6. THE PRINCIPLES OF DYNAMICS

6.1. D'Alembert's Principle for a Particle and a System

Consider a particle *M* moving along a given fixed curve or surface (Fig. 72). The resultant of all the active forces applied to the particle is denoted by the symbol $\boldsymbol{F}^a$. If the action of the constraint is replaced by its reaction $\boldsymbol{N}$, the particle can be considered as a free one moving under the action of forces $\boldsymbol{F}^a$ and $\boldsymbol{N}$. Let us see what force $\boldsymbol{F}^i$ should be added to the forces $\boldsymbol{F}^a$ and $\boldsymbol{N}$ to balance them. If the resultant of the forces $\boldsymbol{F}^a$ and $\boldsymbol{N}$ is $\boldsymbol{R}$, then, obviously, the required force $\boldsymbol{F}^i = -\boldsymbol{R}$.

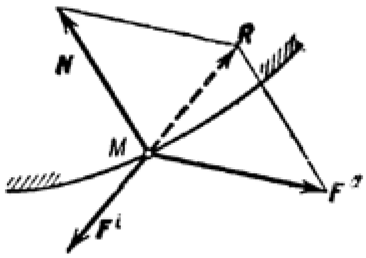

Fig. 72

Let us express force $\boldsymbol{F}^i$ in terms of the acceleration of the moving particle. As, according to the fundamental law of dynamics, $\boldsymbol{R} = m\boldsymbol{a}$, $\boldsymbol{F}^i = -m\boldsymbol{a}$.

The force $\boldsymbol{F}^i$, equal in magnitude to the product of the mass of the particle and its acceleration and directed oppositely to the acceleration, is called the *inertia force* of the particle.

Thus, if to the forces $\boldsymbol{F}^a$ and $\boldsymbol{N}$ is added the inertia force $\boldsymbol{F}^i$, the forces will be balanced, and we will have

$$\boldsymbol{F}^a + \boldsymbol{N} + \boldsymbol{F}^i = 0. \qquad (6.1)$$

This equation states *D'Alembert's principle for a particle: if at any given moment to the active forces and the reactions of the constraints acting on a particle is added the inertia force, the resultant force system will be in equilibrium and all the equations of statics will apply to it.*

D'Alembert's principle provides a method of solving problems of dynamics by developing equations of motion in the form of equations of equilibrium.

In applying D'Alembert's principle it should always be remembered that actually only forces $\boldsymbol{F}^a$ and $\boldsymbol{N}$ are acting on a particle and that the particle is in motion. The inertia force does not act on a moving particle and the concept is introduced for the sole purpose of developing equations of dynamics with the help of the simpler methods of statics.

D'Alembert's Principle for a System. Consider a system of *n* particles. Let us select any particle of mass m_k and denote the resultants of all the external and internal forces applied to it by the symbols $\boldsymbol{F}_k^{ext}$ and $\boldsymbol{F}_k^{int}$. If we add to these forces the inertia force $\boldsymbol{F}_k^i = -m_k\boldsymbol{a}_k$, then according to D'Alembert's principle for a single particle the force system $\boldsymbol{F}_k^{ext}$, $\boldsymbol{F}_k^{int}$, $\boldsymbol{F}_k^i$ will be in equilibrium, and consequently,

$$\boldsymbol{F}_k^{ext} + \boldsymbol{F}_k^{int} + \boldsymbol{F}_k^i = 0.$$

Reasoning similarly for all the particles of the system, we arrive at the following result, which expresses D'Alembert's principle for a system: *if at any moment of time to the effective external and internal forces acting on every particle of a system are* *added the respective inertia forces, the resultant force system will be in equilibrium and all the equations of statics will apply to it.*

We know from statics that the geometrical sum of balanced forces and the sum of their moments with respect to any centre *0* are zero; we know, further, from the principle of solidification, that this holds good not only for forces acting on a rigid body, but for any deformable system. Thus, according to D'Alembert's principle, we must have

$$\sum\left(\boldsymbol{F}_k^{ext} + \boldsymbol{F}_k^{int} + \boldsymbol{F}_k^i\right) = 0,$$
$$\sum\left[\boldsymbol{m}_0(\boldsymbol{F}_k^{ext}) + \boldsymbol{m}_0\left(\boldsymbol{F}_k^{int}\right) + \boldsymbol{m}_0\left(\boldsymbol{F}_k^i\right)\right] = 0.$$

Let us introduce the following notation:

$$\boldsymbol{R}^i = \sum \boldsymbol{F}_k^i, \boldsymbol{M}_0^i = \sum \boldsymbol{m}_0\left(\boldsymbol{F}_k^i\right).$$

The quantities $\boldsymbol{R}^i$and $\boldsymbol{M}_0^i$ are respectively the principal vector of the inertia forces and their principal moment with respect to a centre *0*. Taking into account that the sum of the internal forces and the sum of their moments are each zero we obtain

$$\sum \boldsymbol{F}_k^{ext} + \boldsymbol{R}^i = 0, \sum \boldsymbol{m}_0(\boldsymbol{F}_k^{ext}) + \boldsymbol{M}_0^i = 0. \qquad (6.2)$$

Use of Eqs. (6.2), which follow from D'Alembert's principle, simplifies the process of problem solution because the equations do not contain the internal forces. Actually Eqs. (6.2) are equivalent to the equations expressing the theorems of the change in the momentum and the total angular momentum of a system, differing from them only in form.

6.2. The Principal Vector and the Principal Moment of the Inertia Forces of a Rigid Body

It follows from the Statics that a system of inertia forces applied to a rigid body can be replaced by a single force equal to $\boldsymbol{R}^i$ and applied at the centre *0*, and a couple of moment $\boldsymbol{M}_0^i$. The principal vector of a system, it will be recalled, does not depend on the centre of reduction and can be computed at once. As $\boldsymbol{F}_k^i = -m_k\boldsymbol{a}_k$, then taking into account §5.3, we will have:

$$\boldsymbol{R}^i = -\sum m_k\boldsymbol{a}_k = -M\boldsymbol{a}_c. \qquad (6.3)$$

Thus, *the principal vector of the inertia forces of a moving body is equal to the product of the mass of the body and the acceleration of its centre of mass, and is opposite in direction to the acceleration.*

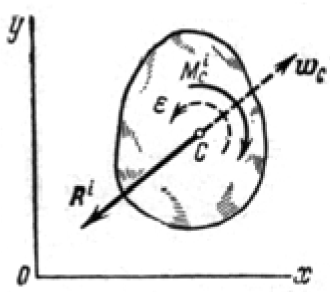

Fig. 73

Let us determine the principal moment of the inertia forces for particular types of motion.

1. Translational Motion. In this case a body has no rotation about its centre of mass *C*, from which we conclude that $\sum \boldsymbol{m}_C(\boldsymbol{F}_k^{ext}) = 0$, and Eq. (6.2) gives $\boldsymbol{M}_C^i = 0$.

Thus, *in translational motion, the inertia forces of a rigid body can be reduced to a single resultant* $\boldsymbol{R}^i$ *through the centre of mass of the body.*

2. Plane Motion. Let a body have a plane of symmetry, and let it be moving parallel to the plane. By virtue of symmetry, the principal vector and the resultant couple of inertia forces lie, together with the centre of mass *C*, in that plane.

Therefore, placing the centre of reduction in point *C*, we obtain from Eq. (6.2) $M_C^i = -\sum m_C(\boldsymbol{F}_k^{ext})$. On the other hand (see § 5.9, 5.10), $\sum m_C(\boldsymbol{F}_k^{ext}) = J_C\varepsilon$. We conclude from this that

$$M_C^i = -J_C\varepsilon. \tag{6.4}$$

Thus, in such motion *a system of inertia forces can be reduced to a resultant force* $\boldsymbol{R}^i$ *(Eq. (6.3)) applied at the centre of mass C (Fig. 73) and a couple in the plane of symmetry of the body whose moment is given by Eq. (6.4).* The minus sign shows that the moment M_C^i is in the opposite direction of the angular acceleration of the body.

