$$

$$J_{xz} = 0, \quad J_{yz} = 0. \quad (107)$$

the rotation does not affect the reactions of bearings A and B.

Equations (106) and (107) express the conditions in which the dynamic reactions on the axle of a rotating body are equal to the static reactions, or the conditions of dynamic balancing of the masses of the body in its rotation about axis z.

Conditions (106) signify that the body's centre of mass must lie on its axis of rotation, and conditions (107), that the axis of rotation should be the principal axis of inertia of the body with respect to the origin of the coordinate system A. When conditions (106) and (107) are satisfied simultaneously, axis Az is the principal central axis of inertia of the body (see § 133). Thus, the dynamic reactions on the axle of a rotating body are equal to the static reactions when the axis of rotation is one of the body's principal central axes of inertia. The conclusion is also valid for the case of a body in nonuniform rotation.

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*) See § 133, Eqs. (10).

The above reasoning also suggests the physical meaning of the quantities J_{xz} and J_{yz}: the products of inertia J_{xz} and J_{yz} are a measure of the degree of dynamic unbalance of the body rotating about axis z.

The *dynamic balancing of masses* is an important engineering problem, which, we have seen, involves the determination of the principal central axes of inertia of a body. In § 133 it was shown that every body has at least three mutually perpendicular principal central axes of inertia.

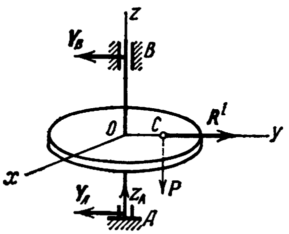

Fig. 367

Let us prove another statement which is no less important in practice: *any axis passing through a body can be made a principal central axis of inertia by adding two point masses to the body.* Let us have a body of mass $\mathscr{M}$, and let the quantities x_C, y_C, J_{xz}, and J_{yz} be known and not equal to zero. Let us further add to the body two masses m_1 and m_2 at points whose coordinates are x_1, y_1, z_1 and x_2, y_2, z_2, respectively. Then, from Eqs. (1) and (10), it follows that

$$\left.\begin{aligned} \mathscr{M}x_C + m_1x_1 + m_2x_2 = 0, \quad J_{xz} + m_1x_1z_1 + m_2x_2z_2 = 0, \\ \mathscr{M}y_C + m_1y_1 + m_2y_2 = 0, \quad J_{yz} + m_1y_1z_1 + m_2y_2z_2 = 0, \end{aligned}\right\} \tag{108}$$

then for the new body we shall have $x'_C = y'_C = J'_{xz} = J'_{yz} = 0$, i.e., axis Oz will be the principal central axis of inertia. The problem is solved by choosing the masses m_1 and m_2 and their positions such that Eqs. (108) will be satisfied. Some of the quantities, of course, must be specified in advance. For instance, we may specify the values of m_1, m_2 and z_1, z_2 (but such that $z_1 \neq z_2$) and find x_1, y_1, x_2, y_2 from Eqs. (108), etc.

This method is widely used in engineering for balancing crankshafts, cranks, coupling rods, etc. Final balancing is done on special balancing machines.

When it is necessary to determine the pressures acting on an axle, the ready equations (105) are usually not used, and D'Alembert's principle is applied for each specific case.

Problem 157. The axis of rotation of a disc is perpendicular to the plane of the disc (Fig. 367). The disc weighs P and its angular velocity is ω = const. Determine the dynamic reactions of bearings A and B if $OA = OB = h$ and the centre of mass of the disc is located at a distance $OC = a$ from the axis.

Solution. Draw the axes $Oxyz$, which rotate together with the body, so that axis y passes through the centre of mass C. As plane Oxy is the disc's plane of symmetry, axis z is the principal axis of inertia with respect to O. Hence, $J_{xz} = J_{yz} = 0$, and from Eqs. (104)

and the condition that ω = const. it follows that $M_O^i = 0$. Consequently, the inertia forces can be reduced to a resultant through O along OC (axis y). In magnitude $R^i = Mw_{Cn} = \frac{P}{g} a\omega^2$. As forces $\boldsymbol{P}$ and $\boldsymbol{R}^i$ lie in plane Oyz, the reactions of the bearings lie in that plane, too, i.e., they have components $\boldsymbol{Y}_A$ and $\boldsymbol{Z}_A$ at point A and $\boldsymbol{Y}_B$ at point B. Then, applying D'Alembert's principle to write the equations for all the effective and inertia forces in terms of the projections on axes Oy and Oz, and the equation of moments with respect to A, we obtain:

$$R^i - Y_A - Y_B = 0, \quad Z_A - P = 0,$$
$$Y_B 2h - Pa - R^i h = 0.$$

Solving these equations, we find:

$$Y_B = \frac{Pa\omega^2}{2g} + \frac{a}{2h} P,$$
$$Y_A = \frac{Pa\omega^2}{2g} - \frac{a}{2h} P, \quad Z_A = P.$$

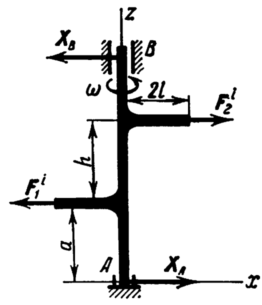

Fig. 368

The reactions $\boldsymbol{Y}_A$ and $\boldsymbol{Y}_B$ remain continuously in plane Oyz, which rotates together with the body.

Problem 158. Two equal bars of length $2l$ and mass m each are welded at right angles to a vertical shaft AB of length b at a distance h from each other (Fig. 368). Determine the dynamic pressures acting on the shaft rotating with a constant angular velocity ω if the force of gravity is negligible.

Solution. The reaction of the bearings and the inertia force according to D'Alembert's principle form a balanced system. The centrifugal inertia forces in each rod are equal in magnitude:

$$F_1^i = F_2^i = ml\omega^2,$$

and they make a couple which, apparently, is balanced by the couple $\boldsymbol{X}_A$, $\boldsymbol{X}_B$. The moments of these couples are equal in magnitude. Consequently, $X_A b = F_1^i h$, whence

$$X_A = X_B = \frac{F_1^i h}{b} = \frac{mlh}{b} \omega^2.$$

The couple is continuously in the Axz plane, which rotates with the body.

Let us show that the same result can be obtained from Eqs. (105). In our case, $x_C = y_C = 0$, and $J_{yz} = 0$, as the plane Axz is the plane of symmetry. The quantity $J_{xz} = J'_{xz} + J''_{xz}$, i.e., is equal to the sum of the products of inertia of each bar. For the lower bar all $z_k = a$, and for the upper one all $z_k = a + h$. Then from Eq. (10),

$$J'_{xz} = \left(\sum m_k x_k\right) a = m x'_C a = -mla, \quad J''_{xz} = m x''_C (a+h) = ml(a+h),$$

whence $J_{xz} = mlh$.

Substituting all the expressions into Eqs. (105), we find:

$$Y_A = Y_B = 0, \quad X_A = -X_B = \frac{ml}{b}\omega^2.$$

The reason for the "minus" at X_B is that the forces X_B in Figs. 366 and 368 are of opposite sense.

Problem 159. The crankshaft of a one-cylinder gas engine carries two identical flywheels A and B of radius $r = 0.5$ m as shown in Fig. 369. The crank arms and crankpin are equivalent to a weight of $p = 21$ kgf at a distance of $h = 0.2$ m from the axis. If $a = 0.6$ m and $b = 1.4$ m, what correction weights p_A and p_B placed on the rims of the flywheels will balance the system?

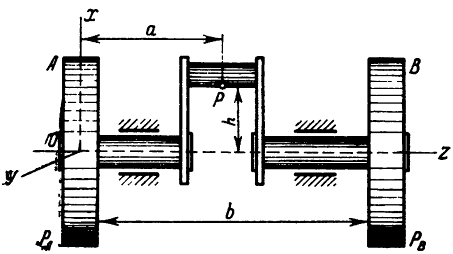

Fig. 369

Solution. Draw a set of coordinate axes rotating with the body, such that the crankpin would lie in the Oxz plane. This plane, then, will be the plane of symmetry, and, consequently, $y_C = 0$ and, as Oy is a principal axis with respect to point O, $J_{yz} = 0$. Furthermore, introducing the notation P for the total weight of the system, we have:

$$x_C = \frac{ph}{P}, \quad J_{xz} = \frac{p}{g} ha.$$

The latter equality follows from the fact that the product of inertia of a system equals the sum of the products of inertia of its components, and the products of inertia J_{xz} of the flywheels and adjacent sections of the crankshaft are zero (Oz is the axis of symmetry).

The coordinates of the correction weights, from Eqs. (108), are $y_A = y_B = 0$, and the weights themselves must satisfy the equations $P x_C + p_A x_A + p_B x_B = 0$, $J_{xz} + \frac{p_A}{g} x_A z_A + \frac{p_B}{g} x_B z_B = 0$.

As the weights are to be placed on the rims of the flywheels, $z_A = 0$, $z_B = b$, and $x_A = x_B = -r$ (the equations have no solutions when the sign is positive, therefore, the weights must be at the bottom). Solving the equations, we find:

$$p_A = \frac{(b-a)h}{rb} p = 4.8 \text{ kgf}, \quad p_B = \frac{ah}{rb} p = 3.6 \text{ kgf}.$$

Adding these weights balances the system and makes Oz the principal central axis of inertia (but not the axis of symmetry).

Chapter 31
The Principle of Virtual Displacements and the General Equation of Dynamics

§ 170. Virtual Displacements of a System. Degrees of Freedom

In this chapter we shall examine another general principle of mechanics, the principle of virtual displacements, which establishes the equilibrium conditions for any mechanical system in the most general form.

In examining the equilibrium of systems of bodies by the methods of so-called geometrical statics (treated in Part One) one has to investigate the equilibrium of each body separately, replacing the constraints with immediately unknown reactions. When a system consists of a large number of bodies, the method becomes unwieldy because of the need to solve a large number of equations with many unknown quantities.

Characteristic of the method based on the principle of virtual work is that the action of constraints is taken into account not by introducing the reaction forces but by investigating the possible displacements of a system as if its equilibrium was disturbed. These displacements are known in mechanics by the name of *virtual displacements.*

Virtual displacements of the particles of a system must satisfy two conditions: (1) they must be infinitesimal, since if a displacement is finite, the system will occupy a new configuration in which the equilibrium conditions may be different; (2) they must be consistent with the constraints of the system, as otherwise we should change the character of the mechanical system under consideration.

For instance, in the crankshaft mechanism in Fig. 370, a displacement of the points of the crank OA into configuration OA_1 cannot be considered as a virtual displacement, as the equilibrium conditions under the action of forces P and Q will have changed. At the same time, even an infinitesimal displacement of point B of the connecting rod along BD would not be a virtual displacement; it would have been possible if the slides at B were replaced by a rocker (see Fig. 188, where C is a rocker), i.e., if it were a different mechanism.

Thus, *we shall define as a virtual displacement of a system the sum total of any arbitrary infinitesimal displacement of the particles of the system consistent with all the constraints acting on the system at the given instant.* We shall denote the virtual displacement of any point by an elementary vector $\delta \boldsymbol{s}$ in the direction of the displacement.

In the most general case, the particles and bodies of a system may have a number of different virtual displacements (not considering $\delta \boldsymbol{s}$ and $-\delta \boldsymbol{s}$ as being different). For every system, however, depending on the type of constraints, we can specify a certain number of

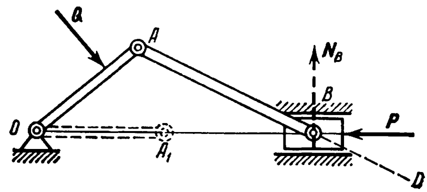

Fig. 370

independent virtual displacements such that any other virtual displacement will be obtained as their geometrical sum. For example, a bead lying on a plane (or surface) can move in many directions on the plane. Nevertheless, any virtual displacement $\delta \boldsymbol{s}$ may be produced as the sum of two displacements $\delta \boldsymbol{s}_1$ and $\delta \boldsymbol{s}_2$ along two mutually perpendicular horizontal axes ($\delta \boldsymbol{s} = \delta \boldsymbol{s}_1 + \delta \boldsymbol{s}_2$).

The number of possible mutually independent displacements of a system is called the number of degrees of freedom of that system. Thus, the bead mentioned above (regarded as a particle) has two degrees of freedom. A crankshaft mechanism, evidently, has one degree of freedom. A free particle has three degrees of freedom (three independent displacements along mutually perpendicular axes). A free rigid body has six degrees of freedom (three translational displacements along orthogonal axes and three rotations about those axes).

