7. LAGRANGIAN DYNAMICS

7.1. Generalized Coordinates, Velocities and Accelerations

As the problems in dynamics become more complex it, naturally, becomes increasingly difficult to work out the solutions. This difficulty is associated not only with the solution of the equations of motion, but with their formulation as well. In fact, the derivation of the basic differential equations of motion in a form suitable for a particular complicated problem may well be the most difficult part of the investigation. A number of methods, more powerful than those hitherto considered in this manual, have been developed for deriving the equations for these more involved situations. Perhaps the most generally useful of these more advanced methods for engineering problems is that of Lagrange, who has put the basic equations of motion in such a form that the simplifying features of a particular problem can be utilized most advantageously. In the present chapter we shall derive Lagrange's equations.

One of the principal advantages of Lagrange's method is that one uses for each problem that coordinate system which most conveniently describes the motion. We have already seen that the position of a particle can be described in a large number of different ways, and we have found in the problems already discussed that the choice of a proper coordinate system may introduce a considerable simplification into the solution of a problem. In general, the requirement for a system of coordinates is that the specification of the coordinates must locate completely the position of each part of the system. This means that there must be one coordinate associated with each degree of freedom of the system. More exactly, there must be at least one coordinate associated with each degree of freedom. So called *non-holonomic s*ystems exist, for which, because of the particular geometrical constraints involved, more coordinates are required than there are degrees of freedom. Such systems are not often encountered and will not be considered here. We shall restrict the following treatment to systems whose coordinates are independent, in the sense that a change can be given to any one of the coordinates without changing any of the other coordinates. By the *generalized coordinates* $(q_1, q_2, \dots q_s)$ we shall mean a set of independent coordinates, equal in number to the *s* degrees of freedom of the system. We use the word "generalized" to emphasize the fact that such coordinates are not necessarily of the type of the simple (x, y, z) or (r, θ, φ) systems and to indicate that they are not necessarily lengths or angles, but may be any quantity appropriate to the description of the position of the system.

The (x_k, y_k, z_k) coordinates of a point k are expressible in terms of the generalized coordinates $(q_1, q_2, \dots q_s)$ by functional relations:

$$\begin{aligned} x_k &= x_k(q_1, q_2, \dots, q_s), \\ y_k &= y_k(q_1, q_2, \dots, q_s), \\ z_k &= z_k(q_1, q_2, \dots, q_s). \end{aligned} \tag{7.1}$$

For example, if (q_1, q_2, q_3) are the cylindrical coordinates of a point (r, θ, φ), the foregoing equations become:

$$x = r\cos\theta,\ \ y = r\sin\theta,\ \ z = z.$$

We have supposed that the relation between the coordinate systems does not involve time. In the more general treatment in which $x = x(q_1, q_2, \dots q_s, t)$ analysis can proceed along essentially the same lines. The equations of motion in generalized coordinates for any particular system could always be obtained by writing the equations first in an (x, y, z) system, and then transforming to the *q*'s by Eqs.(7.1). This procedure usually leads to involved algebraic manipulations, and it is better to make the transformation in general terms and to write the equations of motion directly in generalized coordinates.

Thus, the parameters of any dimensions $(q_1, q_2, \dots q_s)$ describing the configuration of the system in space are called *generalized coordinates*. Their first derivatives with respect to time $(\dot{q}_1, \dot{q}_2, \dots \dot{q}_s)$ are called *generalized velocities* and the second derivatives $(\ddot{q}_1, \ddot{q}_2, \dots \ddot{q}_s)$ are said to be *generalized accelerations.*

7.2. Generalized Forces

Let the q_i, for $i = 1, s,$ be a set of generalized coordinates which uniquely specifies the instantaneous position of some dynamical system which has *s* degrees of freedom. Here, it is assumed that each of q_i can vary independently. Since the generalized coordinates are independent, their elemental increments $(\delta q_1, \delta q_2, \dots \delta q_s)$ are also independent. Each of these quantities defines a virtual displacement of the system. Let the system be under the action of the active forces $\boldsymbol{F_1}, \boldsymbol{F_2}, \dots, \boldsymbol{F_n}$. Since the radius-vector of any point is a function of their coordinates $\boldsymbol{r}_k = x_k\boldsymbol{i} + y_k\boldsymbol{j} + z_k\boldsymbol{k}$, one can write

$$\boldsymbol{r_k} = \boldsymbol{r_k}(q_1, q_2, \dots q_s). \tag{7.2}$$

We now calculate the virtual (elementary) work in terms of displacements of the *n* particles assumed to make up the system and the forces $\boldsymbol{F_1}, \boldsymbol{F_2}, \dots, \boldsymbol{F_n}$ acting on them. The virtual work is

$$\sum \delta A_k^a = \sum_{k=1}^{n} \boldsymbol{F}_k \cdot \delta \boldsymbol{r}_k. \tag{7.3}$$

Now, since $\boldsymbol{r}_k = x_k\boldsymbol{i} + y_k\boldsymbol{j} + z_k\boldsymbol{k}$, we can write:

$$\delta \boldsymbol{r}_k = \sum_{\boldsymbol{i=1}}^{\boldsymbol{s}} \frac{\partial \boldsymbol{r_k}}{\partial q_i} \delta q_i \cdot \tag{7.4}$$

for *k*=1,*n*.

Substituting (7.4) into (7.3), one can obtain

$$\sum \delta A_k^a = \sum_{k=1}^{n} \boldsymbol{F}_k \sum_{\boldsymbol{i=1}}^{\boldsymbol{s}} \frac{\partial \boldsymbol{r_k}}{\partial q_i} \delta q_i.$$

The above expression can be rearranged to give

$$\sum \delta A_k^a = \sum_{i=1}^{s} Q_i \cdot \delta q_i, \tag{7.5}$$

where

$$Q_i = \sum_{k=1}^{n} \boldsymbol{F}_k \frac{\partial \boldsymbol{r_k}}{\partial q_i}. \tag{7.6}$$

Here the Q_i are called *generalized forces*. Note that generalized forces do not necessarily have the dimensions of force. However, the product· $Q_i \delta q_i$ must have the dimension of work. Thus, if particular q_i is a lineal parameter, then the associated Q_i is a force. Conversely, if q_i is an angle, then Q_i is a torque.