3. Rotation about an Axis through the Centre of Mass. Let a body have a plane of symmetry, and let the axis of rotation *Cz* be normal to the plane through the centre of mass. This case will thus be a particular case of the previous motion. But here $\boldsymbol{a}_c = 0$, and consequently, $\boldsymbol{R}^i = 0$.

Thus, in this case *a system of inertia forces can be reduced to a couple in the plane of symmetry of the body of moment*

$$M_z^i = -J_C\varepsilon.$$

In applying Eqs. (6.3) and (6.4) to problem solutions, the magnitudes of the respective quantities are computed and the directions are shown in a diagram.

6.3. Virtual Displacements of a System. Degrees of Freedom

In determining the equilibrium conditions of a system by the methods of so-called graphical statics we had to consider the equilibrium of every body separately, replacing the action of all applied constraints by the unknown reaction forces. When the number of bodies in a system is large, this method becomes cumbersome, involving the solution of a large number of equations with many unknown quantities.

Now we shall make use of a number of kinematical and dynamical concepts to investigate a more general method for the solution of problems of statics, which makes it possible to determine at once, the equilibrium conditions for any mechanical system. The basic difference between this method and the methods of geometrical statics is that the action of constraints is taken into account not by introducing the reaction forces but by investigating the possible displacements of a system if its equilibrium were disturbed. These displacements are known in mechanics by the name of *virtual displacements.*

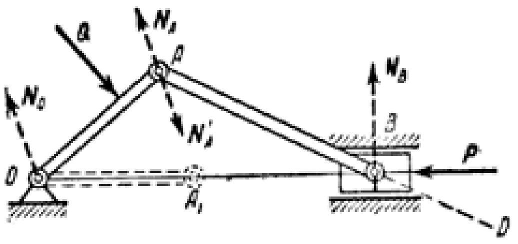

Fig. 74

Virtual displacements of the particles of a system must satisfy two conditions: 1) they must be infinitesimal, since if a displacement is finite the system will occupy a new configuration in which the equilibrium conditions may be different; 2) they must be consistent with the constraints of the system, as otherwise we should change the character of the mechanical system under consideration. For instance, in the crankshaft mechanism in Fig. 74, a displacement of the points of the crank *OA* into configuration OA_1 cannot considered as a virtual displacement, as the equilibrium conditions under the action of forces P and Q will be have changed. At the same time, even an infinitesimal displacement of point *B* of the connecting rod along *BD* would not be a virtual displacement: it would have been possible if the slides at *B* were replaced by a rocker, i.e., if it were a different mechanism.

Thus, we shall define as a virtual displacement of a system the sum total of any arbitrary infinitesimal displacements of the particles of the system consistent with all the constraints acting on the system at the given instant. We shall denote the virtual displacement of any point by an elementary vector $\delta\boldsymbol{s}$ in the direction of the displacement.

In the most general case, the particles and bodies of a system may have a number of different virtual displacements (not considering $\delta\boldsymbol{s}$ and — $\delta\boldsymbol{s}$ as being different). For every system, however, depending on the type of constraints, we can specify a certain number of independent virtual displacements such that any other virtual displacements will be obtained as their geometrical sum. For example, a bead lying on a horizontal plane can move in many directions on the plane. Nevertheless, any virtual displacement $\delta\boldsymbol{s}$ may be produced as the sum of two displacements $\delta\boldsymbol{s}_1$ and $\delta\boldsymbol{s}_2$ along two mutual perpendicular horizontal axes ($\delta\boldsymbol{s} = \delta\boldsymbol{s}_1 + \delta\boldsymbol{s}_2$).

The number of possible mutually independent displacements of a system is called the number of degrees of freedom of that system. Thus, a bead on a plane (regarded as a particle) has two degrees of freedom. A crankshaft mechanism, evidently, has one degree of freedom. A free particle has three degrees of freedom (three independent displacements along mutually perpendicular axes). A free rigid body has six degrees of freedom (three translational displacements along orthogonal axes and three rotations about those axes).

Ideal Constraints. If a particle has for a constraint a smooth surface, the reaction $\boldsymbol{N}$ of the constraint is normal to the surface and the elementary work done by the force $\boldsymbol{N}$ in any virtual displacement of the particle is zero. It was shown that if we neglect rolling friction, the sum of the work done by the reaction forces $\boldsymbol{N}$ and $\boldsymbol{F}_{fr}$ in any virtual displacement of a rolling body is also zero. The internal forces of any non-deformable system also possess this property.

Let us introduce the following notation: the elementary work done by an active force $\boldsymbol{F}^a$ in any virtual displacement $\delta\boldsymbol{s}$ — the virtual work—shall be denoted by the symbol δA^a ($\delta A^a = F^a \delta s \cos\alpha$, where α is the angle between the directions of the force and the displacement), and the virtual work done by the reaction $\boldsymbol{N}$ of a constraint, by the symbol δA^N. Then for all the constraints considered here,

$$\sum \delta A_k^N = 0. \tag{6.5}$$

Constraints, in which the sum of the virtual work produced by all the reaction forces in any virtual displacement of a system is zero, are called *ideal constraints.*

We have seen that to such constraints belong all frictionless constraints along which a body slides and all rough constraints when a body rolls along them, neglecting rolling friction.

6.4. The Principle of Virtual Work

Consider a system of material particles in equilibrium under the action of the applied forces and constraints, assuming all the constraints imposed on the system to be ideal. Let us take an arbitrary particle belonging to the system and denote the resultant of all the applied active forces (both external and internal) by the symbol $\boldsymbol{F}_k^a$, and the resultant of all the reactions of the constraints (also external and internal) by the symbol $\boldsymbol{N}_k$. Then, since this particle is in equilibrium together with the system, $\boldsymbol{F}_k^a + \boldsymbol{N}_k = 0 \; or \; \boldsymbol{N}_k = -\boldsymbol{F}_k^a$.

Consequently, in any virtual displacement of the particle the virtual work δA_k^a and δA_k^N done by the forces $\boldsymbol{F}_k^a$ and $\boldsymbol{N}_k$ are equal in magnitude and opposite in sense and therefore vanish, i.e., we have:

$$\delta A_k^a + \delta A_k^N = 0.$$

Reasoning in the same way we obtain similar equations for all the particles of a system, adding which we obtain

$$\sum \delta A_k^a + \sum \delta A_k^N = 0.$$

But from the property of ideal constraints (6.5), the second summation is zero, whence

$$\sum \delta A_k^a = 0, \tag{6.6}$$

or

$$\sum (F_k^a \delta s_k \cos \alpha_k) = 0. \tag{6.7}$$

We have thus proved that if a mechanical system with ideal constraints is in equilibrium, the active forces applied to it satisfy the condition (6.6). The reverse is also true, i.e., if the active forces satisfy the condition (6.6), the system is in equilibrium. From this follows the principle of virtual work: *the necessary and sufficient conditions for the equilibrium of a system subjected to ideal constraints is that the total virtual work done by all the active forces is equal to zero for any and all virtual displacements consistent with the constraints.* Mathematically the necessary and sufficient condition for the equilibrium of any mechanical system is expressed by Eq. (6.6).