§ 171. The Principle of Virtual Displacements

By virtual work is meant the elementary work that could have been done by a force acting on a material particle in a displacement coinciding with the particle's virtual displacement. Let us denote the virtual work done by an active force $\boldsymbol{F}^a$ by the symbol δA^a ($\delta A^a = F^a \delta s \cos \alpha$, where α is the angle between the direction of the force and the displacement), and the virtual work done by the reaction of a constraint $\boldsymbol{N}$ by the symbol δA^r.

In § 149 we introduced the important concept of systems with ideal constraints. It follows from Eq. (52) in § 149 that the constraints imposed on a system are ideal if the sum of the elementary works done by the reactions of those constraints in any virtual displacement of the system is zero, i.e.,

$$\sum \delta A_k^{\mathrm{r}} = 0. \tag{109}$$

Examples of such ideal constraints were given in § 149.

Consider a system of material particles in equilibrium under the action of the applied forces and constraints, assuming all the constraints imposed on the system to be ideal. Let us take an arbitrary particle B_k belonging to the system and denote the resultant of all the applied active forces (both external and internal) by the symbol $\boldsymbol{F}_k^{\mathrm{a}}$, and the resultant of all the reactions of the constraints (also external and internal) by the symbol $\boldsymbol{N}_k$. Since point B_k is in equilibrium together with the system, $\boldsymbol{F}_k^{\mathrm{a}} + \boldsymbol{N}_k = 0$, or $\boldsymbol{N}_k = -\boldsymbol{F}_k^{\mathrm{a}}$. Consequently, in any virtual displacement of point B_k, the virtual works δA_k^{a} and δA_k^{r} done by the forces $\boldsymbol{F}_k^{\mathrm{a}}$ and $\boldsymbol{N}_k$ are equal in magnitude and opposite in sense and therefore vanish, i.e., we have

$$\delta A_k^{\mathrm{a}} + \delta A_k^{\mathrm{r}} = 0.$$

Reasoning in the same way we obtain similar equations for all the particles of a system, adding which we obtain:

$$\sum \delta A_k^{\mathrm{a}} + \sum \delta A_k^{\mathrm{r}} = 0.$$

But if the constraints imposed on the system are ideal, the second summation, by Eq. (109), is zero, whence

$$\sum \delta A_k^{\mathrm{a}} = 0, \tag{110}$$

or

$$\sum (F_k^{\mathrm{a}} \delta s_k \cos \alpha_k) = 0. \tag{110'}$$

We have thus proved that if a mechanical system with ideal constraints is in equilibrium, the active forces applied to it satisfy the condition (110). The reverse is also true, i.e., if the active forces satisfy the condition (110), the system is in equilibrium. From this follows the principle of virtual displacements*): *The necessary and sufficient conditions for the equilibrium of a system subjected to ideal constraints is that the total virtual work done by all the active forces is equal to zero for any and all virtual displacements consistent with the constraints.* Mathematically the necessary and sufficient condi-

*) This principle was formulated very like the present definition but without proof, by the celebrated Swiss mathematician and mechanic John Bernoulli (1667-1748). In general form the principle was enunciated and proved by Lagrange in 1788. The principle was generalised for the case of unilateral constraints (constraints from which a body can escape) by M. V. Ostrogradsky in his works of 1838-1842.

tion for the equilibrium of any mechanical system is expressed by Eq. (110), which is also called the *equation of virtual work*.

In analytical form this condition can be expressed as follows (see § 112):

$$\sum (F^a_{kx}\delta x_k + F^a_{ky}\delta y_k + F^a_{kz}\delta z_k) = 0. \qquad (111)$$

In Eq. (111) δx_k, δy_k, δz_k are the projections of the virtual displacements $\delta \mathbf{s}_k$ of point B_k on the coordinate axes. They are equal to the infinitesimal increments to the position coordinates of the point in its displacement and are computed in the same way as the differentials of coordinates.

The principle of virtual work provides in general form the equilibrium conditions of any mechanical system, whereas the methods of geometrical statics require the consideration of the equilibrium of every body of the system separately. Furthermore, application of the principle of virtual work requires that only the active forces be considered and this makes it possible to ignore all the unknown reactions of constraints when the constraints are ideal.

§ 172. Solution of Problems

For a system with one degree of freedom, Eqs. (110) and (111) immediately give the equilibrium conditions. If a system has several degrees of freedom, then the conditions (110) or (111) must be developed for each of the independent displacements of the system separately, i.e., the number of equilibrium equations is equal to the number of degrees of freedom of a system.

In problem solutions the number of degrees of freedom of plane mechanisms is conveniently determined as follows. Imagine a mechanism to be moving. If by stopping a translation or rotation of any unit we stop the whole mechanism, then it has one degree of freedom. If, when the translation or rotation of any unit is stopped, the mechanism continues to move, and if it can be stopped by additionally preventing the displacement of some other part, then it has two degrees of freedom, etc.

For solving problems by the *geometrical method* we must: (1) depict all the active forces applied to the system under consideration; (2) impart a virtual displacement to the system and denote by vectors $\delta \mathbf{s}_k$ the elementary displacements of the points of application of the forces, or by angles $\delta\varphi_k$ the elementary rotations of the bodies subjected to forces (if a system has several degrees of freedom, it must be given one of the independent displacements); (3) compute the virtual work done by all the active forces in the given displacement according to the formulas

$$\delta A_k^{\text{a}} = F_{k\tau}^{\text{a}} \delta s_k \quad \text{or} \quad \delta A_k^{\text{a}} = m_O (F_k^{\text{a}}) \, \delta \varphi_k \tag{112}$$

and write condition (110); (4) establish the dependence between the quantities δs_k and $\delta \varphi_k$ in Eq. (110) and express all of them in terms of one quantity, which can always be done as the system was given an independent virtual displacement.

When all the quantities δs_k or $\delta \varphi_k$ in Fig. (110) are expressed in terms of one of them, we obtain an equation from which we can find the required quantity or relation.

When a system has several degrees of freedom, the foregoing procedure must be repeated for each independent displacement.

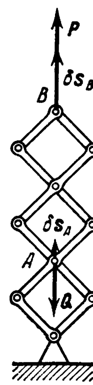

Fig. 371

The relation between the quantities δs_k and $\delta \varphi_k$ can be found either (a) from purely geometrical considerations (problems 160 and 165) or (b) kinematically, i.e., first determine the relation between the corresponding linear velocity v_k and angular velocity ω_k which the system would have if it were in motion, and then take into account that $\delta s_k = v_k \, dt$ and $\delta \varphi_k = \omega_k \, dt$ (Problems 161 and 162 and others).

In the *analytical method* the equilibrium equation is written in the form (111). For this the coordinate axes are taken with reference to a body which remains stationary during the virtual displacement of the system. Then the projections of all the active forces on these axes and the position coordinates x_k, y_k, z_k of all the points of application of the forces must be computed, the coordinates being expressed in terms of any one parameter (angle, for instance). Then the quantities δx_k, δy_k, δz_k are found by differentiating the coordinates x_k, y_k, z_k with respect to that parameter.

If it is impossible to express all the coordinates x_k, y_k, z_k in terms of one parameter immediately, several parameters should be introduced, such that the relation between them can later be established.

In conclusion, note that conditions (110) and (111) can be applied when there are frictional forces, if they are included among the active forces. Similarly, the reactions of constraints can be determined if the corresponding constraints are removed and their reactions are included among the active forces.

Problem 160. Find the relation between forces $\boldsymbol{P}$ and $\boldsymbol{Q}$ in the mechanism shown in Fig. 371 if it is in equilibrium.

Solution. If we give the system a virtual displacement, the diagonals of all the parallelograms formed by the rods will receive the same increment δs. We have $\delta s_A = \delta s$, and $\delta s_B = 3\delta s$. Writing condition (110), we obtain: $P\delta s_B - Q\delta s_A = 0$ or $(3P - Q)\,\delta s = 0$, whence $Q = 3P$. The solution, we see, is very simple.

Problem 161. A beam of weight Q is supported on two cylindrical rollers of weight P each. What force F must be applied to the beam to maintain equilibrium on a plane making an angle α with the horizontal if there is no slipping between either the rollers and the inclined plane or the rollers and the beam (Fig. 372)?

Solution. If we neglect rolling friction, the plane will be an ideal constraint for both rollers. If a virtual displacement is given to the system, we obtain from condition (110):

$$F\delta s_B - Q \sin \alpha \delta s_B - 2P \sin \alpha \delta s_C = 0,$$

where δs_B is the virtual displacement of the point of the beam coincident with point B. The point of contact K is the instantaneous centre of zero velocity of the roller, therefore $v_B = 2v_C$ and $\delta s_B =$

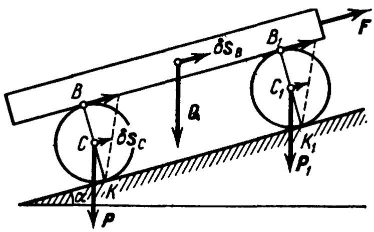

Fig. 372

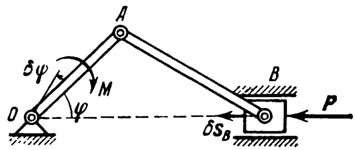

Fig. 373

$= 2\delta s_C$, since $\delta s_B = v_B\, dt$ and $\delta s_C = v_C\, dt$. Substituting this expression for δs_B into the foregoing equation, we obtain finally:

$$F = (Q + P) \sin \alpha.$$

Problem 162. Find the relation between the moment M of the couple acting on the crankshaft mechanism in Fig. 373 and the pressure P on the piston when the system is in equilibrium. The crank is of length $OA = r$ and the connecting rod is of length $AB = l$; $\angle AOB = \varphi$.

Solution. Equilibrium conditions (110) give

$$-M\delta\varphi + P\delta s_B = 0, \quad \text{or} \quad M\omega_{OA} = Pv_B,$$

since $\delta\varphi = \omega_{OA}\, dt$ and $\delta s_B = v_B\, dt$. The solution requires that the relation be found between v_B and ω_{OA}. This kinematic problem was solved in Problem 71, § 82. Referring to the result obtained there, we find

$$M = Pr\left(1 + \frac{r\cos\varphi}{\sqrt{l^2 - r^2\sin^2\varphi}}\right)\sin\varphi.$$

Problem 163. For the reduction gear considered in Problem 91, § 95, find the relation between the torque M_A applied to the driving shaft A and the resistance moment M_B applied to the driven shaft B when both shafts are rotating uniformly.

Solution. The relation between M_A and M_B will be the same in uniform rotation as in equilibrium. Therefore, from Eq. (110) we have:

$$M_A \delta\varphi_A - M_B \delta\varphi_B = 0, \quad \text{or} \quad M_A \omega_A = M_B \omega_B,$$

as $\delta\varphi_A = \omega_A \, dt$ and $\delta\varphi_B = \omega_B \, dt$. Hence, referring to the result obtained in Problem 91, we find:

$$M_A = \frac{\omega_B}{\omega_A} M_B = \frac{n_B}{n_A} M_B = 2.8 M_B$$

Problem 164. Find the relation between forces P and Q acting on the jack in Fig. 374, whose parts are housed in the box K, if it

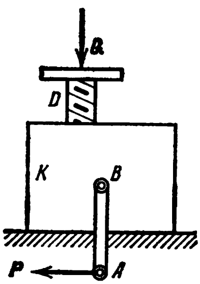

Fig. 374

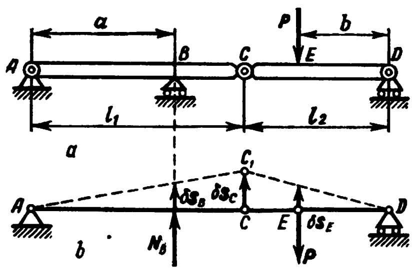

Fig. 375

is known that in one revolution of the crank handle $AB = l$ the screw D moves out by h.

Solution. From Eq. (110) we have:

$$Pl\delta\varphi_{AB} - Q\delta s_D = 0.$$

Assuming that when the handle is rotated uniformly the screw also moves up uniformly, we have:

$$\frac{\delta\varphi_{AB}}{2\pi} = \frac{\delta s_D}{h}, \quad \text{or} \quad \delta\varphi_{AB} = \frac{2\pi}{h} \delta s_D.$$

Substituting this expression for $\delta\varphi_{AB}$ into the foregoing equation, we obtain:

$$Q = \frac{2\pi l}{h} P.$$

Note that this simple problem cannot be solved by the methods of geometrical statics as the parts of the mechanism are unknown.