Formula (7.5) represents the elementary work of the acting forces in terms of generalized coordinates. This definition of the Q_i indicates the way in which they can be determined in specific problems. To find Q_i, the total work done by all of external forces during an infinitesimal displacement δq_i of one of the coordinates is calculated, and Q_i is then obtained by dividing this total work by δq_i.

Thus, the generalized forces are coefficients of the increments of the generalized coordinates in the expression for the total elementary work of all forces applied to the system. It is obvious that the number of generalized forces is equal to the number of degrees of freedom.

If the forces acting on the system are conservative, their total elementary work can be written $\delta A = -\delta\Pi$, where Π is a potential energy of the system which is a function of the coordinates x_k, y_k, z_k. But these coordinates are the functions of generalized coordinates. Thus, $\Pi = \Pi(q_1, q_2, \dots, q_s)$. Calculating a total differential of this function, one can obtain

$$\delta A = -\delta\Pi = -\left[\frac{\partial\Pi}{\partial q_1}\delta q_1 + \frac{\partial\Pi}{\partial q_2}\delta q_2 + \cdots + \frac{\partial\Pi}{\partial q_s}\delta q_s\right].$$

Comparing this expression with equation (7.5) we have

$$Q_1 = -\frac{\partial\Pi}{\partial q_1}, Q_2 = -\frac{\partial\Pi}{\partial q_2}, \dots, Q_s = -\frac{\partial\Pi}{\partial q_s}. \tag{7.7}$$

Therefore, when the forces applied to the system are conservative, generalized forces are the partial derivatives of the potential energy of the system with respect to correspondent generalized coordinates taken with sign minus.

7.3. Conditions of Equilibrium in Terms of Generalized Coordinates

In accordance with principle of virtual works the necessary and sufficient conditions for the equilibrium of a system subjected to ideal constraints is that the total virtual work done by all the active forces is equal to zero for any and all virtual displacements consistent with the constraints, so $\sum_{k=1}^{n} \delta A_k = 0$. In terms of generalized coordinates this condition, taking into account equation (7.5), can be written

$$\sum_{i=1}^{s} Q_i \cdot \delta q_i = 0.$$

Since δq_i, as independent variables, can not be equal to 0, the generalized forces Q_i must disappear in an equilibrium position, i.e., $Q_i = 0$, i=1,2,...,s.

Upon solving the above equations with respect to s unknown generalized coordinates $q_{i,}$ one may always obtain all possible system's equilibrium positions.

Hence, *a holonomic system with perfect constraints is in its equilibrium only if all generalized forces corresponding to generalized coordinates are equal to zero.*

For conservative systems, taking into account Eq.(7.7), we have

$$\frac{\partial\Pi}{\partial q_1} = 0, \frac{\partial\Pi}{\partial q_2} = 0, \frac{\partial\Pi}{\partial q_s} = 0.$$

7.4. Lagrange's Equations of Motion

Let us consider the general equation of dynamics:

$$\sum \delta A_k^a + \sum \delta A_k^{in} = 0, \tag{7.8}$$

where δA_k^a and δA_k^{in} are virtual works of applied impressed forces and inertial forces respectively.

By analogy with Eq.(7.5) one can write

$$\sum \delta A_k^{in} = \sum_{i=1}^{s} Q_i^{in} \cdot \delta q_i,$$

where $Q_i^{in}$are generalized forces of inertia which can be defined as

$$Q_i^{in} = \sum_{k=1}^{n} \boldsymbol{F_k^{in}} \frac{\partial \boldsymbol{r_k}}{\partial q_i}. \tag{7.9}$$

Then from Eq.(7.8) we have for i=1,...,s

$$Q_i + Q_i^{in} = 0. \tag{7.10}$$

Let us express Q_i^{in} in terms of kinetic energy T. Since $\boldsymbol{F_k^{in}} = -m_k \boldsymbol{a_k}$= $=-m_k \frac{d\boldsymbol{v_k}}{dt}$, Eq.(7.9) yields

$$-Q_i^{in} = \sum_{k=1}^{n} m_k \frac{d\boldsymbol{v_k}}{dt} \frac{\partial \boldsymbol{r_k}}{\partial q_i}. \tag{7.11}$$

Notice that

$$\frac{d\boldsymbol{v_k}}{dt} \frac{\partial \boldsymbol{r_k}}{\partial q_i} = \frac{d}{dt}\left(\boldsymbol{v_k} \cdot \frac{\partial \boldsymbol{r_k}}{\partial q_i}\right) - \boldsymbol{v_k} \frac{d}{dt}\left(\frac{\partial \boldsymbol{r_k}}{\partial q_i}\right). \tag{7.12}$$

Further,

$$\frac{d}{dt}\left(\frac{\partial \boldsymbol{r_k}}{\partial q_i}\right) = \frac{\partial}{\partial q_i}\left(\frac{d\boldsymbol{r_k}}{dt}\right) = \frac{\partial \boldsymbol{v_k}}{\partial q_i}. \tag{7.13}$$

Any position vector associated with holonomic system has the form of Eq.(7.2). Since the generalized coordinates are themselves functions of time, the first derivative of the position vector with respect to time is

$$\dot{\boldsymbol{r}}_{\boldsymbol{k}} = \sum_{i=1}^{s} \frac{\partial \boldsymbol{r_k}}{\partial q_i} \dot{q}_i, \tag{7.14}$$

where $\dot{\boldsymbol{r}}_{\boldsymbol{k}} = \frac{d\boldsymbol{r_k}}{dt}$, $\dot{q}_i = \frac{dq_i}{dt}$.

