5. GENERAL THEOREMS OF DYNAMICS

In solving many problems of dynamics it will be found that the so-called general theorems, representing corollaries of the fundamental law of dynamics, are more conveniently applied than the method of integration of differential equations of motion.

The importance of the general theorems is that they establish visual relationships between the principal dynamic characteristics of motion of material bodies, thereby presenting broad possibilities for analyzing the mechanical motion widely employed in practical engineering. Furthermore, the general theorems make it possible to study for practical purposes specific aspects of a given phenomenon without investigating the phenomenon as a whole. Finally, the use of the general theorems makes it unnecessary to carry out for every problem the operations of integration performed once and for all in proving the theorems, which simplifies the solution.

5.1. Momentum of a Particle and a System

One of the basic dynamic characteristics of particle motion is momentum (or linear momentum).

The momentum of a particle is defined as a vector quantity $m\boldsymbol{v}$ *equal to the product of the mass of the particle and its velocity.* The vector $m\boldsymbol{v}$ is directed in the same direction as the velocity, i.e., tangent to the path of the particle.

The linear momentum, or simply the momentum, of a system is defined as the vector quantity $\boldsymbol{Q}$ *equal to the geometric sum (the principal vector) of the moments of all the particles of the system* (Fig 28):

$$\boldsymbol{Q} = \sum m_k \boldsymbol{v}_k. \tag{5.1}$$

It can be seen from the diagram that, irrespective of the velocities of the particles (provided that they are not parallel) the momentum vector can take any value, or even be zero when the polygon constructed with the vectors $m_k\boldsymbol{v}_k$ as its sides is closed. Consequently, the quantity $\boldsymbol{Q}$ does not characterize the motion of the system completely. Let us develop a formula with which it is much more convenient to compute $\boldsymbol{Q}$ and also to explain its meaning. It follows from Eq. (4.5) that

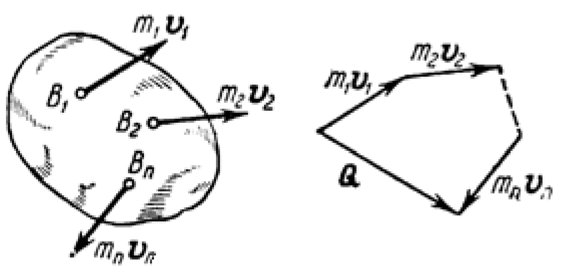

Fig. 28

$$\sum m_k \boldsymbol{r}_k = M\boldsymbol{r}_C.$$

Differentiating both sides with respect to time, we obtain

$$\sum m_k \frac{d\boldsymbol{r}_k}{dt} = M\frac{d\boldsymbol{r}_C}{dt} \text{ or } \sum m_k \boldsymbol{v}_k = M\boldsymbol{v}_C,$$

whence we find that

$$\boldsymbol{Q} = M\boldsymbol{v}_C, \tag{5.2}$$

i.e., the momentum of a system is equal to the product of the mass of the whole system and the velocity of its center of mass. This equation is especially convenient in computing the momentum of rigid bodies.

It follows from Eq. (5.2) that, if the motion of a body (or a system) is such that the center of mass remains motionless, the momentum of the body is zero. Thus, the momentum of a body rotating about a fixed axis through its center of mass is zero (the polygon in Fig. 28 is closed).

If, on the other hand, a body has relative motion, the quantity $\boldsymbol{Q}$ will not characterize the rotational component of the motion about the center of mass. Thus, for a rolling wheel, $\boldsymbol{Q} = M\boldsymbol{v}_c$, regardless of how the wheel rotates about its center of mass *C*.

We see, therefore, that momentum characterizes only the translational motion of a system, which is why it is often called linear momentum. In relative motion, the quantity $\boldsymbol{Q}$ characterizes only the translational component of the motion of a system together with its center of mass.

5.2. Impulse of a Force

The concept of impulse (or linear impulse) of a force is used to characterize the effect on a body of a force acting during a certain interval of time. First let us introduce the concept of elementary impulse, i.e., impulse in an infinitesimal time interval dt. Elementary impulse is defined as a vector quantity $d\boldsymbol{S}$ equal to the product of the vector of the force $\boldsymbol{F}$ and the time element dt:

$$d\boldsymbol{S} = \boldsymbol{F}\,dt.$$

The elementary impulse is directed along the action line of the force. The impulse $\boldsymbol{S}$ of any force $\boldsymbol{F}$ during a finite time interval t_1 is computed as the integral sum of the respective elementary impulses:

$$\boldsymbol{S} = \int_0^{t_1} \boldsymbol{F}\,dt. \tag{5.3}$$

Thus, the impulse of a force in any time interval t_1 is equal to the integral of the elementary impulse over the interval from zero to t_1.

In the special case when the force $\boldsymbol{F}$ is of constant magnitude and direction ($\boldsymbol{F} = const.$), we have $\boldsymbol{S} = \boldsymbol{F}t_1$. In the general case the magnitude of an impulse can be computed from its projections. We can find the projections of an impulse on a set of coordinate axes if we remember that an integral is the limit of a sum, and the projection of a vector sum on an axis is equal to the sum of the projections of the component vectors on the same axis. Hence,

$$S_x = \int_0^{t_1} F_x\,dt,\ S_y = \int_0^{t_1} F_y\,dt,\ S_z = \int_0^{t_1} F_z\,dt.$$

With these projections we can construct the vector $\boldsymbol{S}$ and find its magnitude and the angles it makes with the coordinate axes. The dimension of impulse in the international system of units is $[S] = N \cdot sec$.