Problem 165. Two beams are hinged together at C and loaded as shown in Fig. 375*a*. Neglecting the weight of the beams, determine the pressure on support B.

Solution. Replace the support at B by a force N_B which is equal in magnitude to the required pressure (Fig. 375*b*). For a virtual displacement of the system Eq. (110) gives

$$N_B \delta s_B - P \delta s_E = 0.$$

The relation between δs_B and δs_E is found from the proportions

$$\frac{\delta s_B}{a} = \frac{\delta s_C}{l_1}, \quad \frac{\delta s_E}{b} = \frac{\delta s_C}{l_2},$$

whence

$$\delta s_E = \frac{bl_1}{al_2} \delta s_B,$$

and consequently

$$N_B = \frac{bl_1}{al_2} P.$$

If we were to use the methods of geometrical statics we should have to consider the equilibrium of each beam separately, introduce

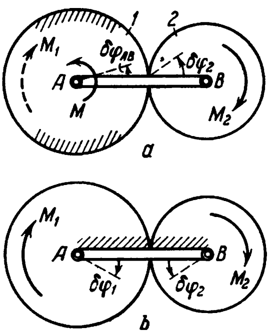

Fig. 376

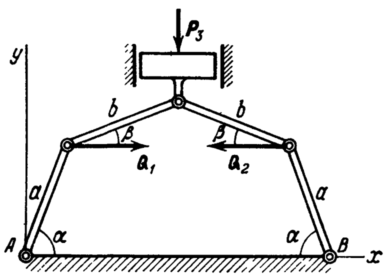

Fig. 377

the reactions of the other supports, and then eliminate them from the obtained set of equilibrium equations.

Problem 166. The epicyclic gear train in Fig. 376 (see also § 95) consists of a gear *1* of radius r_1, an arm AB mounted on axle A independently of the gear, and a gear *2* of radius r_2 mounted on the arm at B as shown. Acting on the arm is a torque M, and acting on the gears are resistance moments M_1 and M_2, respectively. Determine the values of M_1 and M_2 at which the mechanism is in equilibrium.

Solution. The mechanism has two degrees of freedom, since it has two possible independent displacements: the rotation of the arm AB when gear *1* is at rest, and the rotation of gear *1* when the arm is at rest. First consider a virtual displacement of the system in which gear *1* remains at rest (Fig. 376*a*). For this displacement Eq. (110)

gives

$$M\delta\varphi_{AB} - M_2\delta\varphi_2 = 0.$$

But when gear *1* is at rest, the contact point of the gears will be the instantaneous centre of zero velocity of gear *2*, and consequently $v_B = \omega_2 r_2$. At the same time $v_B = \omega_{AB}(r_1 + r_2)$. Hence, $\omega_2 r_2 = \omega_{AB}(r_1 + r_2)$, or $\delta\varphi_2 r_2 = \delta\varphi_{AB}(r_1 + r_2)$, and we obtain:

$$M_2 = \frac{r_2}{r_1 + r_2} M.$$

Now consider a virtual displacement in which the arm AB remains at rest (Fig. 376*b*). For this displacement Eq. (110) gives

$$M_1\delta\varphi_1 - M_2\delta\varphi_2 = 0.$$

But when the arm is at rest,

$$\frac{\delta\varphi_2}{\delta\varphi_1} = \frac{\omega_2}{\omega_1} = \frac{r_1}{r_2} \quad \text{and} \quad M_1 = \frac{r_1}{r_2} M_2.$$

We finally obtain:

$$M_1 = \frac{r_1}{r_1 + r_2} M, \quad M_2 = \frac{r_2}{r_1 + r_2} M.$$

Problem 167. Determine the relation between the forces Q_1 and Q_2 and the force P_3 when the press in Fig. 377 is in equilibrium ($Q_1 = Q_2 = Q$ and $P_3 = P$) if angles α and β are known. Neglect the weight of the rods.

Solution. To give an example of the analytical method of solution let us take equilibrium condition (111). Placing the origin of a coordinate system in the fixed point A and drawing axes x and y as shown, we obtain:

$$Q_{1x}\delta x_1 + Q_{2x}\delta x_2 + P_{3y}\delta y_3 = 0, \tag{a}$$

since all the other projections of the forces vanish.

To find δx_1, δx_2, δy_3, compute the coordinates x_1, x_2, y_3 of the points of application of the forces, expressing them in terms of the angles α and β. Denoting the lengths of the rods by a and b, we obtain:

$$x_1 = a\cos\alpha, \quad x_2 = a\cos\alpha + 2b\cos\beta, \quad y_3 = b\sin\beta + a\sin\alpha.$$

Differentiating, we find:

$$\delta x_1 = -a\sin\alpha\,\delta\alpha, \quad \delta x_2 = -(a\sin\alpha\,\delta\alpha + 2b\sin\beta\,\delta\beta),$$
$$\delta y_3 = b\cos\beta\,\delta\beta + a\cos\alpha\,\delta\alpha.$$

Substituting these expressions into equation (a) and taking into account that $Q_{1x} = Q$, $Q_{2x} = -Q$, and $P_{3y} = -P$, we have:

$$2Qb\sin\beta\,\delta\beta - P(b\cos\beta\,\delta\beta + a\cos\alpha\,\delta\alpha) = 0. \tag{b}$$

To find the relation between $\delta\alpha$ and $\delta\beta$ we make use of the fact that AB = const. Therefore, $2(a\cos\alpha + b\cos\beta)$ = const. Differentiating this equation, we obtain:

$$a\sin\alpha\,\delta\alpha + b\sin\beta\,\delta\beta = 0 \quad \text{and} \quad \delta\alpha = -\frac{b}{a}\frac{\sin\beta}{\sin\alpha}\delta\beta.$$

Substituting the expression for $\delta\alpha$ into equation (b) and taking into account that $\delta\beta \neq 0$, we have:

$$2Q\sin\beta - P(\cos\beta - \cot\alpha\sin\beta) = 0,$$

whence

$$P = \frac{2Q}{\cot\beta - \cot\alpha}.$$

At an angle β very close to α the pressure P will be very large.

§ 173. The General Equation of Dynamics

The principle of virtual displacements gives a general method for solving problems of statics. On the other hand, D'Alembert's principle makes it possible to employ the methods of statics in solving dynamical problems. It seems obvious that by combining these principles we can develop a general method for the solution of problems of dynamics.

Consider a system of material particles subjected to ideal constraints. If we add to all the particles subjected to active forces $\boldsymbol{F}_k^{\mathrm{a}}$ and reaction forces $\boldsymbol{N}_k$ the corresponding inertia forces $\boldsymbol{F}_k^{\mathrm{i}} = -m_k \boldsymbol{w}_k$, then by D'Alembert's principle the resulting force system will be in equilibrium. If we now apply the principle of virtual displacements, we obtain:

$$\sum \delta A_k^{\mathrm{a}} + \sum \delta A_k^{\mathrm{i}} + \sum \delta A_k^{\mathrm{r}} = 0.$$

But from Eq. (109) the last sum is zero, and we finally obtain:

$$\sum \delta A_k^{\mathrm{a}} + \sum \delta A_k^{\mathrm{i}} = 0. \qquad (113)$$

Equation (113) represents the general equation of dynamics. It states the following **principle of D'Alembert-Lagrange**: *In a moving system with ideal constraints the total virtual work done by all the active forces and all the inertia forces in any virtual displacement is zero at any instant.*

In analytical form Eq. (113) gives

$$\sum [(F_{kx}^{\mathrm{a}} + F_{kx}^{\mathrm{i}})\,\delta x_k + (F_{ky}^{\mathrm{a}} + F_{ky}^{\mathrm{i}})\,\delta y_k + (F_{kz}^{\mathrm{a}} + F_{kz}^{\mathrm{i}})\,\delta z_k] = 0. \qquad (114)$$

Equations (113) and (114) make it possible to develop the equations of motion for any mechanical system.

If a system consists of a number of rigid bodies, the relevant equations can be developed if to the active forces applied to each body are added a force equal to the principal vector of the inertia forces applied at any centre, and a couple with a moment equal to the principal moment of the inertia forces with respect to that centre. Then the principle of virtual work can be used.

Problem 168. A pendulum governor consists of two balls A_1 and A_2 of weight p each (Fig. 378). The slide C_1C_2 weighs Q, the governor rotates about the vertical axis with a uniform angular velocity ω. Neglecting the weight of the rods, determine angle α if $OA_1 = OA_2 = l$ and $OB_1 = OB_2 = B_1C_1 = B_2C_2 = a$.

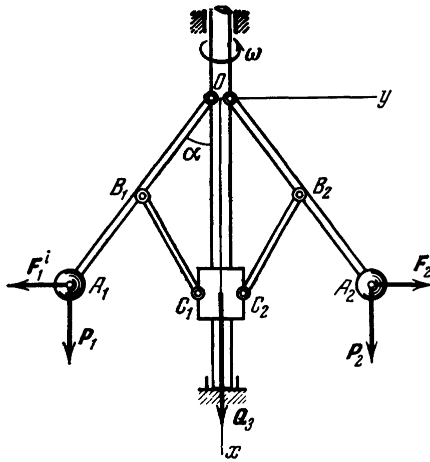

Fig. 378

Solution. Adding to the active forces p_1, p_2, and Q_3 the centrifugal inertia forces F_1^i and F_2^i (the inertia force of the slide will, evidently, be zero), we write the general equation of dynamics in the form (114). Computing the projections of all the forces on the coordinate axes, we have:

$$p_1\delta x_1 + p_2\delta x_2 - F_1^i\delta y_1 + F_2^i\delta y_2 + Q_3\delta x_3 = 0. \quad \text{(a)}$$

We also have:

$$Q_3 = Q, \quad p_1 = p_2 = p, \quad F_1^i = F_2^i = \frac{p}{g} w_A = \frac{p}{g}\omega^2 l \sin\alpha.$$

The coordinates of the points of application of the forces are

$$x_1 = x_2 = l\cos\alpha, \quad y_2 = -y_1 = l\sin\alpha, \quad x_3 = 2a\cos\alpha.$$

Differentiating these expressions, we find:

$$\delta x_1 = \delta x_2 = -l\sin\alpha\,\delta\alpha, \quad \delta y_2 = -\delta y_1 = l\cos\alpha\,\delta\alpha, \quad \delta x_3 = -2a\sin\alpha\,\delta\alpha.$$

Substituting all these expressions into equation (a), we obtain:

$$\left(-2pl\sin\alpha + 2\frac{p}{g}l^2\omega^2\sin\alpha\cos\alpha - 2Qa\sin\alpha\right)\delta\alpha = 0,$$

whence we finally have:

$$\cos\alpha = \frac{pl + Qa}{pl^2\omega^2}g.$$

As $\cos\alpha \leqslant 1$, the balls will move apart when

$$\omega^2 > \frac{pl + Qa}{pl^2}g.$$

Angle α increases with ω and tends to 90° when $\omega \to \infty$.

Problem 169. In the hoist mechanism in Fig. 379, a torque M is applied to gear *2* of weight P_2 and radius of gyration ρ_2. Determine the acceleration of the lifted load A of weight Q neglecting the weight of the string and the friction in the axles. The drum on which the string winds and the gear *1* attached rigidly to it have a total weight P_1 and a radius of gyration ρ_1. The radii of the gears are r_1 and r_2, and of the drum r.