Since all position vectors do not depend on $\dot{q}_i$, the partial derivative $\frac{\partial \boldsymbol{r_k}}{\partial q_i}$ do not depend on $\dot{q}_i$ either. Hence, $\frac{\partial}{\partial \dot{q}_i}\left(\frac{\partial \boldsymbol{r_k}}{\partial q_i}\right) = 0$.

Therefore, differentiation of Eq. (7.14) with respect to $\dot{q}_i$ yields

$$\frac{\partial \boldsymbol{r_k}}{\partial q_i} = \frac{\partial \dot{\boldsymbol{r}}_{\boldsymbol{k}}}{\partial \dot{q}_i} = \frac{\partial \boldsymbol{v_k}}{\partial \dot{q}_i}. \tag{7.15}$$

Making use of formulas (7.13) and (7.15), expression (7.12) can be written

$$\frac{d\boldsymbol{v_k}}{dt} \cdot \frac{\partial \boldsymbol{r_k}}{\partial q_i} = \frac{d}{dt}\left(\boldsymbol{v_k} \cdot \frac{\partial \boldsymbol{v_k}}{\partial \dot{q}_i}\right) - \boldsymbol{v_k} \cdot \frac{\partial \boldsymbol{v_k}}{\partial q_i} = \frac{d}{dt}\left(\frac{1}{2} \frac{\partial \boldsymbol{v_k^2}}{\partial \dot{q}_i}\right) - \frac{1}{2} \frac{\partial \boldsymbol{v_k^2}}{\partial q_i}.$$

Taking into account that mass is constant, the Eq. (7.11) yields

$$-Q_i^{in} = \frac{d}{dt}\left[\frac{\partial}{\partial \dot{q}_i}\left(\sum_{k=1}^{n} \frac{m_k v_k^2}{2}\right)\right] - \frac{\partial}{\partial q_i}\left(\sum_{k=1}^{n} \frac{m_k v_k^2}{2}\right) = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i}, \tag{7.16}$$

where $T = \sum_{k=1}^{n} \frac{m_k v_k^2}{2}$ is kinetic energy of a system.

Hence, from Eq.(7.10) we have finally

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} = Q_i, \qquad (7.17)$$

where *i=1,...,s.*

Eqs. (7.17) are called *Lagrange's equations*. They represent the differential equations of motion of a system in terms of generalized coordinates. It is obvious that their number is equal to the number of degrees of freedom of a system.

If all impressed forces acting on a system are conservative, one can obtain taking into account formulas (7.7)

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial \Pi}{\partial q_i} = 0 \text{ or } \frac{d}{dt}\left(\frac{\partial (T-\Pi)}{\partial \dot{q}_i}\right) - \frac{\partial (T-\Pi)}{\partial q_i} = 0. \qquad (7.18)$$

The last equation is valid since a potential energy Π depends only on generalized coordinates and does not depend on generalized velocities. Hence,

$$\frac{\partial \Pi}{\partial \dot{q}_i} = 0.$$

It is helpful to introduce a function *L,* called the *Lagrangian,* which is defined as the difference between the kinetic and potential energies of the dynamical system under investigation: *L=T-Π.* Then, from (7.18), we have

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0. \qquad (7.19)$$

7.5. Solution of Problems

Problem 62. Resolve problem 61 by means of Lagrange's equations.

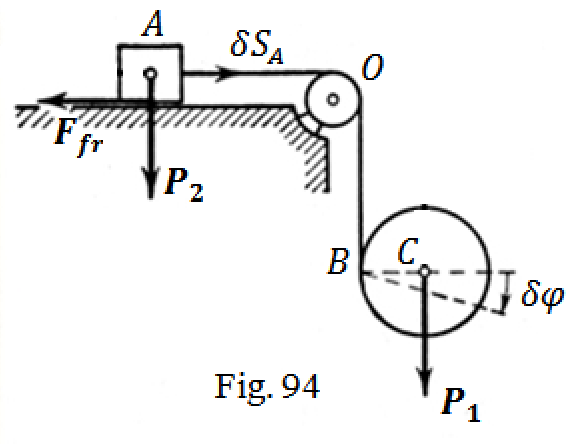

Fig. 94

Solution. Mechanical system has two degrees of freedom (the rotation of the cylinder with respect to the thread, when the load A is at rest, and the displacement of the load when the cylinder does not rotate, fig. 94).

Let us choose as generalized coordinates displacement of bodies with thread S_A and angle of rotation of cylinder φ. Then we have two independent virtual displacements δs_A and $\delta\varphi$. Now consider a virtual displacement δs_A of the system in which the cylinder does not rotate and is translated together with the load. The force $\boldsymbol{P_2}$ does no work in this displacement. There are forces $\boldsymbol{P_1}$ and $\boldsymbol{F_{fr}}$ which produce work on elemental displacement δs_A. It equals

$$\delta A = \left(-F_{fr} + P_1\right)\delta s_A,$$

whence, as $F_{fr} = P_2 f,$ we find generalized force corresponding to generalized coordinate S_A,

$$Q_{S_A} = P_1 - fP_2.$$

Consider the other independent virtual displacement in which the load A remains at rest while the cylinder turns about point B (which in this displacement is the instantaneous centre of rotation) through angle $\delta\varphi$. For this displacement there is only one force of the weight of the cylinder which produces work on elemental displacement $\delta\varphi$. It equals

$$\delta A = P_1 \delta s_C = P_1 r \delta\varphi,$$

where r is the radius of the cylinder. So we have found generalized force corresponding to generalized coordinate φ,

$$Q_\varphi = P_1 r.$$

Now determine kinetic energy of a system. It equals $T = T_1 + T_2$, where T_1 is a kinetic energy of a cylinder and T_2 is kinetic energy of a load.