In analytical form this condition can be expressed as follows:

$$\sum \left(F_{kx}^a \delta x_k + F_{ky}^a \delta y_k + F_{kz}^a \delta z_k \right) = 0. \tag{6.8}$$

In Eq. (6.8) $\delta x_k, \delta y_k, \delta z_k$ are the projections of the virtual displacements $\delta \boldsymbol{s}_k$ of point on the coordinate axes. They are equal to the infinitesimal increments to the position coordinates of the point in its displacement and are computed in the same way as the differentials of coordinates.

The principle of virtual work provides in general form the equilibrium conditions of any mechanical system, whereas the methods of geometrical statics require the consideration of the equilibrium of every body of the system separately.

Furthermore, application of the principle of virtual work requires that only the active forces be considered and makes it possible to ignore all the unknown reactions of constraints, when the constraints are ideal.

6.5. The General Equation of Dynamics

The principle of virtual work gives a general method for solving problems of statics. On the other hand, D'Alembert's principle makes it possible to employ the methods of statics in solving dynamical problems. It seems obvious that by combining both these principles we can develop a general method for the solution of problems of dynamics.

Consider a system of material particles subjected to ideal constraints. If we add to all the particles subjected to active forces $\boldsymbol{F}_k^a$ and the reaction forces $\boldsymbol{N}_k$ the corresponding inertia forces $\boldsymbol{F}_k^i = -m_k \boldsymbol{a}_k$, then by D'Alembert's principle the resulting force system will be in equilibrium. If we now apply the principle of virtual work, we obtain

$$\sum \delta A_k^a + \sum \delta A_k^i + \sum \delta A_k^N = 0.$$

But from Eq. (6.5) the last summation is zero, and we finally obtain

$$\sum \delta A_k^a + \sum \delta A_k^i = 0. \qquad (6.9)$$

Equation (6.9) represents the general equation of dynamics. It states that *in a moving system with ideal constraints the total virtual work done by all the active forces and all the inertia forces in any virtual displacement is zero at any instant.*

In analytical form Eq. (6.9) gives

$$\sum\left[\left(F_{kx}^a + F_{kx}^i\right)\delta x_k + \left(F_{ky}^a + F_{ky}^i\right)\delta y_k + \left(F_{kz}^a + F_{kz}^i\right)\delta z_k\right] = 0. \quad (6.10)$$

Equation (6.9) and (6.10) make it possible to develop the equations of motion for any mechanical system.

If a system consists of a number of rigid bodies, the relevant equations can be developed if to the active forces applied to each body are added a force equal to the principal vector of the inertia forces applied at any center, and a couple of moment equal to the principal moment of the inertia forces with respect to that center. Then the principle of virtual work can be used.

6.6. Solution of Problems

Problem 44. When a train accelerates, a load attached to a string hanging from the ceiling of a carriage deflects by an angle α from the vertical (Fig. 75). Determine the acceleration of the carriage.

Solution. Acting on the load is the force of gravity $\boldsymbol{P}$ and the reaction of the thread $\boldsymbol{T}$. Applying D'Alembert's principle, add to these

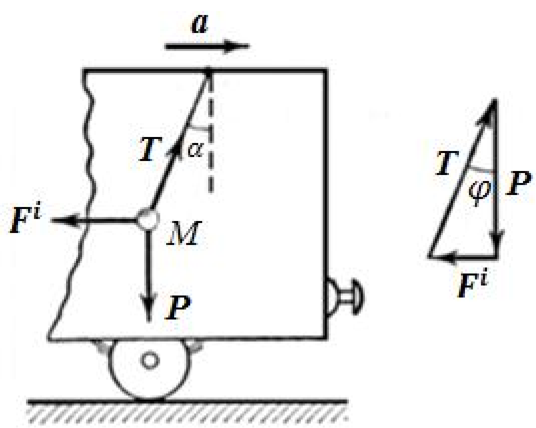

Fig. 75

forces the inertia force $\boldsymbol{F^i}$ directed opposite to the acceleration $\boldsymbol{a}$ of the carriage. In magnitude $F^i = ma = \frac{P}{g}a$. The forces $\boldsymbol{P}$, $\boldsymbol{T}$ and $\boldsymbol{F^i}$ are balanced. Constructing a closed force triangle and taking into account that $\varphi = \alpha$, we find

$$F^i = P\,tan\,\alpha \text{ or } \frac{P}{g}a = P\,tan\,\alpha.$$

Hence, the acceleration of the carriage is $a = g\,tan\,\alpha$.

Problem 45. Neglecting the mass of all the rotating parts of the centrifugal-type governor in Fig. 76 as compared with the mass of the balls B and D, determine the angle α defining the position of relative equilibrium of rod AB of length l if the governor rotates with a constant angular velocity ω.

Solution. In order to determine the position of relative equilibrium (with respect to a set of axes rotating together with the governor) add, according to Eq. (6.1), to the force of gravity $\boldsymbol{P}$ and the reaction $\boldsymbol{N}$ acting on ball B the transport inertia force $\boldsymbol{F^i_{tr}}$. As ω const, $a_{tr} = a^n_{tr} = BC\omega^2 = l\omega^2\,sin\,\alpha$, whence $F^i_{tr} = ml\omega^2\,sin\,\alpha$. Writing the equilibrium equation for the projections on axis $B\tau$, which is perpendicular to AB, we have

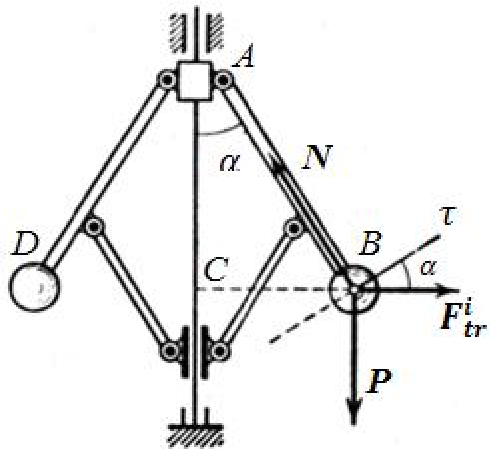

Fig. 76

$$-P\,sin\,\alpha + F^i_{tr}\,cos\,\alpha = 0.$$

Hence, substituting F^i_{tr} for its expression and eliminating $sin\,\alpha$ (not considering the solution for $\alpha = 0$), we obtain

$$-g + l\omega^2\,cos\,\alpha = 0,$$

whence

$$cos\,\alpha = \frac{g}{l\omega^2}.$$

As $cos\,\alpha \leq 1$, equilibium at $\alpha \neq 0$ is possible only when $\omega^2 > \frac{g}{l}$.

Problem 46. The semicircle BCD of radius R in Fig. 77 rotates about a vertical axis with a constant angular velocity ω. A ring M starts slipping along it without friction from a point B slightly off the axis of rotation. Determine the relative velocity v_1 of the ring at point C if its initial velocity $v_0 = 0$.