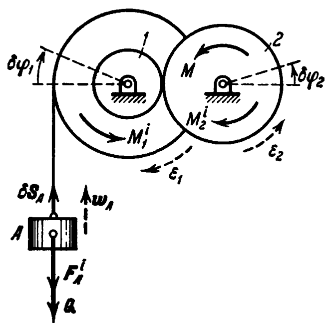

Fig. 379

Solution. Depict the active force Q and torque M (forces P_1 and P_2 do no work) and add to them the inertia force F_A^i of the load and the couples of moments M_1^i and M_2^i to which the inertia forces of the rotating bodies are reduced (see § 167). In magnitude these quantities are:

$$F_A^i = \frac{Q}{g} w_A, \quad |M_1^i| = \frac{P_1}{g} \rho_1^2 \varepsilon_1, \quad |M_2^i| = \frac{P_2}{g} \rho_2^2 \varepsilon_2.$$

The directions of all the vectors are shown in the diagram. Writing Eq. (113) for a virtual displacement of the system, we obtain:

$$-(Q + F_A^i)\, \delta s_A - M_1^i \delta\varphi_1 + (M - M_2^i)\, \delta\varphi_2 = 0.$$

Expressing all the displacements in terms of $\delta\varphi_1$, we have:

$$\delta s_A = r\delta\varphi_1, \quad \frac{\delta\varphi_2}{\delta\varphi_1} = \frac{\omega_2}{\omega_1} = \frac{r_1}{r_2}, \quad \text{or} \quad \delta\varphi_2 = \frac{r_1}{r_2} \delta\varphi_1.$$

Finally the equation of motion takes the form

$$Q\left(1 + \frac{w_A}{g}\right) r + \frac{P_1}{g} \rho_1^2 \varepsilon_1 + \frac{P_2}{g} \rho_2^2 \varepsilon_2 \frac{r_1}{r_2} - M \frac{r_1}{r_2} = 0.$$

Now express the quantities ε_1 and ε_2 in terms of the required acceleration w_A. Taking into account that ε_1 and ε_2 are related in the same way as ω_1 and ω_2, we obtain:

$$\varepsilon_1 = \frac{w_A}{r}, \quad \varepsilon_2 = \frac{r_1}{r_2} \varepsilon_1 = \frac{r_1}{r_2} \frac{w_A}{r},$$

and finally we have:

$$w_A = \frac{\frac{r_1}{r_2} M - rQ}{rQ + \frac{\rho_1^2}{r} P_1 + \frac{\rho_2^2}{r} \frac{r_1^2}{r_2^2} P_2} g.$$

An alternate solution of this problem is with the help of the theorem of the change in kinetic energy (see § 150).

Problem 170. One end of a thread is wound on a uniform cylinder of weight p_1 (Fig. 380). The thread passes over a pulley O, and its other end is attached to a load A of weight p_2 which slides on a horizontal plane, the coefficient of friction being f. Neglecting the mass of the pulley and the string, determine the acceleration of the load and of the centre C of the cylinder.

Solution. If motion starts from rest, the centre of the cylinder C will move vertically and the system has two degrees of freedom (the rotation of the cylinder with respect to the thread when the load is at rest and the displacement of the load when the cylinder does not rotate).

Add to the acting forces $\boldsymbol{p}_1$, $\boldsymbol{p}_2$, and $\boldsymbol{F}_{fr}$ the inertia forces of the cylinder reduced to a principal vector $\boldsymbol{R}_1^i$ and a couple with a moment M_C^i (see § 167), and the inertia force $\boldsymbol{F}_A^i$ of the load. In magnitude

$$F_A^i = \frac{p_2}{g} w_A, \quad R_1^i = \frac{p_1}{g} w_C, \quad |M_C^i| = $$
$$= J_C \varepsilon = \frac{p_1}{2g} r^2 \frac{w_C - w_A}{r}.$$

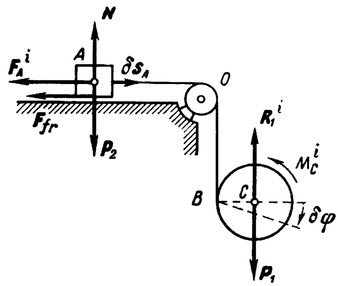

Fig. 380

The last equality follows from the fact that if point C of the cylinder has a velocity v_C and point B (together with the thread) a velocity $v_B = v_A$, then the angular velocity of the cylinder $\omega = \frac{v_C - v_A}{r}$ [see § 81, Eq. (56)], and, consequently, $\varepsilon = \frac{w_C - w_A}{r}$; furthermore, for a cylinder $J_C = 0.5mr^2$, where r is the radius of the cylinder.

Now consider a virtual displacement δs_A of the system in which the cylinder does not rotate and is translated together with the load. The moment M_C^i does no work in this displacement, and from Eq. (113) we obtain:

$$(-F_{fr} - F_A^i - R_1^i + p_1)\,\delta s_A = 0,$$

whence, as $F_{fr} = fp_2$, we find:

$$\frac{p_1}{g} w_C + \frac{p_2}{g} w_A = p_1 - fp_2. \tag{a}$$

Consider the other independent virtual displacement in which the load A remains at rest while the cylinder turns about point B (which in this displacement is the instantaneous centre of rotation) through angle $\delta\varphi$. For this displacement, Eq. (113) gives

$$(p_1 - R_1^i)\, r\delta\varphi - M_C^i \delta\varphi = 0.$$

Substituting the expressions for R_1^i and M_C^i, we obtain:

$$3w_C - w_A = 2g. \tag{b}$$

Solving equations (a) and (b) simultaneously, we obtain the required accelerations:

$$w_A = \frac{p_1 - 3fp_2}{p_1 + 3p_2} g, \quad w_C = \frac{p_1 + (2-f)\,p_2}{p_1 + 3p_2} g.$$

The result shows that the motion considered is possible when $f \leqslant p_1/3p_2$. If the coefficient of friction is greater, load A will remain at rest, the displacement δs_A will not be possible, and there will be no equation (a). The motion of the centre of the cylinder will be described by equation (b) if in the latter w_A is taken to be zero. Consequently, at $f > p_1/3p_2$ load A will be at rest while the centre C of the cylinder will fall with an acceleration $w_C = 2g/3$.

Note that for systems with more than one degree of freedom the method for developing equations described in § 150 is useless and the general equation of dynamics must be applied.

Chapter 32*

Equilibrium Conditions and Equations of Motion of a System in Generalised Coordinates

§ 174. Generalised Coordinates and Generalised Velocities

The number of coordinates (parameters) defining the configuration of a mechanical system depends on the number of particles (or bodies) comprising the system and on the number and type of the constraints imposed on it. We shall consider only systems with *geometrical constraints*, i.e., constraints that restrict only the configuration of the particles of the system in space, but not their velocities. The number of degrees of freedom of a system is, it will be recalled, given by the number of possible mutually independent displacements of the system (§ 170). Geometrical constraints identically reduce both the number of independent virtual displacements of a system and the number of mutually independent coordinates defining its configuration. For example, if a point B_k with coordinates x_k, y_k, z_k is connected with a fixed point A (x_A, y_A, z_A) by a rigid rod of length l (a geometrical constraint), the number of virtual displacements of the system is reduced by one, as displacement of the particle along AB_k becomes impossible. Simultaneously, the particle's coordinates will all the time satisfy the equation $(x_A - x_k)^2 + (y_A - y_k)^2 + (z_A - z_k)^2 = l^2$, which defines the constraint in mathematical terms; consequently, the number of mutually independent coordinates of the system will also decrease by one. We thus find that the number of independent coordinates defining the configuration of a system with geometrical constraints equals the number of degrees of freedom of that system. Any parameters of any dimension and of any geometrical (or physical) meaning can be chosen for such coordinates, notably sections of straight lines, arcs, angles, areas, etc.

Independent parameters of any dimension, the number of which is equal to the number of degrees of freedom and which uniquely define a system's configuration, are called the *generalised coordinates* of the system. We shall denote generalised coordinates by the symbol q. As a free particle has three degrees of freedom, a system comprising n material particles (whose coordinates must, by virtue of the geometrical constraints imposed on the system, satisfy k equations describing those constraints) has $s = 3n - k$ degrees of freedom, and its configuration is given by s generalised coordinates

$$q_1, q_2, \ldots, q_s. \tag{115}$$

Conversely, if it is established that a given configuration of a system is uniquely defined by any s mutually independent parameters, the system has s degrees of freedom.

Insofar as generalised coordinates are mutually independent, the elementary increments of those coordinates

$$\delta q_1, \delta q_2, \ldots, \delta q_s \tag{116}$$

are also mutually independent. And each of the quantities in (116) defines a corresponding virtual displacement of the system, which is independent of all other such displacements.

The same as in any transition from one coordinate system to another, the Cartesian coordinates x_k, y_k, z_k of any particle of a given

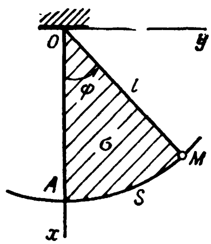

Fig. 381

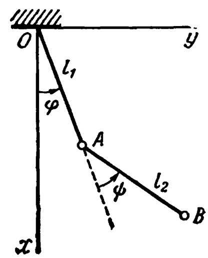

Fig. 382

physical system can be expressed in terms of the generalised coordinates by equations of the form $x_k = x_k(q_1, q_2, \ldots, q_s)$, etc. Consequently, for the particle's radius vector $\boldsymbol{r}_k$, defined by its projections, i.e., the coordinates x_k, y_k, z_k ($\boldsymbol{r}_k = x_k\boldsymbol{i} + y_k\boldsymbol{j} + z_k\boldsymbol{k}$), we have*):

$$\boldsymbol{r}_k = \boldsymbol{r}_k(q_1, q_2, \ldots, q_s). \tag{117}$$

Example 1. It is apparent that the plane simple pendulum in Fig. 381 has one degree of freedom ($s = 1$); consequently its configuration is determined by one generalised coordinate q. We can select this coordinate to be either angle φ or the length S of arc $\overset{\frown}{AM}$ or the area σ of sector OAM, always indicating the positive and negative

*) To simplify the equations we assume that the constraints do not alter with time (otherwise $\boldsymbol{r}_k$ would also depend on the argument t). This assumption does not affect the form of the final equations (§ 177), and they are also valid for constraints that change with time.

directions for each of these coordinates. It would be wrong to take the abscissa x of point M as the generalised coordinates as it does not give the position of M uniquely (for any given value x the pendulum may be deflected either to the right or to the left of the vertical).

If we take angle φ as the generalised coordinate, the virtual displacement of the pendulum can be obtained by giving the angle an increment $\delta\varphi$. The Cartesian coordinates x and y of point M can be expressed in terms of φ in the form $x = l \cos \varphi$, $y = l \sin \varphi$, where $l = OM$. Then, by Eq. (117), $\boldsymbol{r} = \boldsymbol{r}(\varphi)$.

Example 2. It is apparent that the plane double pendulum in Fig. 382 has two degrees of freedoms, and we can choose angles φ and ψ as the generalised coordinates ($q_1 = \varphi$, $q_2 = \psi$); the angles are mutually independent, as angle φ can be altered without altering angle ψ, and vice versa. The quantities $\delta\varphi$ and $\delta\psi$ give the mutually independent virtual displacements of the system. The Cartesian coordinates of points A and B in terms of the generalised coordinates are given by equations of the form $x_A = l_1 \cos \varphi$, $x_B = l_1 \cos \varphi + l_2 \cos(\varphi + \psi)$, etc., where $l_1 = OA$ and $l_2 = AB$. Consequently, by Eq. (117), $\boldsymbol{r}_A = \boldsymbol{r}_A(\varphi)$, $\boldsymbol{r}_B = \boldsymbol{r}_B(\varphi, \psi)$.

In motion the generalised coordinates change continuously with time, and the law of motion is given by the equations:

$$q_1 = f_1(t),\ q_2 = f_2(t),\ \ldots,\ q_s = f_s(t). \tag{118}$$

Equations (118) are the *kinematic equations of motion of a system in generalised coordinates.*

The derivatives of the generalised coordinates with respect to time are called the *generalised velocities* of the system. We shall denote generalised velocities by the symbols

$$\dot{q}_1,\ \dot{q}_2,\ \ldots,\ \dot{q}_s,$$

where $\dot{q}_1 = \frac{dq_1}{dt}$, etc. The dimension of generalised velocity depends on the dimension of the corresponding generalised coordinate. If q is a linear quantity, $\dot{q}$ is linear velocity; if q is an angle, $\dot{q}$ is angular velocity; if q is an area, $\dot{q}$ is sectoral velocity, etc. Thus, the concept of generalised velocity embraces all concepts of velocity previously encountered in kinematics.