Cylinder is in a resultant motion which consists of the transport motion (this is a motion of a tread with velocity v_A) and relative motion (this is a motion of a cylinder with respect to the thread, i.e., plane motion). Therefore,

$$T_1 = \frac{P_1 v_C^2}{2g} + \frac{I_C \omega^2}{2},$$

where v_C is absolute velocity of the center C, J_C is a moment of inertia of cylinder and ω is its relative angular velocity since transport motion is a translational one).

From the problem 61 we have $J_C = 0{,}5 m_1 r^2$. Calculate v_C:

$$v_C = v_A + r\omega.$$

Taking into account that $v_A = \dot{s}_A$ and $\omega = \dot{\varphi}$, we have

$$T_1 = \frac{P_1(\dot{s}_A + r\dot{\varphi})^2}{2g} + \frac{P_1 r^2 \dot{\varphi}^2}{4g}.$$

Load *A* is in translational motion, then

$$T_2 = \frac{P_2 v_A^2}{2g} = \frac{P_2 \dot{s}_A^2}{2g}.$$

Thus,

$$T = \frac{P_1(\dot{s}_A + r\dot{\varphi})^2}{2g} + \frac{P_1 r^2 \dot{\varphi}^2}{4g} + \frac{P_2 \dot{s}_A^2}{2g}.$$

Motion of the system considered is governed by Lagrange's equations of the following form:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{s}_A}\right) - \frac{\partial T}{\partial s_A} = Q_{s_A},$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\varphi}}\right) - \frac{\partial T}{\partial \varphi} = Q_\varphi.$$

Therefore,

$$\frac{\partial T}{\partial \dot{s}_A} = \frac{P_1(\dot{s}_A + r\dot{\varphi})}{g} + \frac{P_2 \dot{s}_A}{g}, \qquad \frac{\partial T}{\partial \dot{\varphi}} = \frac{P_1(\dot{s}_A + r\dot{\varphi}) r}{g} + \frac{P_1 r^2 \dot{\varphi}}{2g},$$

$$\frac{\partial T}{\partial s_A} = \frac{\partial T}{\partial \varphi} = 0.$$

Hence, we have following Lagrange's equations:

$$P_1(\ddot{s}_A + r\ddot{\varphi}) + P_2 \ddot{s}_A = g(P_1 - f P_2),$$

$$2\ddot{s}_A + 3r\ddot{\varphi} = 2g.$$

But $\ddot{s}_A$ is acceleration of the load *A,* i.e., a_A. At the same time it is a transport acceleration of the cylinder. The product $r\ddot{\varphi} = r\varepsilon$ represents relative acceleration of the point *C*. Since relative and transport accelerations have the same sense, $\ddot{s}_A + r\ddot{\varphi} = a_C$. Then we obtain

$$P_1 a_C + P_2 a_A = g(P_1 - fP_2),$$
$$3a_C - a_A = 2g.$$

This set of equations gives

$$a_A = \frac{P_1 - 3fP_2}{P_1 + 3P_2} g,\ a_C = \frac{P_1 + (2-f)P_2}{P_1 + 3P_2} g.$$

Problem 63. An uniform and thin bar *2* of mass m and length l is hinged to link *1* which rotates with a constant angular speed ω (Fig. 95). Derive the differential equation of motion of link *2* by means of Lagrange's equations. Neglect the mass of the link *1*.

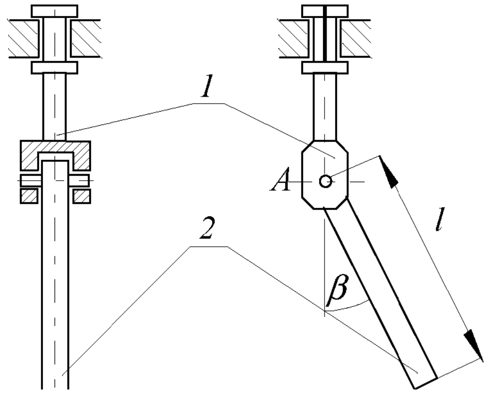

Fig. 95

Solution. Assume that β is the generalized coordinate. Absolute angular velocity of the link *2* is a sum of the transport angular velocity of the link *1* $\boldsymbol{\omega}$ and the relative velocity of the link *2* with respect to the link *1*(Fig. 96).

$$\boldsymbol{\omega}_2 = \boldsymbol{\omega} + \boldsymbol{i}_2\dot{\beta} = \boldsymbol{k}_1\omega + \boldsymbol{i}_2\dot{\beta}.$$

Since

$$\boldsymbol{k}_1 = \boldsymbol{j}_2 \sin\beta + \boldsymbol{k}_2 \cos\beta,$$

the absolute angular velocity of the link 2 is

$$\boldsymbol{\omega}_2 = \boldsymbol{i}_2\dot{\beta} + \boldsymbol{j}_2\omega\sin\beta + \boldsymbol{k}_2\omega\cos\beta.$$

Its components are

$$\omega_{2x} = \dot{\beta}, \omega_{2y} = \omega\sin\beta, \omega_{2z} = \omega\cos\beta.$$

The link *2* performs rotational motion about point *O*. The moment of inertia of the link *2* about *z* axis is zero. Hence, its total kinetic energy is

$$T = \frac{1}{2} I\left(\omega_{2x}^2 + \omega_{2y}^2\right) = \frac{1}{2} I\left(\dot{\beta}^2 + \omega^2 \sin^2\beta\right).$$

There is only one force of the weight of the link *2* which produces work on elemental displacement $\delta\beta$. It equals $\delta A = -\frac{1}{2} mgl \sin\beta\delta\beta$. Hence, generelized force is $Q_\beta = -\frac{l}{2} mgl \sin\beta$.