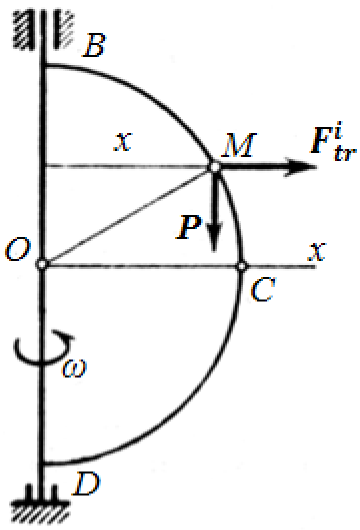

Fig. 77

Solution. The velocity v_1, can be determined from the theorem of the change in kinetic energy. In order to write Eq. (5.36), which expresses the theorem, compute the work done by forces $\boldsymbol{P}$ and $\boldsymbol{F^i_{tr}}$, where $F^i_{tr} = m\omega^2 x$ (the work done by the reaction $\boldsymbol{N}$ is zero). Assuming approximately $x_B = 0$, we obtain

$$A_{(BC)}(\boldsymbol{F}_{tr}^i) = \int_{(B)}^{(C)} F_{tr\,x}^i\,dx = m\omega^2 \int_0^R x\,dx = \frac{1}{2}m\omega^2 R^2.$$

Furthermore, $A_{(BC)}(\boldsymbol{P}) = PR$. Substituting these expressions into Eq. (5.36) and taking into account that $v_0 = 0$, we have

$$\frac{mv_1^2}{2} = mR\left(g + \frac{1}{2}\omega^2 R\right),$$

whence

$$v_1 = \sqrt{2gR\left(1 + \frac{\omega^2 R}{2g}\right)}.$$

*Problem 47.*Two weights P_1 and P_2 are connected by a thread and move along a horizontal plane under the action of a force $\boldsymbol{Q}$ applied to the first weight (Fig. 78*a*). The coefficient of friction of the weights on the plane is f. Determine the accelerations of the weights and the tension in the thread.

Solution. Denote all the external forces acting on the system and add to them the inertia forces of the weights. As both weights are translated with the same acceleration $\boldsymbol{a}$, then in magnitude

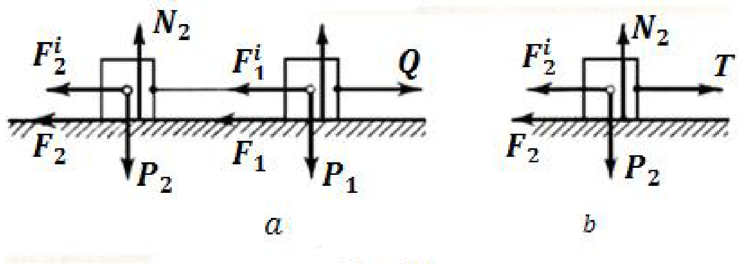

Fig. 78

$F_1^i = \frac{P_1}{g}a$ and $F_2^i = \frac{P_2}{g}a.$

The forces are directed as shown. Then frictional forces are

$$F_1 = fP_1,\ F_2 = fP_2.$$

According to D'Alembert's principle, the force system must be in equilibrium. Writing the equilibrium equation in terms of the projections on horizontal axis, we find

$$Q - f(P_1 + P_2) - \frac{1}{g}(P_1 + P_2)a = 0,$$

whence

$$a = \left(\frac{Q}{P_1 + P_2} - f\right)g.$$

Evidently, the weights will move if $f < \frac{Q}{P_1+P_2}$.

In our force system the required tension in the thread is an internal force. To determine it we divide the system and apply D'Alembert's principle to one of the weight, say the second (Fig. 78*b*). Acting on it is force $\boldsymbol{P}_2$, the normal reaction $\boldsymbol{N}_2$, the frictional force $\boldsymbol{F}_2$, and tension $\boldsymbol{T}$ in the thread. Add to them the inertia force $\boldsymbol{F}_2^{in}$ and write the equilibrium in terms of the projection on horizontal axis. We have

$$T - fP_2 - \frac{P_2}{g}a = 0.$$

Substituting the earlier found value of a, we obtain finally

$$T = \frac{QP_2}{P_1 + P_2}.$$

It is interesting to note that the tension in the thread does not depend on the friction and, given the same total weight of the system, it decreases with the reduction of the second (rear) weight. That is why, for example, in making up a goods train it is better to place the heavier vans closer to the locomotive.

Problem 48. Solve problem 22 with the help of D'Alembert's principle and also determine the tension in the thread.

Solution. 1) Considering the drum and the load as a single system, we add to the bodies of the system inertia forces (Fig. 79). Load A is in translational motion and for it $R^i = \frac{Q}{g}a_A = \frac{Q}{g}r\varepsilon$. The inertia forces of the drum can be reduced to a couple of moment M_0^i equal in magnitude to $J_0\varepsilon = \frac{P}{g}\varrho^2\varepsilon$ and directed opposite the rotation. Writing now for all the forces the equilibrium conditions in the form $\sum m_0(\boldsymbol{F}_k) = 0$, we obtain

$$|M_0^i| + R^i r - Qr = 0,$$

or

$$\frac{P}{g}\varrho^2\varepsilon + \frac{Q}{g}r^2\varepsilon - Qr = 0,$$

from which we find

$$\varepsilon = \frac{Qgr}{P\varrho^2 + Qr^2}.$$

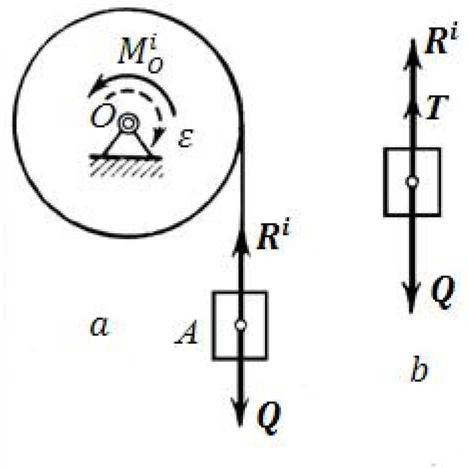

Fig. 79

2) Considering now load A separately and adding to the active forces $\boldsymbol{Q}$ and $\boldsymbol{T}$ the inertia force $\boldsymbol{R^i}$, we obtain from the equilibrium conditions that the tension in the thread

$$T = Q - R^i = Q\left(1 - \frac{r\varepsilon}{g}\right) = \frac{PQ\varrho^2}{P\varrho^2 + Qr^2}.$$

Problem 49. Determine the forces acting on a spinning flywheel, assuming its mass to be distributed along the rim. The weight of the flywheel is P, its radius r and its angular velocity ω.

Solution. The required force is an internal one. In order to determine it, cut the rim into two and apply D'Alembert's principle to one portion (Fig. 80). We denote the action of the separated half by two equal forces $\boldsymbol{F}'$ equal in magnitude to the required force $\boldsymbol{F}$. For each element of the rim, the inertia force (a centrifugal force) is directed along the radius. These concurrent forces intersecting at O have a resultant equal to the principal vector $\boldsymbol{R^i}$ of the inertia forces directed, by virtue of symmetry, along axis Ox. By Eq. (6.3), $R^i = ma_C = mx_C\omega^2$, where m is the mass of half the rim and x_C is the coordinate of the mass centre of the semicircular arc, which is equal to $\frac{2r}{\pi}$. Therefore,

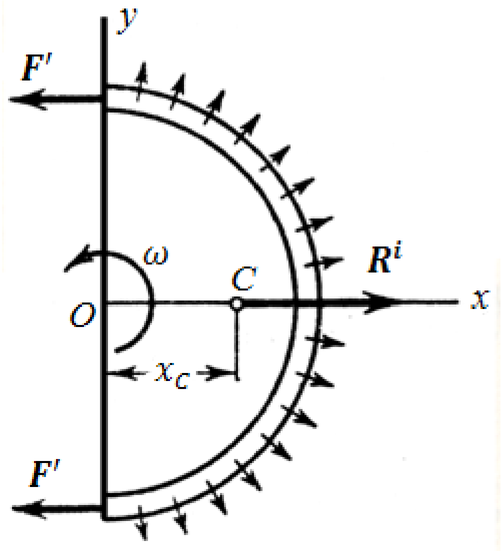

Fig. 80

$$m = \frac{P}{2g}, R^i = \frac{Pr\omega^2}{\pi g}.$$

The equilibrium conditions give $2F = R^i$, and finally

$$F = \frac{Pr\omega^2}{2\pi g}.$$

This formula can be used to determine the limiting angular velocity beyond which a flywheel made of a specific material may be torn apart.