§ 175. Generalised Forces

Consider a physical system of n material particles subject to forces $\boldsymbol{F}_1, \boldsymbol{F}_2, \ldots, \boldsymbol{F}_n$. Let the system have s degrees of freedom and its configuration be given by the generalised coordinates (115), and let us impart it an independent virtual displacement such that coordinate q_1 receives an increment δq_1 and the other coordinates remain unchanged. Then each of the radius vectors $\boldsymbol{r}_k$ of the system's particles will receive an elementary increment $(\delta \boldsymbol{r}_k)_1$ *). As by Eq. (117) $\boldsymbol{r}_k = \boldsymbol{r}_k (q_1, q_2, \ldots, q_s)$, and as only coordinate q_1 changes in the given displacement (the others remain constant), $(\delta \boldsymbol{r}_k)_1$ is computed as a *partial differential* and, consequently,

$$(\delta \boldsymbol{r}_k)_1 = \frac{\partial \boldsymbol{r}_k}{\partial q_1} \delta q_1. \tag{119}$$

Now let us compute the sum of the elementary works done by all acting forces in the stated displacement, denoting it by the symbol δA_1. Applying Eq. (36′) from § 112 and Eq. (119), we obtain:

$$\delta A_1 = \boldsymbol{F}_1 \cdot (\delta \boldsymbol{r}_1)_1 + \boldsymbol{F}_2 \cdot (\delta \boldsymbol{r}_2)_1 + \ldots + \boldsymbol{F}_n \cdot (\delta \boldsymbol{r}_n)_1 =$$

$$= \boldsymbol{F}_1 \cdot \frac{\partial \boldsymbol{r}_1}{\partial q_1} \delta q_1 + \boldsymbol{F}_2 \cdot \frac{\partial \boldsymbol{r}_2}{\partial q_1} \delta q_1 + \ldots + \boldsymbol{F}_n \cdot \frac{\partial \boldsymbol{r}_n}{\partial q_1} \delta q_1.$$

(The dot here is a symbol of scalar multiplication of two vectors.) Taking the common multiplier δq_1 outside the parentheses, we finally obtain:

$$\delta A_1 = Q_1 \delta q_1, \tag{120}$$

where

$$Q_1 = \sum \boldsymbol{F}_k \cdot \frac{\partial \boldsymbol{r}_k}{\partial q_1}. \tag{121}$$

By analogy with the equation $\delta A = F_\tau \delta s$, which gives the elementary work done by force $\boldsymbol{F}$, the quantity Q_1 is called the *generalised force* corresponding to coordinate q_1.

Imparting to the system another independent virtual displacement in which only coordinate q_2 changes, we obtain the following expression for the elementary work done by all active forces in that displacement:

$$\delta A_2 = Q_2 \delta q_2, \tag{122}$$

where

$$Q_2 = \sum \boldsymbol{F}_k \cdot \frac{\partial \boldsymbol{r}_k}{\partial q_2}. \tag{123}$$

The quantity Q_2 is the generalised force corresponding to coordinate q_2, etc.

Apparently, if the system is given an elementary displacement in which all its generalised coordinates change simultaneously, the sum of the elementary works done by the applied forces in that displacement will be given by the equation

$$\sum \delta A_k = Q_1 \delta q_1 + Q_2 \delta q_2 + \ldots + Q_s \delta q_s. \tag{124}$$

Equation (124) gives the *total elementary work* done by all forces acting on a system in generalised coordinates. It can be observed

*) The symbol $(\delta \boldsymbol{r}_k)_1$ indicates that the elementary increment of the radius vector $\boldsymbol{r}_k$ is the result only of the change of coordinate q_1 by the value δq_1.

from the equation that generalised forces are quantities equal to the coefficients before the increments of the generalised coordinates in the expression for the total elementary work done by the forces acting on the system.

If all the constraints imposed on a system are ideal, the work in the virtual displacements is done only by the active forces, and the quantities $Q_1, Q_2, \ldots, Q_s$ represent the *generalised active forces* of the system.

The dimension of a generalised force depends on the respective generalised coordinate. As the product $Q\delta q$, and consequently Qq, has the dimension of work,

$$[Q]=\frac{[A]}{[q]}, \qquad (125)$$

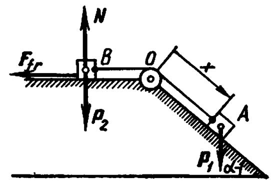

Fig. 383

i.e., the dimension of generalised force equals the dimension of work divided by the dimension of the corresponding generalised coordinate. Hence, if q is a linear quantity, Q has the dimension of ordinary force (kgf in the mkg(f)s system); if q is an angle (a nondimensional quantity), Q is measured in kgf-m, i.e., it has the dimension of moment; if q is volume (e.g., the position of a piston in a cylinder can be defined in terms of the volume of the space back of the piston), Q is measured in kgf/m^2, i.e., the dimension of pressure, etc. We thus see that, by analogy with generalised velocity, the concept of generalised force includes all quantities examined before as measures of the mechanical interactions of material bodies (force, moment, pressure).

Generalised forces are computed with the help of equations of the form (120) and (122) used to compute the elementary virtual work (see § 172). First it is necessary to establish the number of degrees of freedom of the system, to select the generalised coordinates, and to depict all the active forces applied to the system, as well as the friction forces (if they do work). Then, to determine Q_1, give the system a virtual displacement in which only coordinate q_1 changes, compute from Eqs. (112) the sum of the elementary works done in that displacement by all acting forces, and represent the obtained expression in the form (120). The factor for δq_1 gives the required value of Q_1. $Q_2, Q_3, \ldots$, etc. are computed similarly.

Example 1. Compute the generalised force for the system in Fig. 383, where load A of weight P_1 moves along a smooth inclined plane, and load B of weight P_2 moves along a rough horizontal surface with coefficient of friction f. The loads are joined by a thread passing over pulley O; the masses of the thread and pulley can be neglected. Obviously, the system has one degree of freedom and its configuration is given by the coordinate $q_1 = x$ (the positive direction of x is indicated by an arrow). To determine Q_1, give the system a virtual displacement δx and compute the elementary work done in that displacement by forces $\boldsymbol{P}_1$ and $\boldsymbol{F}_{fr}$ (the other forces do no work). As $F_{fr} = fN = fP_2$,

$$\delta A = (P_1 \sin\alpha - fP_2)\,\delta x.$$

Consequently,

$$Q_1 = P_1 \sin\alpha - fP_2.$$

Example 2. Neglecting friction, determine the generalised forces for the system depicted in Fig. 384: a homogeneous rod AB of length l and weight P can rotate in the vertical plane about a pin at A. Sliding on the rod is a bead M of weight p and a spring AM whose relaxed length is a and whose stiffness is c.

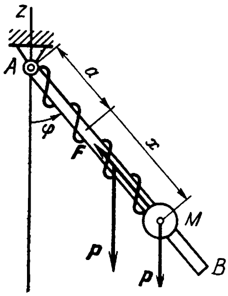

Fig. 384

The system has, obviously, two degrees of freedom (the displacement of the bead along the rod and rotation of the rod about A are independent). Take angle φ and the distance x of the bead from the end of the relaxed spring as the generalised coordinates; the positive directions of the coordinates are indicated by the arrows.

First give the system a virtual displacement in which φ receives an increment $\delta\varphi$ ($\delta\varphi > 0$) and $x = \text{const}$. In this displacement work is done by forces $\boldsymbol{P}$ and $\boldsymbol{p}$. From the second of Eqs. (112) we find:

$$\delta A_1 = \left[-P\frac{l}{2}\sin\varphi - p(a+x)\sin\varphi\right]\delta\varphi.$$

(The "minus" because the moment is directed opposite to $\delta\varphi$.) Consequently,

$$Q_1 = -\left[P\frac{l}{2} + p(a+x)\right]\sin\varphi.$$

Now give the system a virtual displacement in which only coordinate x changes, receiving an increment $\delta x > 0$, and $\varphi = \text{const}$. In this displacement work is done by the force of gravity $\boldsymbol{p}$ and the elastic force of magnitude $F = cx$. Then,

$$\delta A_2 = (p\cos\varphi - cx)\,\delta x,$$

and

$$Q_2 = p\cos\varphi - cx.$$

As $q_1 = \varphi$, the generalised force Q_1 has the dimension of moment, and force Q_2 has the dimension of conventional force.

Conservative Forces. As we know, if all the forces acting on a system are conservative, there exists a *force function* U, dependent on the coordinates x_k, y_k, z_k of the particles of the system, such that the sum of the elementary works done by the acting forces equals the full differential of the function, i.e., $\sum \delta A_k = \delta U$ [§ 151, Eq. (62)]. But in going over to the generalised coordinates $q_1, q_2, \ldots, q_s$, all the x_k, y_k, z_k coordinates can be expressed in terms of the generalised coordinates, whence $U = U(q_1, q_2, \ldots, q_s)$. Consequently, computing δU as the full differential with respect to the function $U(q_1, q_2, \ldots, q_s)$, we obtain:

$$\sum \delta A_k = \delta U = \frac{\partial U}{\partial q_1}\delta q_1 + \frac{\partial U}{\partial q_2}\delta q_2 + \ldots + \frac{\partial U}{\partial q_s}\delta q_s.$$

Comparing this expression with Eq. (124), we conclude that in this case

$$Q_1 = \frac{\partial U}{\partial q_1}, \quad Q_2 = \frac{\partial U}{\partial q_2}, \quad \ldots, \quad Q_s = \frac{\partial U}{\partial q_s}, \tag{126}$$

or, since the potential energy $V = -U$,

$$Q_1 = -\frac{\partial V}{\partial q_1}, \quad Q_2 = -\frac{\partial V}{\partial q_2}, \quad \ldots, \quad Q_s = -\frac{\partial V}{\partial q_s}. \tag{127}$$

Consequently, if all the forces acting on a system are conservative, the generalised forces equal the partial derivatives of the force function (or the partial derivatives of the potential energy taken with the opposite sign) with respect to the corresponding generalised coordinates.

Example 3. All the forces acting on the system in Fig. 384 are conservative. Directing the coordinate axis Az vertically up and taking into account that $V = -U$, for the whole system we obtain, by Eqs (59) and (59'):

$$V = -P\frac{l}{2}\cos\varphi - p(a+x)\cos\varphi + \frac{1}{2}cx^2,$$

where the generalised coordinates $q_1 = \varphi$, $q_2 = x$. Then,

$$Q_1 = -\frac{\partial V}{\partial \varphi} = -\left[P\frac{l}{2} + p(a+x)\right]\sin\varphi, \quad Q_2 = -\frac{\partial V}{\partial x} = p\cos\varphi - cx,$$

which coincides with the result obtained in Example 2.

§ 176. Equilibrium Conditions for a System in Generalised Coordinates

According to the principle of virtual displacements, the necessary and sufficient condition for a mechanical system to be in equilibrium is for the sum of the elementary works done by all the active forces (including the forces of friction, if they do work) in any virtual displacement of the system to be zero, i.e., $\sum \delta A_k = 0$. In generalised coordinates this condition, by Eq. (124), yields:

$$Q_1\delta q_1 + Q_2\delta q_2 + \ldots + Q_s\delta q_s = 0. \qquad (128)$$

As all the quantities $\delta q_1, \delta q_2, \ldots, \delta q_s$ are mutually independent, Eq. (128) can be satisfied only if each of the factors for $\delta q_1, \delta q_2, \ldots, \delta q_s$ is separately zero, i.e.,

$$Q_1 = 0,\ Q_2 = 0,\ \ldots,\ Q_s = 0. \qquad (129)$$

For, if one of these quantities, for instance Q_1, is not zero, the system can be imparted a virtual displacement in which $\delta q_1 \neq 0$, while $\delta q_2 = \delta q_3 = \ldots = \delta q_s = 0$, which contradicts the condition (128).

Thus, for a mechanical system to be in equilibrium it is necessary and sufficient for all the generalised forces corresponding to the generalised coordinates selected for the system to be zero. The number of equilibrium conditions (129) is, we see, equal to the number of generalised coordinates, i.e., to the number of degrees of freedom of the system.

Comparing the method of computing generalised forces described in § 175 with the principle of virtual displacements used in § 172, we see that the latter essentially involved computations of the respective generalised forces, which were then equated to zero.

Let us consider two more examples.

Example 1. The equilibrium condition for the system in Fig. 383 is $Q_1 = 0$, or $P_1 = fP_2 \csc\alpha$. Since in computing Q_1 it was assumed that $F_{fr} = fN = F_l$, the condition $Q_1 = 0$ gives the highest value of P_1 at which load A does not descend, i.e., it defines the limiting equilibrium position (see § 39). The system will also be in equilibrium at $P_1 < fP_2 \csc\alpha$.

Example 2. For the system in Fig. 384, the equilibrium conditions $Q_1 = 0$ and $Q_2 = 0$ yield the obvious result: in equilibrium $\varphi = 0$, $x = p/c = \delta_{st}$.