Motion of the system considered is governed by Lagrange's equations of the following form:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\beta}}\right) - \frac{\partial T}{\partial \beta} = Q_\beta,$$

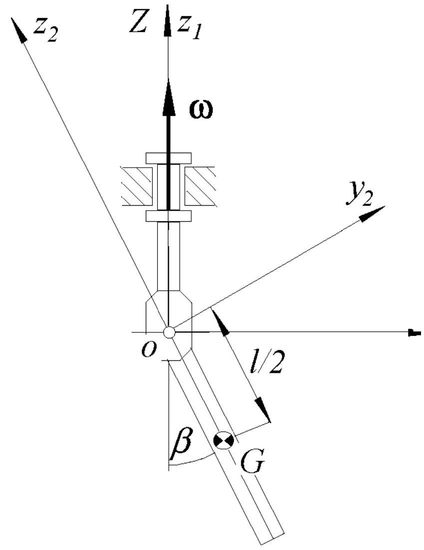

Fig. 96

where

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\beta}}\right) = I\ddot{\beta},\ \frac{\partial T}{\partial \beta} = I\omega^2 \sin\beta\cos\beta.$$

Then one can obtain

$$I\ddot{\beta} - I\omega^2 \sin\beta\cos\beta + \frac{1}{2} mgl \sin\beta = 0.$$

Since $I=\frac{1}{3}ml^2$ the final form of equation of motion is

$$\ddot{\beta}-\omega^2\sin\beta\cos\beta+\frac{3}{2l}g\sin\beta=0.$$

Problem 64. The bead 1 which can be considered as a particle of mass m, may slides without any friction along the slide *2*. The slide *2* rotates with the constant angular velocity ω about the vertical axis Z (Fig. 97). By means of Lagrange's equations derive equation of motion of the bead *1* and determine all possible equilibrium positions.

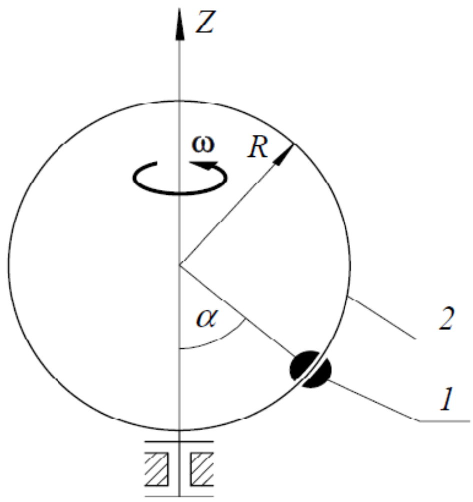

Fig. 97

Given are: $R=25\,cm,\ \omega=10\,rad/s.$

Solution. The angle α can be considered as the generalized coordinate.

In Fig. 98 the inertial system of coordinates is denoted by *XYZ*. System of coordinates *xyz* is rigidly attached to the slide and rotates with the angular velocity ω about axis *Z*. This is a transport angular velocity of the bead. The relative velocity of the bead is its velocity along slide. Vector of the absolute velocity of the bead *1* is

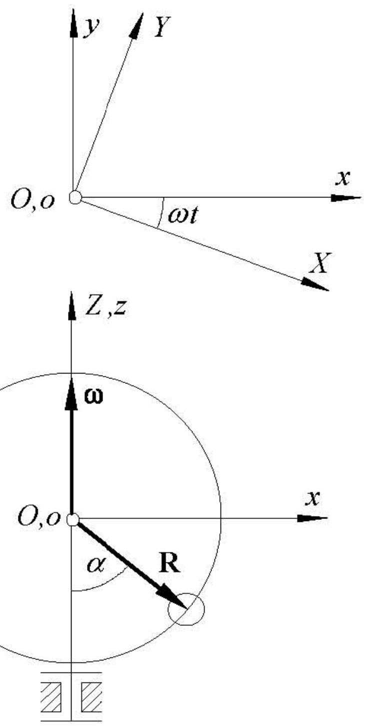

Fig. 98.

$$\boldsymbol{v}_a=\boldsymbol{v}_r+\boldsymbol{v}_{tr},$$

where $v_r=R\dot{\alpha}, v_{tr}=R\omega\sin\alpha$. Since $\boldsymbol{v}_r\perp\boldsymbol{v}_{tr}, v_a^2=\dot{\alpha}^2R^2+\omega^2R^2\sin^2\alpha.$

Hence,

$$T=\frac{m\,v_a^2}{2}=\frac{1}{2}m(\dot{\alpha}^2R^2+\omega^2R^2\sin^2\alpha).$$

By analogy with problem 63 $Q_\alpha=-mgR\sin\alpha$. Lagrange's equations may be taken in the following form:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{\alpha}}\right)-\frac{\partial T}{\partial\alpha}=Q_\alpha,$$

where

$$\frac{\partial T}{\partial\dot{\alpha}}=mR^2\dot{\alpha},\ \frac{d}{dt}\left(\frac{\partial T}{\partial\dot{\alpha}}\right)=mR^2\ddot{\alpha},$$

$$\frac{\partial T}{\partial\alpha}=mR^2\omega^2\sin\alpha\cos\alpha.$$

Hence, these expressions yield equation of motion

$$mR^2\ddot{\alpha}-m\omega^2R^2\sin\alpha\cos\alpha+mgR\sin\alpha=0.$$

Since for the static equilibrium position $\dot{\alpha}=\ddot{\alpha}=0$, then, according to the last equation, we have

$$\omega^2R\sin\alpha\cos\alpha=g\sin\alpha.$$

Hence, the possible equilibrium positions are

$$\alpha_0 = 0,\ \alpha_0 = \pi,\ \alpha_0 = \cos^{-1}\frac{g}{\omega^2 R} = \cos^{-1}\frac{9.81}{10^2 \cdot 0.25} = \pm 66.89°.$$

Problem 65. Fig. 99 shows an arm of a robot operating in the horizontal plane. Motion of the arm is controlled by two actuators installed at joints 0_1 and 0_2. The actuators produce moments M_1 and M_2. Derive differential equations of motion of the robot's arm.