Problem 50. A homogeneous rod AB of length l and weight P is hinged at A to a vertical shaft rotating with an angular velocity ω (Fig. 81). Determine the tension T in the horizontal thread securing the rod at an angle α to the shaft.

Solution. Applying D'Alembert's principle, we add to the external forces $\boldsymbol{P}, \boldsymbol{T}, \boldsymbol{X_A}$ and $\boldsymbol{Y_A}$ acting on the rod the inertia forces. For each element of the rod of mass Δm the centrifugal inertia force is $\Delta m\omega^2 x$, where x is the distance of the element from the axis of rotation Ay. The resultant of these parallel forces distributed according to a linear law passes through the centre of gravity of triangle ABE, i.e., at a distance $h = \frac{2}{3} l \cos\alpha$ from the Ax axis. As this resultant is equal to the principal vector of the inertia forces, then, by Eq. (6.3),

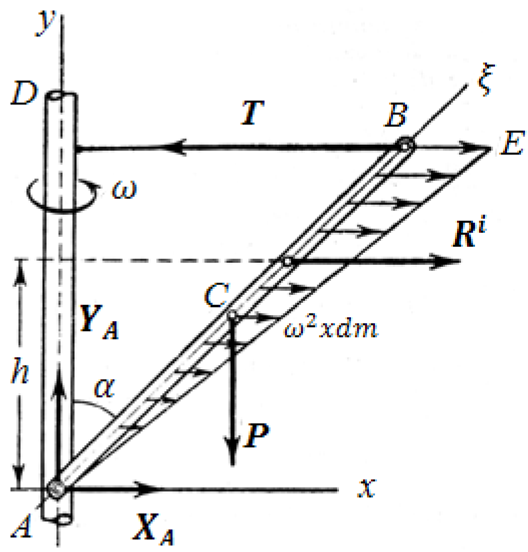

Fig. 81

$$R^i = ma_C = m\omega^2 x_C = \frac{P}{g}\omega^2 \frac{1}{2} \sin\alpha$$

(here x_C is the coordinate of the centre of gravity of the rod).

Writing now the statics equation $\sum m_A(\boldsymbol{F}_k) = 0$, we obtain

$$Tl\cos\alpha - R^i h - P\frac{l}{2}\sin\alpha = 0.$$

Substituting the values of R^i and h into this equation, we obtain finally

$$T = P\left(\frac{l\omega^2}{3g}\sin\alpha + \frac{1}{2}\tan\alpha\right).$$

Problem 51. A homogeneous rod AB of mass m and length l rotates about an axis perpendicular to it with an angular velocity ω and an angular acceleration ε (Fig. 82). Determine the stresses generated by the rotation in a cross section of the rod at a distance x from the axis.

Solution. The required forces are internal. To determine them, cut the rod into

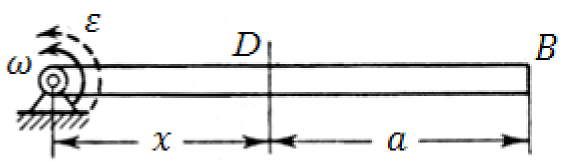

Fig. 82

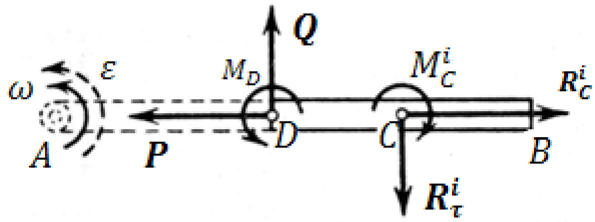

Fig. 83

two and consider the motion of portion DB of length $a = 1 - x$ (Fig. 83). The action of the removed portion AD is replaced by a force applied at the centre D of the cross section, which we shall represent by its components $\boldsymbol{P}$ and $\boldsymbol{Q}$, and a couple of moment M_D. The quantities $\boldsymbol{P}, \boldsymbol{Q}$ and M_D will specify the required stresses in section D of the rod, i.e., the forces with which portions AD and DB act on one another. To compute these quantities we use D'Alembert's principle. First let us find the principal vector $\boldsymbol{R^i}$ of the inertia forces of portion DB, and their principal moment M_C^i with respect to the centre of mass C of the portion. The mass of portion DB and the coordinate $x_C = AC$ of its centre of mass are

$$m_1 = \frac{1-x}{l}m,\ x_C\frac{1+x}{2}.$$

Then, form Eqs. (6.3), we find magnitude of vectors $\boldsymbol{R}_\tau^i$ and $\boldsymbol{R}_C^i$:

$$|\boldsymbol{R}_\tau^i| = m_1|a_{c\tau}| = m_1 x_C|\varepsilon| = m|\varepsilon|\frac{l^2 - x^2}{2l},$$

$$R_C^i = m_1 a_{cn} = m_1 x_C \omega^2 = m\omega^2\frac{l^2 - x^2}{2l}.$$

Furthermore, from Eq. (6.4), $M_C^i = -J_C\varepsilon$. As in this case $J_C = \frac{ma^2}{12}$, then

$$|M_C^i| = m\frac{(l-x)^3}{12l}|\varepsilon|.$$

The forces $\boldsymbol{R}_\tau^i$ and $\boldsymbol{R}_C^i$ and the moment M_C^i are directed as shown in the Fig.83.

Thus, all the inertia forces of portion DB of the rod are replaced by a force $\boldsymbol{R}^i$ applied at C, whose components are $\boldsymbol{R}_\tau^i$ and $\boldsymbol{R}_C^i$ and a couple of moment M_C^i. Writing now the equilibrium conditions $\sum F_{kx} = 0,\ \sum F_{ky} = 0,\ \sum m_D(\boldsymbol{F}_k) = 0$ for the active forces and the inertia forces applied to portion DB, we obtain

$$R_C^i - P = 0; Q - |R_\tau^i| = 0;\ M_D - |M_C^i| - |R_\tau^i|\frac{a}{2} = 0.$$

From this we find finally that acting at section D of the rod are 1) a tensile force $P = R_n^i$ and 2) a transverse force $Q = |R_\tau^i|$, and 3) a couple with a bending moment

$$M_D = |M_C^i| + |R_\tau^i|\frac{a}{2} = \frac{m|\varepsilon|}{12l}[(l-x)^3 + 3(l+x)(l-x)^2],$$

or finally

$$M_D = \frac{m|\varepsilon|}{6l}(l - x)^2(2l + x).$$

The value of forces P, Q and the bending moment will be greatest at the cross section $x = 0$.

Problem 52. Two equal bars of length l and weight p each are welded at right angles to a vertical shaft of length b at istance h from each other (Fig. 84). Determine the dynamical pressures acting on the shaft if it rotates with a constant angular velocity ω.

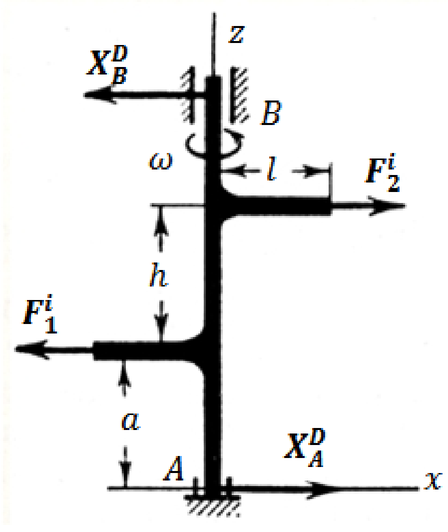

Fig. 84

Solution. The centrifugal inertia forces in each rod are equal in magnitude:

$$F_1^i = F_2^i = \frac{P}{g}\frac{1}{2}\omega^2,$$

and they make a couple which, apparently, is balanced by the couple $\boldsymbol{X_A^D}, \boldsymbol{X_B^D}$. The moments of these couples are equal in magnitude. Consequently, $X_A^D b = F_1^i h$, whence

$$X_A^D = X_B^D = \frac{F_1^i h}{b} = \frac{plh}{2gb}\omega^2.$$

The couple is continuously in the Axz plane, which rotates with the body.