Conservative Forces. In this case the equilibrium conditions (129), taking into account Eqs. (126) and (127), yield

$$\frac{\partial U}{\partial q_1} = 0,\ \frac{\partial U}{dq_2} = 0,\ \ldots,\ \frac{\partial U}{\partial q_s} = 0, \qquad (130)$$

or

$$\frac{\partial V}{\partial q_1} = 0,\ \frac{\partial V}{\partial q_2} = 0,\ \ldots,\ \frac{\partial V}{\partial q_s} = 0. \qquad (130')$$

It follows that in equilibrium the total differential of the functions U and V is zero, i.e.,

$$dU(q_1, q_2, \ldots, q_s) = 0 \quad \text{and} \quad dV(q_1, q_2, \ldots, q_s) = 0. \qquad (131)$$

Thus, a system acted upon by conservative forces is in equilibrium in those configurations in which the force function or potential energy of the system have limiting values (in particular, minimum or maximum).

§ 177. Lagrange's Equations

To establish the equations of motion for a mechanical system with geometrical constraints in terms of generalised coordinates, we refer to the general equation of dynamics (113), which yields:

$$\sum \delta A_k + \sum \delta A_k^{\mathrm{i}} = 0. \tag{132}$$

For extension to general cases, we shall not assume all the constraints of the system to be ideal, hence the first summation may include work done by active forces as well as, for example, work done by friction forces.

Let a system have s degrees of freedom and its configuration be given by the generalised coordinates (115). Then, by Eq. (124),

$$\sum \delta A_k = Q_1\delta q_1 + Q_2\delta q_2 + \ldots + Q\,\delta q_s. \tag{133}$$

Obviously, as in the case of the forces $\boldsymbol{F}_k$ in § 175, the elementary work done by the inertia forces $\boldsymbol{F}_k^{\mathrm{i}}$ can be expressed in terms of generalised coordinates yielding

$$\sum \delta A_k^{\mathrm{i}} = Q_1^{\mathrm{i}}\delta q_1 + Q_2^{\mathrm{i}}\delta q_2 + \ldots + Q_s^{\mathrm{i}}\delta q_s, \tag{133'}$$

where $Q_1^{\mathrm{i}}, Q_2^{\mathrm{i}}, \ldots, Q_s^{\mathrm{i}}$ are the *generalised inertia forces*, which according to Eqs. (121) and (123) are:

$$Q_1^{\mathrm{i}} = \sum \boldsymbol{F}_k^{\mathrm{i}}\cdot\frac{\partial \boldsymbol{r}_k}{\partial q_1},\quad Q_2^{\mathrm{i}} = \sum \boldsymbol{F}_k^{\mathrm{i}}\cdot\frac{\partial \boldsymbol{r}_k}{\partial q_2},\quad \ldots,\quad Q_s^{\mathrm{i}} = \sum \boldsymbol{F}_k^{\mathrm{i}}\cdot\frac{\partial \boldsymbol{r}_k}{\partial q_s}. \tag{134}$$

Substituting the quantities (133) and (133') into Eq. (132), we find that

$$(Q_1 + Q_1^{\mathrm{i}})\,\delta q_1 + (Q_2 + Q_2^{\mathrm{i}})\,\delta q_2 + \ldots + (Q_s + Q_s^{\mathrm{i}})\,\delta q_s = 0.$$

As $\delta q_1, \delta q_2, \ldots, \delta q_s$ are mutually independent, the obtained equation can be satisfied only if each of the factors of $\delta q_1, \delta q_2, \ldots, \delta q_s$ is separately zero and, reasoning as in the case of equations (129), we accordingly must have:

$$Q_1 + Q_1^{\mathrm{i}} = 0,\quad Q_2 + Q_2^{\mathrm{i}} = 0,\quad \ldots,\quad Q_s + Q_s^{\mathrm{i}} = 0. \tag{135}$$

These equations can be applied directly to the solution of problems of dynamics. However, they can be simplified if all the generalised inertia forces are expressed in terms of the system's kinetic energy. First transform the quantity Q_1^{i}. As the inertia force of any particle of the system $F_k^i = -m_k w_k = -m_k \frac{dv_k}{dt}$, the first of Eqs. (134) yields

$$-Q_1^i = \sum m_k \frac{dv_k}{dt} \cdot \frac{\partial r_k}{\partial q_1}. \tag{136}$$

To express Q_1^i in terms of the kinetic energy of the system, the right-hand part of Eq. (136) must be transformed so as to include only the velocities v_k of the system's particles. For this, first note that

$$\frac{dv_k}{dt} \cdot \frac{\partial r_k}{\partial q_1} = \frac{d}{dt}\left(v_k \cdot \frac{\partial r_k}{\partial q_1}\right) - v_k \cdot \frac{d}{dt}\left(\frac{\partial r_k}{\partial q_1}\right). \tag{137}$$

The validity of Eq. (137) can be easily verified by differentiating the product in the parentheses on the right. Also, take into account that

$$\frac{dr_k}{dt} = \dot{r}_k = v_k \text{ and } \frac{dq_1}{dt} = \dot{q}_1,$$

where v_k is the velocity of a particle with the radius vector r_k and $\dot{q}_1$ is the generalised velocity corresponding to coordinate q_1. Then for the derivatives with respect to r_k in Eq. (137) the following two results are valid.

(1) The operations of total differentiation with respect to t and partial differentiation with respect to q_1 are commutative which yields:

$$\frac{d}{dt}\left(\frac{\partial r_k}{\partial q_1}\right) = \frac{\partial}{dq_1}\left(\frac{dr_k}{dt}\right) = \frac{\partial v_k}{\partial q_1}. \tag{138}$$

(2) The partial derivative of r_k with respect to q_1 is the limit of the ratio of the partial increment $(\Delta r_k)_1$ to the increment Δq_1, whence, in accordance with the known L'Hospital rule*):

$$\frac{dr_k}{\partial q_1} = \frac{\partial \dot{r}_k}{\partial \dot{q}_1} = \frac{\partial v_k}{\partial \dot{q}_1}. \tag{139}$$

Using equations (138) and (139), Eq. (137) can be represented in the form

$$\frac{dv_k}{dt} \cdot \frac{\partial r_k}{\partial q_1} = \frac{d}{dt}\left(v_k \cdot \frac{\partial v_k}{\partial \dot{q}_1}\right) - \left(v_k \cdot \frac{\partial v_k}{\partial q_1}\right) = \frac{d}{dt}\left(\frac{1}{2}\frac{\partial v_k^2}{\partial \dot{q}_1}\right) - \frac{1}{2}\frac{\partial v_k^2}{\partial q_1}.$$

*) For, denoting for brevity the partial increment of r_k by the symbol Δr_k and taking into account that the derivative of a difference equals the difference between the derivatives, we have:

$$\frac{\partial r_k}{\partial q_1} = \lim \frac{\Delta r_k}{\Delta q_1} = \lim \frac{\frac{d(\Delta r_k)}{dt}}{\frac{d(\Delta q_1)}{dt}} = \lim \frac{\Delta\left(\frac{dr_k}{dt}\right)}{\Delta\left(\frac{dq_1}{dt}\right)} = \lim\left(\frac{\Delta \dot{r}_k}{\Delta \dot{q}_1}\right) = \frac{\partial \dot{r}_k}{\partial \dot{q}_1}.$$

Then, remembering that mass is a constant quantity and the sum of derivatives equals the derivative of a sum, expression (136) takes the form

$$-Q_1^i = \frac{d}{dt}\left[\frac{\partial}{\partial \dot{q}_1}\left(\sum \frac{m_k v_k^2}{2}\right)\right] - \frac{\partial}{\partial q_1}\left(\sum \frac{m_k v_k^2}{2}\right) = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_1}\right) - \frac{\partial T}{\partial q_1},$$

where

$$T = \sum \frac{m_k v_k^2}{2}$$

is the kinetic energy of the system.

Similar expressions are obtained for all the other generalised inertia forces, and Eqs. (135) finally yield

$$\left.\begin{aligned} &\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_1}\right) - \frac{\partial T}{\partial q_1} = Q_1, \\ &\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_2}\right) - \frac{\partial T}{\partial q_2} = Q_2, \\ &\dots\dots\dots\dots\dots \\ &\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_s}\right) - \frac{\partial T}{\partial q_s} = Q_s. \end{aligned}\right\} \qquad (140)$$

Eqs. (140) are *the differential equations of motion of a system in generalised coordinates, or Lagrange's equations.* We see that the number of equations equals the number of degrees of freedom of the system.

Lagrange's equations offer a uniform and fairly simple method of solving problems of dynamics. An important advantage is that their form and number depend neither on the number of particles (or bodies) comprising the system, nor on the manner in which they are moving; the number of equations depends only on the number of degrees of freedom. Furthermore, in the case of ideal constraints, the right-hand parts of Eqs. (140) include only *generalised active forces* and, consequently, the unknown reactions of the constraints need not be considered.

The principal problem of dynamics in generalised coordinates is as follows: knowing the generalised forces $Q_1, Q_2, \ldots, Q_s$ and the initial conditions, determine the law of motion of the system in the form (118), i.e., determine the generalised coordinates $q_1, q_2, \ldots, q_s$ as functions of time. As the kinetic energy T depends on the generalised velocities $\dot{q}_i$, differentiation of the first members in the left-hand parts of Eqs. (140) with respect to t yields the second time-derivatives $\ddot{q}_i$ of the required coordinates. Consequently, Lagrange's equations are ordinary differential equations of the second order with respect to the generalised coordinates $q_1, q_2, \ldots, q_s$.

Conservative Forces. If all the forces acting on a system are conservative, the first of Eqs. (140), using Eqs. (127), can be represented in the form

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_1}\right)-\frac{\partial T}{\partial q_1}+\frac{\partial V}{\partial q_1}=0, \text{ or } \frac{d}{dt}\left[\frac{\partial (T-V)}{\partial \dot{q}_1}\right]-\frac{\partial (T-V)}{\partial q_1}=0.$$

The last equation is valid because potential energy depends only on the coordinates $q_1, q_2, \ldots, q_s$ and does not depend on the generalised velocities: $\partial V/\partial \dot{q}_1 = 0$.

All the other of Eqs. (140) transform similarly. Let us introduce the function

$$L = T - V. \tag{141}$$

The function L of the generalised coordinates and generalised velocities is equal to the difference between the system's kinetic and potential energies; it is called *Lagrange's function* or the *kinetic potential.* Then, in the case of conservative forces, Lagrange's equations take the form

$$\left.\begin{array}{l}\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_1}\right)-\frac{\partial L}{\partial q_1}=0,\\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_2}\right)-\frac{\partial L}{\partial q_2}=0,\\ \cdots\cdots\cdots\cdots\\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_s}\right)-\frac{\partial L}{\partial q_s}=0.\end{array}\right\} \tag{142}$$

It follows that the state of a mechanical system subject only to conservative forces is described by Lagrange's function and knowing it one can write the differential equations of motion of the system.

By appropriate generalisations of concepts, functions analogous to Lagrange's can be used to describe the state of other physical systems (a continuous medium, gravitational or electromagnetic field, etc.). That is why Lagrange's equations of the form (142) play an important part in a number of fields in physics.

§ 178. Solution of Problems

Lagrang's equations can be used to investigate the motion of any mechanical system with geometrical constraints, regardless of the number of bodies (or particles), of how the bodies are moving, or of the type of motion (absolute or relative).

To write Lagrange's equations for any given mechanical system one must: (1) determine the number of degrees of freedom and select the generalised coordinates (§ 174); (2) depict the system in an arbitrary configuration and draw all the acting forces (for a system with ideal constraints, only the active forces); (3) compute the generalised forces Q_i as described in § 175 (to prevent mistakes in the signs, each virtual displacement of the system should be such that the increment of the respective coordinate is positive); (4) compute the system's kinetic energy T of its absolute motion and express it *in terms of the generalised coordinates* q_i *and the generalised velocities* $\dot{q}_i$; and (5) compute the respective partial derivatives of T with respect to q_i and $\dot{q}_i$ and substitute all the computed quantities into Eqs. (140).

Knowing the acting forces and initial conditions and integrating the obtained equations, we can find the law of motion of the system in the form (118). If the law of motion is given, the equations can be used to determine the acting forces.

When all the forces acting on a system are conservative, Lagrange's equations can be written in the form (142). Then, instead of computing the generalised forces, determine the potential energy of the system, expressing it in terms of the generalised coordinates, and, after computing the kinetic energy, write Lagrange's function (141).

To emphasise the universal character of Lagrange's function, let us use it in several problems solved before by other methods.

Problem 171. Use the Lagrange method to write the differential equation of the oscillations of a compound pendulum (§ 155).