Given are:

I_1 – moment of inertia of the link *1* about a vertical axis through its centre of gravity G_1.

I_2 – moment of inertia of the link *2* about a vertical axis through its centre of gravity G_2.

m_1, m_2 – masses of the link *1* and *2* respectively.

a_1, a_2, l_1, l_2 – dimensions shown in Fig. 99.

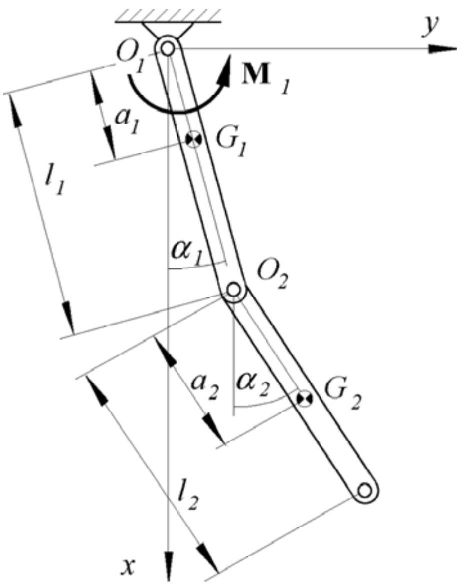

Fig. 99

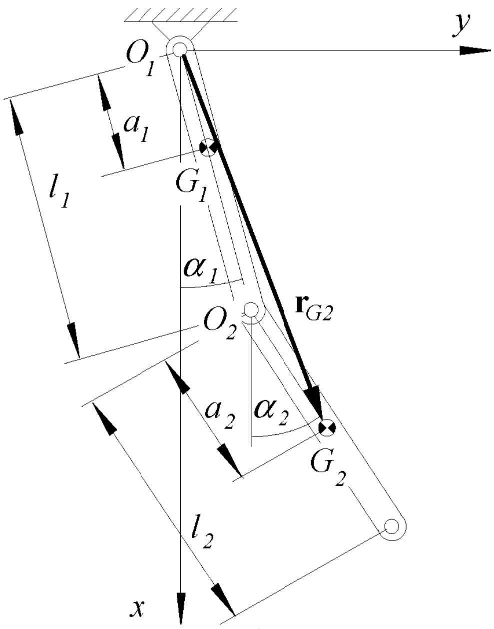

Fig. 100

Solution. The system considered has 2 degrees of freedom and the angles α_1 and α_2 may be considered as the generalized coordinates. Hence, Lagrange's equations for this case can be adopted in the following form:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\alpha}_1}\right) - \left(\frac{\partial T}{\partial \alpha_1}\right) = Q_1,$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\alpha}_2}\right) - \left(\frac{\partial T}{\partial \alpha_2}\right) = Q_2.$$

The kinetic energy T is a sum of kinetic energy of the link 1 and kinetic energy of the link 2.

$$T_1 = \frac{1}{2}(I_1 + m_1 a_1^2)\dot{\alpha}_1^2, \quad T_2 = \frac{1}{2}(m_2 v_{G2}^2 + I_2 \dot{\alpha}_2^2),$$

where v_{G2} is the absolute velocity of the centre of gravity G_2.

The position vector of centre of gravity G_2 is (Fig. 100)

$$\boldsymbol{r}_{G2} = \boldsymbol{i}(l_1 \cos\alpha_1 + a_2 \cos\alpha_2) + \boldsymbol{j}(l_1 \sin\alpha_1 + a_2 \sin\alpha_2).$$

Its first derivative yields the velocity of the centre of gravity G_2

$$\boldsymbol{v}_{G2} = \boldsymbol{i}(-l_1\dot{\alpha}_1 \sin\alpha_1 - a_2\dot{\alpha}_2 \sin\alpha_2) + \boldsymbol{j}(l_1\dot{\alpha}_1 \cos\alpha_1 + a_2\dot{\alpha}_2 \cos\alpha_2).$$

Squared magnitude of the velocity is

$$v_{G2}^2 = (-l_1\dot{\alpha}_1 \sin\alpha_1 - a_2\dot{\alpha}_2 \sin\alpha_2)^2 + (l_1\dot{\alpha}_1 \cos\alpha_1 + a_2\dot{\alpha}_2 \cos\alpha_2)^2$$
$$= l_1^2\dot{\alpha}_1^2 + a_2^2\dot{\alpha}_2^2 + 2l_1a_2\dot{\alpha}_1\dot{\alpha}_2\cos(\alpha_1 - \alpha_2).$$

Upon introducing this expression, one can obtain

$$T_2 = \frac{1}{2}m_2(l_1^2\dot{\alpha}_1^2 + a_2^2\dot{\alpha}_2^2 + 2l_1a_2\dot{\alpha}_1\dot{\alpha}_2\cos(\alpha_1 - \alpha_2)) + \frac{1}{2}I_2\dot{\alpha}_2^2.$$

Hence, the total kinetic energy is

$$T = T_1 + T_2 = \frac{1}{2}(I_1 + m_1a_1^2)\dot{\alpha}_1^2 + \frac{1}{2}m_2(l_1^2\dot{\alpha}_1^2 + a_2^2\dot{\alpha}_2^2 + 2l_1a_2\dot{\alpha}_1\dot{\alpha}_2\cos(\alpha_1 - \alpha_2)) + \frac{1}{2}I_2\dot{\alpha}_2^2.$$

Since the robot operates in the horizontal plane, the only non-conservative forces acting on the system are the driving moments.