Problem 53. Find the relation between the moment M of the couple acting on the crankshaft mechanism in Fig. 85 and the pressure P on the piston when the system is in equilibrium. The crank is of length $OA = r$ and the connecting rod is of length $AB = l$.

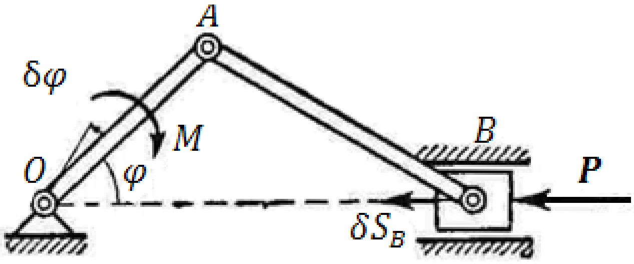

Fig. 85

Solution. Equilibrium conditions (6.6) give

$M\delta\varphi - P\delta s_B = 0$ or $M\omega_{OA} = Pv_B$,

since $\delta\varphi = \omega_{OA}dt$ and $\delta s_B = v_B dt$. The relation between v_B and ω_{OA} can be found by the methods of kinematics:

$$v_B = \omega_{OA} r\left(1 + \frac{r\cos\varphi}{\sqrt{l^2 - r^2\sin^2\varphi}}\right)\sin\varphi.$$

Referring to this result, we find

$$M = Pr\left(1 + \frac{r\cos\varphi}{\sqrt{l^2 - r^2\sin^2\varphi}}\right)\sin\varphi.$$

Problem 54. For the reduction gear (fig. 86), find the relation between the torque M_A applied to the driving shaft A and the resistance moment M_B applied to the driven shaft B when both shafts are rotating uniformly.

The number of teeth in the gears is: $z_1 = 120, z_2 = 40, z_3 = 30, z_4 = 50$.

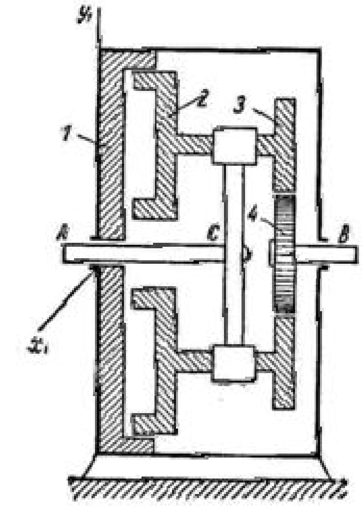

Fig. 86

Solution. The relation between M_A and M_B will be the same in uniform rotation as in equilibrium. Therefore, from Eq. (6.6) we have

$$M_A\delta\varphi_A - M_B\delta\varphi_B = 0 \; or \; M_A\omega_A = M_B\omega_B,$$

as $\delta\varphi_A = \omega_A dt,$ and $\delta\varphi_B = \omega_B dt$. Hence, referring to the kinematic methods, we find

$$\frac{\omega_B}{\omega_A} = 1 + \frac{z_1 z_3}{z_2 z_4}, \; M_A = \frac{\omega_B}{\omega_A} M_B = (1 + \frac{z_1 z_3}{z_2 z_4}) M_B = 2.8\, M_B.$$

Problem 55. Find the relation between forces P and Q in the hoisting mechanism in Fig. 87, whose parts are housed in the box K, if it is known that in one revolution of the crank handle $AB = l$ the screw D moves out by h.

Solution. From Eq. (6.6) we have

$$Pl\delta\varphi_{AB} - Q\delta s_D = 0.$$

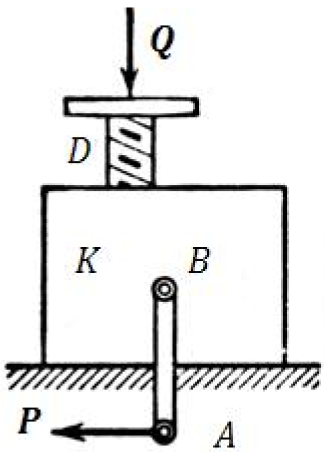

Fig. 87

Assuming that when the handle is rotated uniformly the screw also moves up uniformly, we have

$$\frac{\delta\varphi_{AB}}{2\pi} = \frac{\delta s_D}{h} \text{ or } \delta\varphi_{AB} = \frac{2\pi}{h}\delta s_D.$$

Substituting this expression for $\delta\varphi_{AB}$ into the foregoing equation, we obtain

$$Q = \frac{2\pi l}{h} P.$$

Note that this simple problem could just not be solved by the methods of geometrical statics as the parts of the mechanism are unknown.

Problem 56. Two beams are hinged together at C and loaded as shown in Fig.88a. Neglecting the weight of the beams, determine the pressure on support B.

Solution. Replace the support at B by a force $\boldsymbol{N_B}$, which is equal in magnitude to the required pressure (Fig. 88b). For a virtual displacement of the system Eq. (6.6) gives

$$N_B\delta s_B - P\delta s_E = 0.$$

The relation between δs_B and δs_E is found from the proportions

$$\frac{\delta s_B}{a} = \frac{\delta s_C}{l_1}; \quad \frac{\delta s_E}{b} = \frac{\delta s_C}{l_2},$$

whence

$$\delta s_E = \frac{bl_1}{al_2}\delta s_B,$$

and consequently

$$N_B = \frac{bl_1}{al_2}P.$$

If we used the methods of geometrical statics we would have to consider the equilibrium of each beam separately, introduce the reactions of the other supports, and then eliminate them from the obtained set of equilibrium equations.

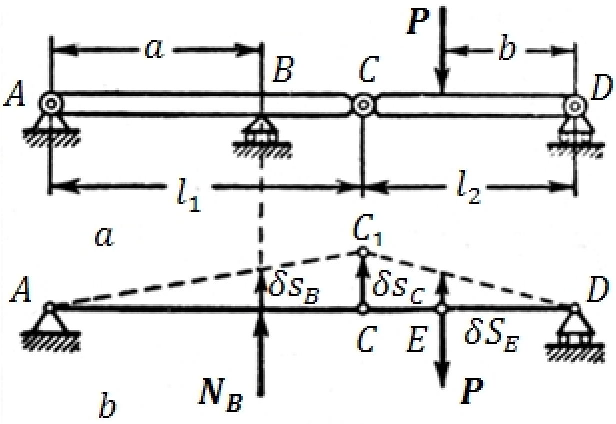

Fig. 88

Problem 57. The epicycles gear train in Fig. 89 consists of a gear *1* of radius r_1, an arm AB mounted on axle A independently of the gear, and a gear *2* of radius r_2 mounted on the arm at B as shown. Acting on the arm is a torque M, and acting on the gears are resistance moments M_1 and M_2, respectively. Determine the values of M_1 and M_2 at which the mechanism is in equilibrium.