Solution. The pendulum has one degree of freedom, and its configuration is determined by angle φ (see Fig. 333). Consequently, $q_1 = \varphi$. Giving angle φ a *positive* increment $\delta\varphi$ we find that work in that displacement is done only by the force of gravity $\boldsymbol{P}$, and $\delta A_1 = (-Pa \sin \varphi)\, \delta\varphi$, where $a = OC$. Hence, $Q_1 = -Pa \sin \varphi$. By Eq. (43), the kinetic energy of the pendulum $T = \frac{1}{2} J_O \omega^2$, or $T = \frac{1}{2} J_O \dot{\varphi}^2$ (remember that T should be expressed in terms of the generalised velocity, and $\omega = \dot{\varphi}$). Lagrange's equation takes the form

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\varphi}}\right) - \frac{\partial T}{\partial \varphi} = Q_1. \qquad \text{(a)}$$

As T does not depend on angle φ, solving the equation yields

$$\frac{\partial T}{\partial \varphi} = 0, \quad \frac{\partial T}{\partial \dot{\varphi}} = J_O \dot{\varphi}, \quad \text{and} \quad \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\varphi}}\right) = J_O \ddot{\varphi}.$$

Substituting the computed quantities into equation (a), we obtain:

$$J_O \ddot{\varphi} = -Pa \sin \varphi,$$

i.e., the same result as in § 155.

As the force of gravity P is a conservative force, Lagrange's equation can be written in the form (142). Directing axis Oz vertically downward, we have $V = -Pz = -Pa\cos\varphi$, and by Eq. (141),

$$L = \frac{1}{2} J_O \dot{\varphi}^2 + Pa\cos\varphi,$$

whence

$$\frac{\partial L}{\partial \dot{\varphi}} = J_O\dot{\varphi}, \quad \frac{\partial L}{\partial \varphi} = -Pa\sin\varphi,$$

and Eq. (142) also yields $J_O\ddot{\varphi} + Pa\sin\varphi = 0$.

Problem 172. Use Lagrange's equations to solve Problem 141 (§ 150).

Solution. The mechanism has one degree of freedom (see Fig. 327), and its configuration is given by the coordinate φ ($q_1 = \varphi$). Giving angle φ an increment $\delta\varphi$ we find that, substituting $\delta\varphi$ for $d\varphi$, the elementary work done in that displacement has the same expression as that of dA^i in Problem 141. Consequently,

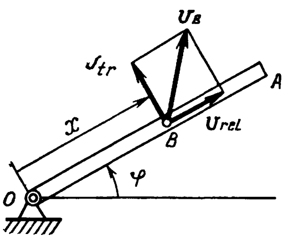

Fig. 385

$$Q_1 = -c\frac{(l-r)^2}{r^2}\varphi.$$

The quantity T for the mechanism was also computed [equation (b) in Problem 141]. Taking into account that $\omega_{cr} = \dot{\varphi}$, we have:

$$T = \frac{1}{12g}(2Q + 9P)\, l^2\dot{\varphi}^2,$$

whence

$$\frac{\partial T}{\partial \varphi} = 0, \quad \frac{\partial T}{\partial \dot{\varphi}} = \frac{1}{6g}(2Q + 9P)\, l^2\dot{\varphi}.$$

Substituting the computed quantities into Lagrange's equation, which has the form of equation (a) in Problem 171, we obtain:

$$\frac{1}{6g}(2Q + 9P)\, l^2\ddot{\varphi} = -c\frac{(1-r)^2}{r^2}\varphi,$$

or $\ddot{\varphi} + k^2\varphi = 0$, the same as in Problem 141.

Note that, *for a system with one degree of freedom, writing the differential equation of motion by the Lagrange method essentially involves the same computations as when applying the theorem of the change in kinetic energy.*

Problem 173. Determine the law of motion of a bead B moving inside a tube OA rotating uniformly in a horizontal plane with an angular velocity ω (Fig. 385). At the initial moment the bead is at distance a from the pin O and its velocity along the tube is zero.

Solution. This simple problem offers an example of how to write the equations of relative motion with the help of Lagrange's equations without introducing the transport or Coriolis inertia forces. The position of the bead in its relative motion along the tube is given by the single coordinate x. Consequently, the motion is described by the single Lagrange equation

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right)-\frac{\partial T}{\partial x}=Q_1. \tag{a}$$

The acting forces do no work in the displacement in which x receives the increment δx; consequently, $\delta A_1 = 0$ and $Q_1 = 0$.

We compute the kinetic energy of the bead in its *absolute motion*. Then $T = 0.5\, mv_B^2$, where v_B is the absolute velocity of the bead; in vector form, $\boldsymbol{v}_B = \boldsymbol{v}_{rel} + \boldsymbol{v}_{tr}$. In the present case, $v_{rel} = \dot{x}$, $v_{tr} = OB \cdot \omega = x\omega$ and $\boldsymbol{v}_{rel}$ and $\boldsymbol{v}_{tr}$ are mutually perpendicular. Consequently,

$$T=\frac{1}{2}m(\dot{x}^2+\omega^2x^2),$$

whence

$$\frac{\partial T}{\partial \dot{x}}=m\dot{x}, \quad \frac{\partial T}{\partial x}=m\omega^2x.$$

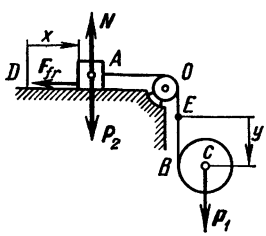

Fig. 386

Substituting the computed values into equation (a) and cancelling m, we obtain the differential equation of the bead's relative motion in the form

$$\ddot{x}-\omega^2x=0.$$

Integrating and determining the integration constants according to the initial conditions of the problem (at $t = 0$ we have $x = a$ and $\dot{x} = 0$), we finally obtain the law of motion of the bead along the tube $x=a\,(e^{\omega t}+e^{-\omega t})/2$.

Problem 174. Solve Problem 170 (§ 173) with the help of Lagrange's equations.

Solution. It was established that, given the stated conditions, the system has two degrees of freedom. Let the generalised coordinates be the distance x of load A from some point D on the plane (Fig. 386) and the distance y of the centre C of the cylinder from some point (a knot) E on the thread ($q_1 = x$, $q_2 = y$). Lagrange's equations for the system are:

$$\left.\begin{aligned}\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right)-\frac{\partial T}{\partial x}=Q_1,\\ \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{y}}\right)-\frac{\partial T}{\partial y}=Q_2.\end{aligned}\right\} \tag{a}$$

Denote the respective forces in the drawing and compute Q_1 and Q_2. First give the system a virtual displacement in which only the x coordinate changes, receiving an increment $\delta x > 0$, and $y = \text{const.}$ In this displacement work is done by forces $\boldsymbol{p}_1$ and $\boldsymbol{F}_{fr}$ ($F_{fr} = fp_2$), and $\delta A_1 = (p_1 - fp_2)\,\delta x$. Then give the system a virtual displacement in which only the y coordinate changes, receiving an increment $\delta y > 0$, and $x = \text{const.}$; in this displacement $\delta A_2 = p_1 \delta y$. Consequently,

$$Q_1 = (p_1 - fp_2) \quad \text{and} \quad Q_2 = p_1. \tag{b}$$

Now compute the kinetic energy T of the system, which is equal to the sum of the energies of the load and the cylinder: $T = T_{\text{load}} + T_{\text{cyl}}$. From Eqs. (42) and (44) and taking into account that $v_A = \dot{x}$, we have:

$$T_{\text{load}} = \frac{1}{2}\frac{p_2}{g}\dot{x}^2 \quad \text{and} \quad T_{\text{cyl}} = \frac{1}{2}\frac{p_1}{g}v_C^2 + \frac{1}{2}J_C\omega^2. \tag{c}$$

The velocity v_C of the centre of the cylinder is a resultant of the relative velocity $\dot{y}$ with respect to the thread, and the transport velocity $\dot{x}$; as both velocities are directed vertically down, $v_C = \dot{x} + \dot{y}$. The angular velocity of the cylinder $\omega = \frac{\dot{y}}{r}$ (r is the radius of the cylinder), as the instantaneous centre of rotation of the cylinder in the relative motion is at point B, and the cylinder does not rotate in any change of coordinate x. Taking into account that $J_C = \frac{1}{2}\frac{p_1}{g}r^2$, we finally obtain:

$$T = \frac{p_2}{2g}\dot{x}^2 + \frac{p_1}{2g}\left[(\dot{x}+\dot{y})^2 + \frac{1}{2}\dot{y}^2\right],$$

whence

$$\left.\begin{aligned} &\frac{\partial T}{\partial x} = 0, \quad \frac{\partial T}{\partial \dot{x}} = \frac{p_2}{g}\dot{x} + \frac{p_1}{g}(\dot{x}+\dot{y}), \\ &\frac{\partial T}{\partial y} = 0, \quad \frac{\partial T}{\partial \dot{y}} = \frac{p_1}{g}\left(\dot{x}+\dot{y}+\frac{1}{2}\dot{y}\right). \end{aligned}\right\} \tag{d}$$

Substituting the quantities (b) and (d) into Lagrange's equation (a), we obtain the differential equations of motion of the system:

$$(p_1 + p_2)\ddot{x} + p_1\ddot{y} = (p_1 - fp_2)\,g, \quad 2\ddot{x} + 3\ddot{y} = 2g.$$

Solving these equations, we obtain:

$$\ddot{x} = \frac{p_1 - 3fp_2}{p_1 + 3p_2}\,g \quad \text{and} \quad \ddot{y} = \frac{2(1+f)\,p_2}{p_1 + 3p_2}\,g. \tag{e}$$

Hence, $w_A = \ddot{x}$, $w_C = \ddot{x} + \ddot{y}$ ($\ddot{y}$ gives only the relative acceleration of the centre of mass of the cylinder); these results, it is easy to calculate, coincide with those obtained in Problem 170.

Let us work with another set of generalised coordinates. Let $q_1 = x$, as before, and let $q_2 = z$, where z is the distance of the centre of the cylinder from a fixed point O. In this case, giving the system a displacement in which x receives an increment $\delta x > 0$ and $z =$ const. (which can be achieved by simultaneously rotating the cylinder), we obtain $\delta A_1 = -fp_2\delta x$. For the second independent displacement ($\delta z > 0$, $x =$ const.), $\delta A_2 = p_1\delta z$. Thus, now

$$Q_1 = -fp_2 \quad \text{and} \quad Q_2 = p_1.$$

As in the present case $v_C = \dot{z}$ and $\omega = \frac{\dot{z} - \dot{x}}{r}$,

$$T = \frac{p_2}{2g}\dot{x}^2 + \frac{p_1}{2g}\left[\dot{z}^2 + \frac{1}{2}(\dot{z} - \dot{x})^2\right],$$

whence

$$\frac{\partial T}{\partial \dot{x}} = \frac{p_2}{g}\dot{x} - \frac{p_1}{2g}(\dot{z} - \dot{x}) \quad \text{and} \quad \frac{\partial T}{\partial \dot{z}} = \frac{p_1}{g}\left[\dot{z} + \frac{1}{2}(\dot{z} - \dot{x})\right],$$

and the derivatives with respect to x and z are zero. Substituting the obtained values into Lagrange's equations, we finally obtain:

$$p_1\ddot{z} - (2p_2 + p_1)\ddot{x} = 2fp_2 g \quad \text{and}$$

$$3\ddot{z} - \ddot{x} = 2g.$$

Solving these equations, we obtain $w_A = \ddot{x}$ and $w_C = \ddot{z}$. We see that at $q_2 = z$ the value of w_C is obtained immediately, and in this respect such a choice of coordinate q_2 is preferable. On the other hand, if it is necessary to determine the relative acceleration of the centre C or the angular acceleration ε of the cylinder, the computations are simpler with the choice of $q_2 = y$.

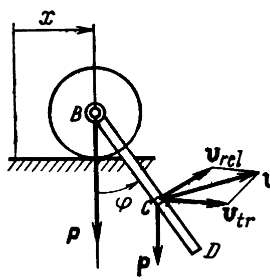

Fig. 387

Thus, a successful choice of generalised coordinates can substantially simplify the solution of a problem.

Problem 175. A homogeneous cylinder of weight P rolls without slipping on a horizontal plane. Hinged to its axle at B is a homogeneous rod BD of length l and weight p (Fig. 387). Write the differential equations of motion of the system and determine the law of small oscillations if at the initial instant the rod is deflected from the equilibrium position through a small angle and released from rest.