The virtual work produced by these forces is $\delta A = (M_1 + M_2)\delta\alpha_1 - M_2\delta\alpha_2$.

Therefore, the generalized forces are

$$Q_1 = M_1 + M_2, \qquad Q_2 = -M_2.$$

Then

$$\frac{\partial T}{\partial \alpha_1} = -2m_2l_1a_2\dot{\alpha}_1\dot{\alpha}_2\sin(\alpha_1 - \alpha_2), \ \frac{\partial T}{\partial \alpha_2} = 2m_2l_1a_2\dot{\alpha}_1\dot{\alpha}_2\sin(\alpha_1 - \alpha_2),$$
$$\frac{\partial T}{\partial \dot{\alpha}_1} = (I_1 + m_1a_1^2)\dot{\alpha}_1 + m_2(l_1^2\dot{\alpha}_1 + l_1a_2\dot{\alpha}_2\cos(\alpha_1 - \alpha_2)), \frac{\partial T}{\partial \dot{\alpha}_2} = m_2(a_2^2\dot{\alpha}_2 + l_1a_2\dot{\alpha}_1\cos(\alpha_1 - \alpha_2)) + I_2\dot{\alpha}_2.$$

Therefore, the final form of equations of motion is

$$[(I_1 + m_1a_1^2) + m_2l_1^2]\ddot{\alpha}_1 + 2m_2l_1a_2\dot{\alpha}_1\dot{\alpha}_2\sin(\alpha_1 - \alpha_2) = M_1 + M_2,$$
$$(I_2 + m_2a_2^2) - 2m_2l_1a_2\dot{\alpha}_1\dot{\alpha}_2\sin(\alpha_1 - \alpha_2) = -M_2.$$

Problem 66. The angle α locates the angular position of the stationary slide *4*

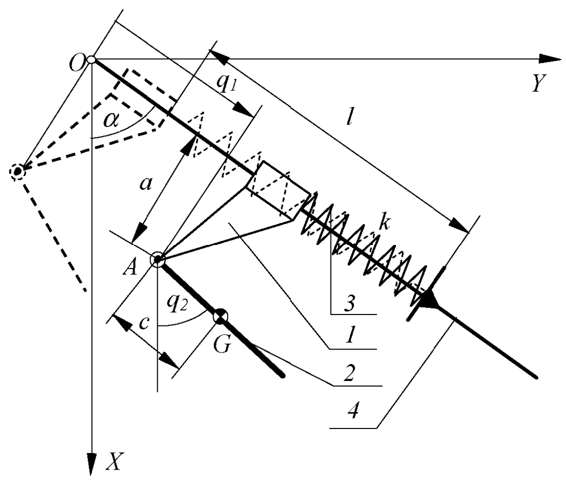

Fig. 101

with respect to the vertical plane *XY* of the inertial system of coordinates *XYZ* (Fig. 101). The massless link 1 is free to move along this slide and is supported by the spring *3* of stiffness *k*. The length of the uncompressed spring is *l*. The link *2* is hinged to the link 1 at the point *A*. The distance *c* locates the position of the centre of gravity *G* of the link *2*. The link *2* possesses mass *m* and its moment of inertia about axis through the centre of gravity *G* is *I*.

Produce the equations of motion of the system and the expressions for the generalized coordinates corresponding to the possible equilibrium positions of the system.

Solution. This system possesses two degrees of freedom and q_1 and q_2 stand for the generalized independent coordinates. In fact q_2 is an angle of rotation of the link *2* about point *A*. Since the link *1* is massless, the total kinetic energy of the system is associated with the link *2* only. The link *2* performs plane motion, hence its kinetic energy is

$$T = \frac{1}{2}mv_G^2 + \frac{1}{2}I\dot{q}_2^2,$$

where v_G stands for the absolute linear velocity of the centre of gravity *G* of the link 2 and $\dot{q}_2$ is its absolute angular velocity.

The velocity v_G can be produced by differentiation of the following absolute position vector

$$\boldsymbol{r}_G = \boldsymbol{I}(q_1\cos\alpha + a\sin\alpha + c\cos q_2) + \boldsymbol{J}(q_1\sin\alpha - a\cos\alpha + c\sin q_2).$$

Hence, the wanted velocity is

$$\boldsymbol{v}_G = \dot{\boldsymbol{r}}_G = \boldsymbol{I}(\dot{q}_1\cos\alpha - c\,\dot{q}_2\sin q_2) + \boldsymbol{J}(\dot{q}_1\sin\alpha + c\,\dot{q}_2\cos q_2).$$

Then

$$v_G^2 = (\dot{q}_1\cos\alpha - c\,\dot{q}_2\sin q_2)^2 + (\dot{q}_1\sin\alpha + c\dot{q}_2\sin q_2)^2.$$

Introduction of these expressions yields the wanted kinetic energy function

$$T = \frac{1}{2}m((\dot{q}_1\cos\alpha - c\,\dot{q}_2\sin q_2)^2 + (\dot{q}_1\sin\alpha + c\,\dot{q}_2\cos q_2)^2) + \frac{1}{2}I\dot{q}_2^2 =$$
$$\frac{1}{2}m\dot{q}_1^2 + mc\dot{q}_1\dot{q}_2\sin(\alpha - q_2) + \frac{1}{2}mc^2\dot{q}_2^2 + \frac{1}{2}I\dot{q}_2^2.$$

The elemental work of the force of weight and elastic force of the spring is

$$\delta A = mg\cos\alpha\,\delta q_1 - kq_1\delta q_1 - mgc\sin q_2\,\delta q_2.$$

Therefore, the generalized forces are

$$Q_1 = mg\cos\alpha - kq_1, Q_2 = -mgc\sin q_2.$$

Since the system is of two degrees of freedom and the generalized coordinates are q_1 and q_2, one can obtain the following Lagrange's equations:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_1}\right) - \frac{\partial T}{\partial q_1} = Q_1,\ \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_2}\right) - \frac{\partial T}{\partial q_2} = Q_2\,.$$

Taking into account the formulas obtained above, we have the equations of motion

$$m\ddot{q}_1 + mc\sin(\alpha - q_2)\,\ddot{q}_2 - mc\cos(\alpha - q_2)\dot{q}_2^2 - mg\cos\alpha + kq_1 = 0,$$
$$(I + mc^2)\ddot{q}_2 + mc\sin(\alpha - q_2)\,\ddot{q}_1 + mgc\sin q_2 = 0.$$

The above set of equations allows the equilibrium position of the system to be determined. If q_1 and q_2 are constant,

$$mg\cos\alpha - kq_1 = 0,$$
$$mgc\sin q_2 = 0.$$

Hence, $q_1 = \frac{mg\cos\alpha}{k}$, $q_2 = 0$.