Solution. The mechanism has two degrees of freedom, since it has two possible independent displacements: the rotation of the arm AB when gear *1* is at rest, and the rotation of gear *1* when the arm is at rest. First consider a virtual displacement of the system in which gear *1* remains at rest (Fig. 89*a*). For this displacement Eq. (6.6) gives

$$M\delta\varphi_{AB} - M_2\delta\varphi_2 = 0.$$

But when gear *1* is at rest the contact point of the gears will be the instantaneous centre of zero velocity of gear 2, and consequently $v_B = \omega_2 r_2$. At the same time,

$$v_B = \omega_{AB}(r_1 + r_2).$$

Hence $\omega_2 r_2 = \omega_{AB}(r_1 + r_2)$ or $\delta\varphi_2 r_2 = \delta\varphi_{AB}(r_1 + r_2)$, and we obtain

$$M_2 = \frac{r_2}{r_1 + r_2}M.$$

Now consider a virtual displacement in which the arm AB remains at rest (Fig. 89*b*). For this displacement Eq. (6.6) gives

$$M_1\delta\varphi_1 - M_2\delta\varphi_2 = 0.$$

But when the arm is at rest

$$\frac{\delta\varphi_2}{\delta\varphi_1} = \frac{\omega_2}{\omega_1} = \frac{r_1}{r_2} \text{ and } M_1 = \frac{r_1}{r_2}M_2.$$

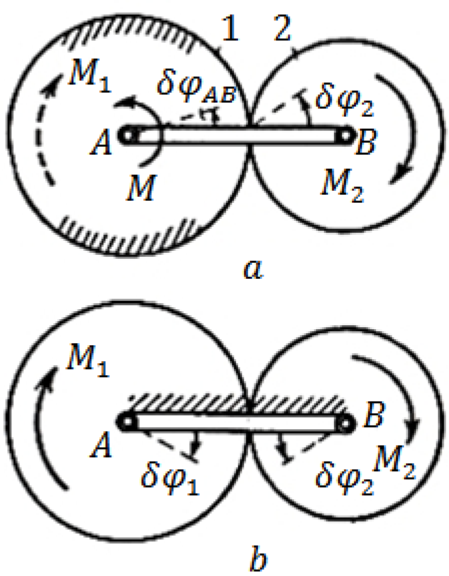

Fig. 89

We finally obtain

$$M_1 = \frac{r_1}{r_1 + r_2}M, \quad M_2 = \frac{r_2}{r_1 + r_2}M.$$

Problem 58. Determine the relation between forces Q and P at which the press in Fig. 90 is in equilibrium if angles α and β are known. Neglect the weight of the rods.

Solution. To give an example of the analytical method of solution, let us take equilibrium condition (6.8). Placing the origin of a coordinate system in the fixed point A and drawing the x and y axes as shown, we obtain

$$Q_{1x}\delta x_1 + Q_{2x}\delta x_2 + P_{3y}\delta y_3 = 0,$$

since all the other projections of the forces vanish.

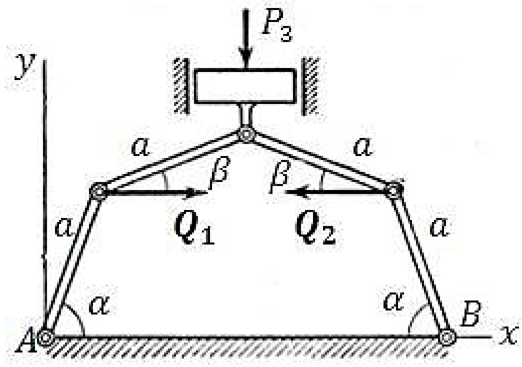

Fig. 90

To find $\delta x_1, \delta x_2, \delta y_3$ compute the coordinates x_1, x_2, y_3 of the points of application of the forces, expressing them in terms of the angles α and β. Denoting the length of each rod by a, we obtain

$$x_1 = a\cos\alpha,\ x_2 = a\cos\alpha + 2a\cos\beta,\ y_3 = a(\sin\beta + \sin\alpha),$$

differentiating which, we find $\delta x_1 = -a\sin\alpha\delta\alpha$, $\delta x_2 = -a(\sin\alpha\delta\alpha + 2\sin\beta\delta\beta)$, $\delta y_3 = a(\cos\beta\delta\beta + \cos\alpha\delta\alpha)$.

Substituting these expressions and taking into account that $Q_{1x} = Q, Q_{2x} = -Q$, and $P_{3y} = -P$, we have

$$2Q\sin\beta\delta\beta - P(\cos\beta\delta\beta + \cos\alpha\delta\alpha) = 0.$$

To find the relation between $\delta\alpha$ and $\delta\beta$ we make use of the fact that $AB = const$. Therefore, $2a(\cos\alpha + \cos\beta) = const$. Differentiating this equation, we obtain

$$\sin\alpha\delta\alpha + \sin\beta\delta\beta = 0 \quad and \quad \delta\alpha = -\frac{\sin\beta}{\sin\alpha}\delta\beta.$$

Substituting the expression for $\delta\alpha$, we have

$$2Q\sin\beta - P(\cos\beta - \cot\alpha\sin\beta) = 0,$$

whence

$$P = \frac{2Q}{\cot\beta - \cot\alpha}.$$

At an angle β very close to α the pressure P will be very large.

Problem 59. A centrifugal-type governor consists of two balls A_1 and A_2 of weight P each (Fig. 91). The slide C_1C_2 weighs Q, the governor rotates about the vertical axis with a uniform angular velocity ω. Neglecting the weight of the rods, determine angle α, if $OA_1 = OA_2 = l$, and $OB_1 = OB_2 = B_1C_1 = B_2C_2 = a$.

Solution. Adding to the active forces P_1, P_2 and Q_3 the centrifugal inertia forces F_1^i and F_2^i (the inertia force of the slide will, evidently, be zero), we write the general equation of dynamics in the form (6.10). Computing the projections of all the forces on the coordinate axes, we have

$$P_1\delta x_1 + P_2\delta x_2 - F_1^i\delta y_1 + F_2^i\delta y_2 + Q_3\delta x_3 = 0.$$

We also have

$$Q_3 = Q; \quad P_1 = P_2 = P\,; \quad F_1^i = F_2^i =$$
$$= \frac{P}{g}a_A = \frac{P}{g}\omega^2 l\sin\varphi.$$

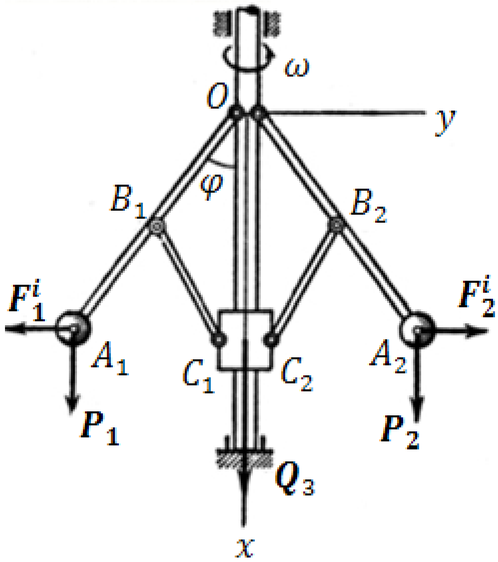

Fig. 91

The coordinates of the points of application of the forces are

$$x_1 = x_2 = l\cos\varphi, \quad y_2 = -y_1 = l\sin\varphi, \quad x_3 =$$
$$= 2a\cos\varphi.$$

Differentiating these expressions, we find

$$\delta x_1 = \delta x_2 = -l\sin\varphi\delta\varphi\,; \quad \delta y_2 = -\delta y_1$$
$$= l\cos\varphi\delta\varphi\,; \quad \delta x_3 = -2a\sin\varphi\delta\varphi.$$

Substituting all these expressions, we obtain

$$\left(-2Pl\sin\varphi + 2\frac{P}{g}l^2\omega^2\sin\varphi\cos\varphi - 2Qa\sin\varphi\right)\delta\varphi = 0,$$

whence we finally have

$$\cos\alpha = \frac{Pl + Qa}{Pl^2\omega^2}g.$$

As $\cos\varphi \le 1$, the balls will move apart when

$$\omega^2 > \frac{Pl + Qa}{Pl^2}g.$$

Angle φ increases with ω and tends to 90º when $\omega \to \infty$.