Solution. The system has, apparently, two degrees of freedom. Selecting as the generalised coordinates the distance x of the centre of the cylinder from its initial position and the angle φ of the deflection of the rod from the vertical ($q_1 = x$, $q_2 = \varphi$), Lagrange's equations for the system will be:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) - \frac{\partial T}{\partial x} = Q_1,$$
$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\varphi}}\right) - \frac{\partial T}{\partial \varphi} = Q_2. \qquad \text{(a)}$$

First compute Q_1 and Q_2. Giving the system a displacement in which x receives an increment $\delta x > 0$ (φ = const.), we find that $\delta A_1 = 0$. For the other independent displacement, in which φ receives an increment $\delta\varphi > 0$ (x = const.), we find that $\delta A_2 = -0.5lp \sin\varphi \cdot \delta\varphi$. Consequently.

$$Q_1 = 0, \quad Q_2 = -0.5lp \sin\varphi.$$

The kinetic energy of the system $T = T_{cyl} + T_{rod}$. The value of T_{cyl} was computed in Problem 134 (§ 147). Taking into account the result obtained there and Eq. (44), we have:

$$T_{cyl} = \frac{3}{4}\frac{P}{g}v_B^2, \quad T_{rod} = \frac{1}{2}\frac{p}{g}v_C^2 + \frac{1}{2}\frac{pl^2}{12g}\dot{\varphi}^2.$$

Here $v_B = \dot{x}$ and $v_C = v_{rel} + v_{tr}$, where in magnitude $v_{rel} = 0.5l\dot{\varphi}$ and $v_{tr} = \dot{x}$. Consequently (see Fig. 387):

$$v_C^2 = \frac{l^2}{4}\dot{\varphi}^2 + \dot{x}^2 + l\dot{\varphi}\dot{x}\cos\varphi.$$

Finally, we obtain the following expression for the system's kinetic energy:

$$T = \frac{3P}{4g}\dot{x}^2 + \frac{p}{2g}\left(\dot{x}^2 + l\dot{x}\dot{\varphi}\cos\varphi + \frac{l^2}{3}\dot{\varphi}^2\right),$$

whence

$$\frac{\partial T}{\partial \dot{x}} = \frac{3P+2p}{2g}\dot{x} + \frac{pl}{2g}\dot{\varphi}\cos\varphi, \quad \frac{\partial T}{\partial x} = 0,$$
$$\frac{\partial T}{\partial \dot{\varphi}} = \frac{p}{g}\left(\frac{l}{2}\dot{x}\cos\varphi + \frac{l^2}{3}\dot{\varphi}\right), \quad \frac{\partial T}{\partial \varphi} = -\frac{p}{2g}l\dot{x}\dot{\varphi}\sin\varphi.$$

Substituting these quantities into the determined values of Q_1 and Q_2 in Eqs. (a), after cancelling we obtain the following differential equations of motion of the system:

$$\left.\begin{aligned} &\frac{d}{dt}[(3P+2p)\dot{x} + pl\dot{\varphi}\cos\varphi] = 0, \\ &\frac{d}{dt}\left(\dot{x}\cos\varphi + \frac{2}{3}l\dot{\varphi}\right) + \dot{x}\dot{\varphi}\sin\varphi = -g\sin\varphi. \end{aligned}\right\} \qquad \text{(b)}$$

Now let us determine the law of small oscillations of the system. We assume angle φ and the displacement x to be small quantities of the same order of magnitude, i.e., $\varphi = \varepsilon f_1(t)$ and $x = \varepsilon f_2(t)$, where ε is a small quantity and $f_1(t)$ and $f_2(t)$ are certain functions with respect to time (limited together with their derivatives) determining the law of oscillations. Obviously, in this case the velocities $\dot{\varphi} = \varepsilon f_1'(t)$ and $\dot{x} = \varepsilon f_2'(t)$ are also small quantities of the order of ε.

To write the differential equations of small oscillations of the system, only members of the order of ε have to be left in equations (b); small quantities of a higher order can be neglected. For that in the component $pl\dot{\varphi}\cos\varphi$ in the first equation assume $\cos\varphi = 1$, and in the second equation assume $\sin\varphi = \varphi$, $\cos\varphi = 1$, and omit the member $\dot{x}\dot{\varphi}\sin\varphi$ as having the order ε^3. As a result we have:

$$\frac{d}{dt}[(3P+2p)\dot{x}+pl\dot{\varphi}]=0, \quad \text{or} \quad \frac{d}{dt}\left(\dot{x}+\frac{2}{3}l\dot{\varphi}\right)+g\varphi=0,$$

whence, computing the derivatives, we obtain the following differential equations of small oscillations of the system:

$$(3P+2p)\ddot{x}+pl\ddot{\varphi}=0 \quad \text{and} \quad \ddot{x}+\frac{2}{3}l\ddot{\varphi}+g\varphi=0. \tag{c}$$

Computing $\ddot{x}$ from the first equation and substituting its value into the second, we obtain:

$$\ddot{\varphi}+k^2\varphi=0, \tag{d}$$

where

$$k^2=\frac{3(3P+2p)g}{(6P+p)l}. \tag{e}$$

Integrating Eq.(d) and determining the integration constants from the initial conditions of the problem (at $t = 0$ we have $\varphi = \varphi_0$ and $\dot{\varphi} = 0$), we obtain finally:

$$\varphi = \varphi_0 \cos kt. \tag{f}$$

Integrating the first of Eqs. (c) and taking into account that at $t = 0$, $x = 0$ and $\dot{x} = 0$, $\varphi = \varphi_0$, $\dot{\varphi} = 0$, we have:

$$(3P + 2p)x + pl(\varphi - \varphi_0) = 0.$$

Substituting the value of φ from Eq. (f), we obtain:

$$x=\frac{p}{3P+2p}l\varphi_0(1-\cos kt). \tag{g}$$

Eqs. (f) and (g) give the law of small oscillations of the system. The frequency k is given by Eq. (e).

This comparatively simple result was obtained because $Q_1 = 0$. In general, however, when $Q_1 \neq 0$ and $Q_2 \neq 0$, the oscillations of a system with two degrees of freedom are much more complex and represent the resultant of oscillations with two different frequencies k_1 and k_2.

Problem 176. Write Lagrange's equations for the motion of a symmetrical gyroscope.

Solution. A gyroscope has three degrees of freedom. Selecting the Euler angles φ, ψ, θ (see Fig. 235) as the generalised coordinates, Lagrange's equations take the form:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\varphi}}\right)-\frac{\partial T}{\partial \varphi}=Q_{\varphi}, \quad \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\psi}}\right)-\frac{\partial T}{\partial \psi}=Q_{\psi}, \quad \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right)-\frac{\partial T}{\partial \theta}=Q_{\theta}. \quad \text{(a)}$$

The gyroscope's kinetic energy is determined from Eq. (75) of § 158. Assuming, as always, axis Oz to be directed along the gyroscope's axis of symmetry, we have $J_x = J_y$, and

$$T=\frac{1}{2}\left[J_x\left(\omega_x^2+\omega_y^2\right)+J_z\omega_z^2\right]. \quad \text{(b)}$$

To express T in terms of the generalised coordinates we apply Euler's kinematic equations [§ 97, Eqs. (98)]:

$$\omega_x=\dot{\psi}\sin\theta\sin\varphi+\dot{\theta}\cos\varphi,$$

$$\omega_y=\dot{\psi}\sin\theta\cos\varphi-\dot{\theta}\sin\varphi,$$

$$\omega_z=\dot{\varphi}+\dot{\psi}\cos\theta.$$

From these equations we find

$$\omega_x^2+\omega_y^2=\dot{\psi}^2\sin^2\theta+\dot{\theta}^2 \quad \text{and} \quad \omega_z^2=(\dot{\varphi}+\dot{\psi}\cos\theta)^2.$$

Substituting the obtained values into Equation (b), we obtain:

$$T=\frac{1}{2}\left[J_x\left(\dot{\psi}^2\sin^2\theta+\dot{\theta}^2\right)+J_z\left(\dot{\varphi}+\dot{\psi}\cos\theta\right)^2\right].$$

Then, taking into account that $J_z(\dot{\varphi}+\dot{\psi}\cos\theta)=J_z\omega_z$, we have:

$$\frac{\partial T}{\partial \dot{\varphi}}=J_z(\dot{\varphi}+\dot{\psi}\cos\theta)=J_z\omega_z, \quad \frac{\partial T}{\partial \varphi}=0;$$

$$\frac{\partial T}{\partial \dot{\psi}}=J_x\dot{\psi}\sin^2\theta+J_z\omega_z\cos\theta, \quad \frac{\partial T}{\partial \psi}=0;$$

$$\frac{\partial T}{\partial \dot{\theta}}=J_x\dot{\theta}, \quad \frac{\partial T}{\partial \theta}=J_x\dot{\psi}^2\sin\theta\cos\theta-J_z\omega_z\dot{\psi}\sin\theta.$$

To compute the generalised forces, refer to Fig. 235. If coordinate φ is given an increment $\delta\varphi$, the gyroscope will perform an elementary rotation about axis Oz. The elementary work done in such a rotation is $\delta A_1 = M_z \delta\varphi$, where M_z is the principal moment of all the acting forces with respect to axis Oz. Consequently, $Q_\varphi = M_z$. Similarly, taking into account that in any change of angle ψ the gyroscope performs a rotation about axis Oz_1, and in any change of angle θ about the line of nodes OK, we obtain that $Q_\psi = M_{z1}$ and $Q_\theta = M_{OK}$.

Substituting all the computed quantities into equations (a), we finally obtain the following *differential equations of gyroscopic motion in the Lagrangean form*:

$$J_z \frac{d\omega_z}{dt} = M_z,$$

$$\frac{d}{dt}(J_x \dot{\psi} \sin^2 \theta + J_z \omega_z \cos \theta) = M_{z1},$$

$$J_x \ddot{\theta} - J_x \dot{\psi}^2 \sin \theta \cos \theta + J_z \omega_z \dot{\psi} \sin \theta = M_{OK},$$

where

$$\omega_z = \dot{\varphi} + \dot{\psi} \cos \theta.$$

Unlike Euler's equations [§ 158, item (3)], these equations define the motion only of a symmetrical body for which $J_x = J_y$. However, they are simpler than the totality of Euler's dynamic and kinematic equations.

In the special case when acting on the gyroscope is only the force of gravity P applied at a point C on axis Oz (see Fig. 234; point C is not shown), and when the distance $OC = a$ and the axis Oz_1 is vertical, we have $M_z = 0$, $M_{z1} = 0$, and $M_{OK} = Pa \sin \theta$.

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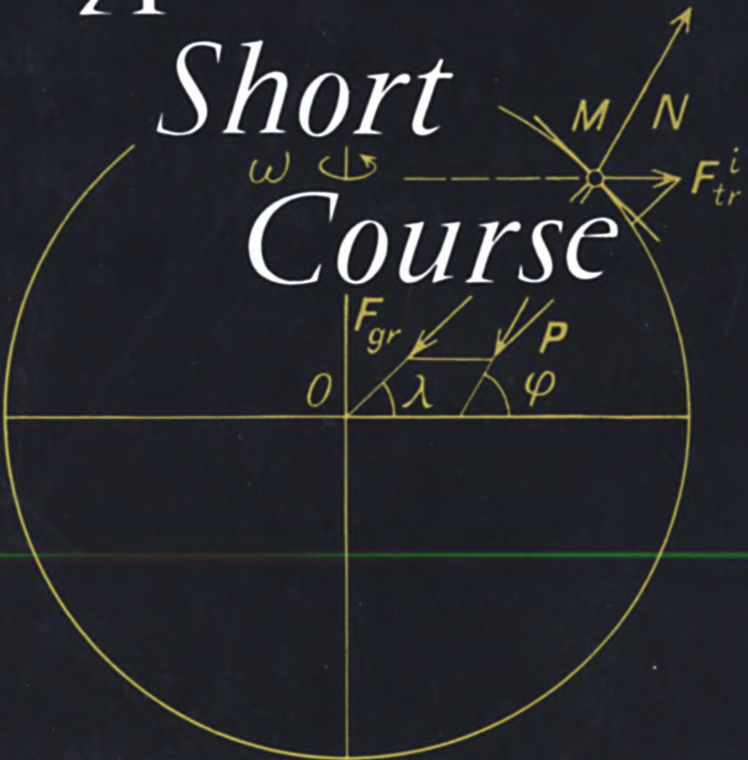

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