Problem 67. The circular slide of radius *R* is free to rotate about the horizontal axis *Y* of the inertial system of coordinates *XYZ* (Fig. 102). Its moment of inertia about that axis is *I*. The body *2*, which can be considered as a particle of mass *m*, can move along the slide without friction. System of coordinates *xyz*, shown in Fig. 102, is rigidly attached to the slide *1*.

By means of Lagrange equations derive the differential equations of motion of the system along the generalized coordinates α and β.

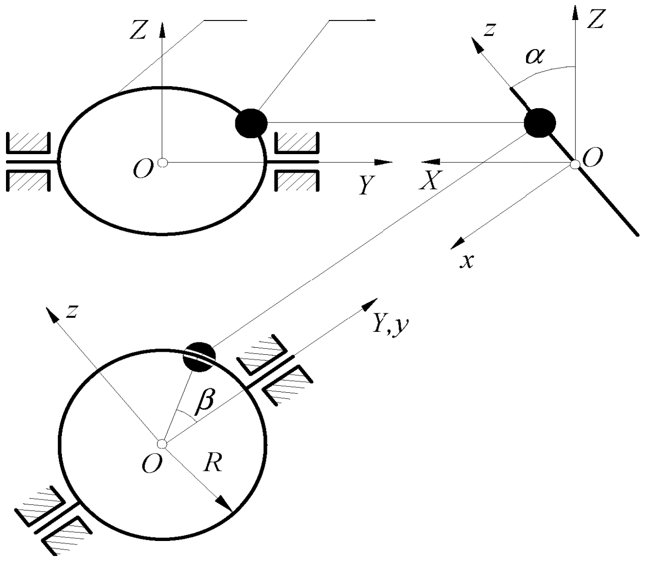

Fig. 102

Solution. The system considered has 2 degrees of freedom and the angles α and β may be considered as the generalized coordinates. Hence, Lagrange's equations for this case can be adopted in the following form:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\alpha}}\right) - \left(\frac{\partial T}{\partial \alpha}\right) = Q_\alpha,$$
$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\beta}}\right) - \left(\frac{\partial T}{\partial \beta}\right) = Q_\beta.$$

The kinetic energy T is a sum of kinetic energy of the body (particle) and kinetic energy of the slide $T = T_1 + T_2$, where T_1 is a kinetic energy of the particle and T_2 is a kinetic energy of the slide.

$T_1 = \frac{mv^2}{2}$, where v is an absolute velocity of a particle. Then $\boldsymbol{v_a} = \boldsymbol{v}_r + \boldsymbol{v}_{tr}$, where $\boldsymbol{v}_r$ is a reletive velocity and $\boldsymbol{v}_{tr}$ is a transport velocity of the particle.

Relative motion is a motion of the particle along the slide. Hence, $v_r = R\dot{\beta}$. Transport motion is a motion of the slide. Therefore, $v_{tr} = R\dot{\alpha}\sin\beta$. But $\boldsymbol{v}_r \perp \boldsymbol{v}_{tr}$, so $v_a^2 = R^2(\dot{\alpha}^2 \sin^2\beta + \dot{\beta}^2)$ and $T_1 = \frac{mR^2(\dot{\alpha}^2 \sin^2\beta + \dot{\beta}^2)}{2}$.

Slide is in rotational motion, then $T_2 = \frac{I\dot{\alpha}^2}{2}$.

The total kinetic energy is

$$T = \frac{mR^2(\dot{\alpha}^2 \sin^2\beta + \dot{\beta}^2)}{2} + \frac{I\dot{\alpha}^2}{2}.$$

By derivation one can obtain

$$\frac{\partial T}{\partial \dot{\alpha}} = I\dot{\alpha} + mR^2\dot{\alpha}\sin^2\beta, \frac{\partial T}{\partial \dot{\beta}} = mR^2\dot{\beta}, \frac{\partial T}{\partial \alpha} = 0, \frac{\partial T}{\partial \beta} = mR^2\dot{\alpha}^2 \sin\beta\cos\beta.$$

Now determine the generalized forces. Considering elemental work done by a gravity force in virtual displacement $\delta\alpha$, we have

$$\delta A_\alpha = mgR\sin\alpha\sin\beta\,\delta\alpha.$$

The elemental work done by a gravity force in virtual displacement $\delta\beta$ is

$$\delta A_\beta = -mgR\cos\alpha\cos\beta\delta\beta.$$

Thus, $Q_\alpha = mgR\sin\alpha\sin\beta$ and $Q_\beta = -mgR\cos\alpha\cos\beta$.

Finally we have the differential equations of motion of the system

$$(I + mR^2\sin^2\beta)\ddot{\alpha} + 2mR^2\dot{\alpha}\dot{\beta}\sin\beta\cos\beta - mgR\sin\alpha\sin\beta = 0,$$
$$mR^2\ddot{\beta} - mR^2\dot{\alpha}^2\sin\beta\cos\beta + mgR\cos\alpha\cos\beta = 0.$$