Problem 60. In the hoist mechanism in Fig. 92, a torque M is applied to gear *2* of weight P_2 and radius of gyration ϱ_2. Determine, the acceleration of the lifted load A of weight Q, neglecting the weight of the string and the friction in the axles. The drum on which the string winds and the gear *1* attached to it have a total weight P_1 and a radius of gyration ϱ_1.The radii of the gears are r_1 and r_2, and of the drum r.

Solution. Draw the active force $\boldsymbol{Q}$ and torque M (forces $\boldsymbol{P_1}$ and $\boldsymbol{P_2}$ do no work) and add to them the inertia force $\boldsymbol{F_A^i}$ of the load and the couples of moments M_1^i and M_2^i to which the inertia forces of the rotating bodies are reduced. In magnitude these quantities are

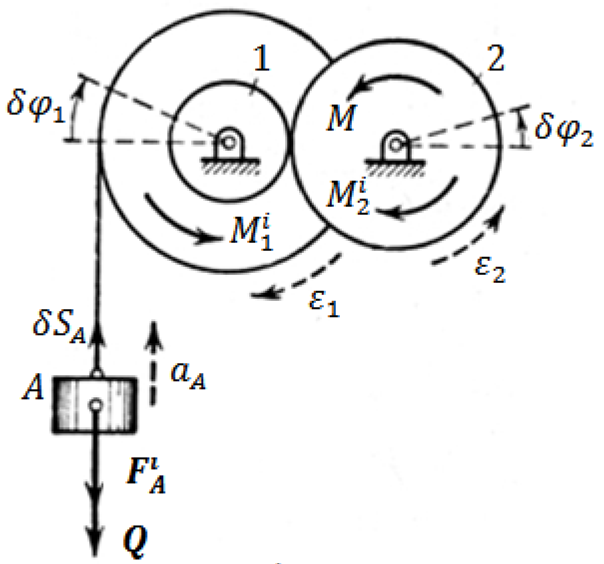

Fig. 92

$$F_A^i = \frac{Q}{g} a_A, \quad |M_1^i| = \frac{P_1}{g} \varrho_1^2 \varepsilon_1,$$
$$|M_2^i| = \frac{P_2}{g} \varrho_2^2 \varepsilon_2.$$

The directions of all the vectors are shown in the Fig 92. Eq. (6.9) for a virtual displacement of the system, we obtain

$$-(Q + F_A^i)\delta s_A - M_1^i \delta\varphi_1 + (M - M_2^i)\delta\varphi_2 = 0.$$

Expressing all the displacements in terms of $\delta\varphi_1$, we have

$$\delta s_A = r\delta\varphi_1, \quad \frac{\delta\varphi_1}{\delta\varphi_2} = \frac{\omega_1}{\omega_2} = \frac{r_2}{r_1}$$

and $\delta\varphi_2 = \frac{r_1}{r_2}\delta\varphi_1$.

Finally the equation of motion takes the form

$$Q\left(1 + \frac{a_A}{g}\right) r + \frac{P_1}{g}\varrho_1^2\varepsilon_1 + \frac{P_2}{g}\varrho_2^2\varepsilon_2\frac{r_1}{r_2} - M\frac{r_1}{r_2} = 0.$$

Now express the quantities ε_1 and ε_2 in terms of the required acceleration a_A Taking into account that ε_1 and ε_2 are related in the same way as ω_1 and ω_2, we obtain

$$\varepsilon_1 = \frac{a_A}{r}, \ \varepsilon_2 = \frac{r_1}{r_2}\varepsilon_1 = \frac{r_1}{r_2}\frac{a_A}{r}.$$

And finally we have

$$a_A = \frac{\frac{r_1}{r_2}M - rQ}{rQ + \frac{\varrho_1^2}{r}P_1 + \frac{\varrho_2^2 r_1^2}{r r_2^2}P_2} g.$$

Problem 61. One end of a thread is wound on a uniform cylinder of weight P_1 (Fig. 93). The thread passes over a pulley O, and its other end is attached to a load A of weight P_2 which slides on a horizontal plane, the coefficient of friction being f. Neglecting the mass of the pulley, and the string, determine the acceleration of the load and of the centre C of the cylinder.

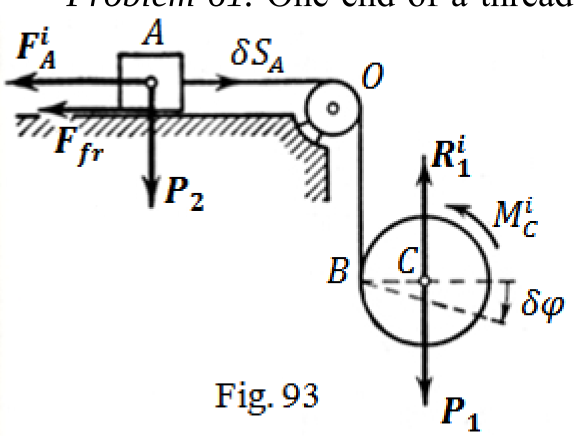

Fig. 93

So1ution. If motion starts from rest, the centre of the cylinder C will move vertically, and the system has two degrees of freedom (the rotation of the cylinder with respect to the thread when the load is at rest and the displacement of the load when the cylinder does not rotate).

Add to the acting forces $\boldsymbol{P_1}, \boldsymbol{P_2}$, and $\boldsymbol{F_{fr}}$ the inertia forces of the cylinder reduced to a principal vector $\boldsymbol{R_1^i}$ and a couple of moment M_C^i, and the inertia force $\boldsymbol{F_A^i}$ of the load. In magnitude

$$F_A^i = \frac{P_2}{g} a_A, R_1^i = \frac{P_1}{g} a_C, \left|M_C^i\right| = J_C \varepsilon = \frac{P_1}{2g} r^2 \frac{a_C - a_A}{r}.$$

The last equality follows from the fact that, if point C of the cylinder has a velocity v_C, and point B (together with the string) a velocity $v_B = v_A$, then the angular velocity of the cylinder $\omega = \frac{v_C - v_A}{r}$, and, consequently, $\varepsilon = \frac{a_C - a_A}{r}$. Furthermore, for the cylinder $J_C = 0.5mr^2$, where r is the radius of the cylinder.

Now consider a virtual displacement δs_A of the system in which the cylinder does not rotate and is translated together with the load. The couple of moment M_C^i does no work in this displacement and from Eq. (6.9) we obtain

$$\left(-F_{fr} - F_A^i - R_1^i + P_1\right)\delta s_A = 0,$$

whence, as $F_{fr} = fP_2$, we find

$$\frac{P_1}{g} a_C + \frac{P_2}{g} a_A = P_1 - fP_2.$$

Consider the other independent virtual displacement in which the load A remains at rest while the cylinder turns about point B (which in this displacement is the instantaneous centre of rotation) through angle $\delta\varphi$. For this displacement, Eq. (6.9) gives

$$\left(P_1 - R_1^i\right) r\delta\varphi - M_C^i \delta\varphi = 0$$

.

Substituting the expressions for R_1^i and M_C^i we obtain

$$3a_C - a_A = 2g.$$

Solving equations simultaneously, we obtain the required accelerations

$$a_A = \frac{P_1 - 3fP_2}{P_1 + 3P_2} g, a_C = \frac{P_1 + (2 - f)P_2}{P_1 + 3P_2} g.$$