To solve the principal problem of dynamics, it is important to establish the forces whose impulses can be computed without knowing the equation of motion of the particle moving under the action of those forces. It is apparent that to these forces belong only constant forces and forces depending on time.

5.3. Theorem of the Motion of Center of Mass

In many cases the nature of the motion of a system (especially of a rigid body) is completely described by the law of motion of its center of mass. To develop this law, let us take the equations of motion of a system (4.10) and add separately their left and right sides. We obtain

$$\sum m_k \boldsymbol{a}_k = \sum \boldsymbol{F}_k^e + \sum \boldsymbol{F}_k^i. \quad (5.4)$$

Let us transform the left side of the equation. For the radius vector of the center of mass we have

$$\sum m_k \boldsymbol{r}_k = M \boldsymbol{r}_C.$$

Taking the second derivative of both sides of this equation with respect to time, and noting that the derivative of a sum equals the sum of the derivatives, we find

$$\sum m_k \frac{d^2 \boldsymbol{r}_k}{dt^2} = M \frac{d^2 \boldsymbol{r}_k}{dt^2}$$

or

$$\sum m_k \, \boldsymbol{a}_k = M \boldsymbol{a}_C,$$

where $\boldsymbol{a}_c$ is the acceleration of the center of mass of the system. As the internal forces of a system give $\sum \boldsymbol{F}_k^i = 0$, by substituting all the developed expressions into Eq. (5.4), we obtain finally:

$$M \boldsymbol{a}_c = \sum \boldsymbol{F}_k^e. \quad (5.5)$$

Eq. (5.5) states the theorem of the motion of the center of mass of a system. Its form coincides with that of the equation of motion of a particle of mass *m=M* where the acting forces are equal to $\boldsymbol{F}_k^e$. We can therefore formulate the theorem of the motion of the centre of mass as follows: *the center of mass of a system moves as if it were a particle of mass equal to the mass of the whole system to which are applied all the external forces acting on the system.* Projecting both sides of Eq. (5.5) on the coordinate axes, we obtain

$$M \frac{d^2 x_c}{dt^2} = \sum F_{kx}^e, \; M \frac{d^2 y_c}{dt^2} = \sum F_{ky}^e, \; M \frac{d^2 z_c}{dt^2} = \sum F_{kz}^e. \quad (5.6)$$

These are the differential equations of motion of the center of mass in terms of the projections on the coordinate axes. The theorem is valuable for the following reasons:

1) It justifies the use of the methods of particle dynamics. It follows from Eqs. (5.6) that the solutions developed on the assumption that a given body is equivalent to a particle define the law of motion of the center of mass of that body. Thus, these solutions have a concrete meaning. In particular, if a body is being translated, its motion is completely specified by the motion of its center of mass, and consequently, a body in translational motion can always be treated as a particle of mass equal to the

mass of the body. In all other cases, a body can be treated as a particle only when the position of its center of mass is sufficient to specify the position of the body.

2) The theorem makes it possible, in developing the equation of motion for the centre of mass of any system, to ignore all unknown internal forces. This is of special practical value.

5.4. The Law of Conservation of Motion of Center of Mass

The following important corollaries arise from the theorem of the motion of center of mass:

1) Let the sum of the external forces acting on a system be zero:

$$\sum \boldsymbol{F}_k^e = 0.$$

It follows, then, from Eq. (5.5) that $\boldsymbol{a}_c = 0$ or $\boldsymbol{v}_c = const.$ Thus, *if the sum of all the external forces acting on a system is zero, the center of mass of that system moves with a velocity of constant magnitude and direction, i.e., uniformly and rectilinearly.* In particular, if the center of mass was initially at rest it will remain at rest. The action of the internal forces, we see, does not affect the motion of the center of mass.

2) Let the sum of the external forces acting on a system be other than zero, but let the sum of their projections on one of the coordinate axes (the *x*- axis, for instance), be zero:

$$\sum F_{kx}^e = 0.$$

The first of Eqs. (5.6), then, gives

$$\frac{d^2x_c}{dt^2} = 0 \text{ or } \frac{dx_c}{dt} = v_{c_x} = const.$$

Thus, *if the sum of the projections on an axis of all the external forces acting on a system is zero, the projection of the velocity of the center of mass of the system on that axis is a constant quantity.* In particular, if at the initial moment $v_{c_x} = 0$, it will remain zero at any subsequent instant, i.e., the center of mass of the system will not move along the *x*-axis ($x_C = const.$).

The above results express the law of conservation of motion of the center of mass of a system.

5.5. Theorem of the Change in the Momentum of a Particle

As the mass of a particle is constant, and its acceleration $\boldsymbol{a} = \frac{dv}{dt}$ equation, which expresses the fundamental law of dynamics, can be expressed in the form:

$$\frac{d(mv)}{dt} = \sum \boldsymbol{F}_k. \tag{5.7}$$

Let a particle of mass *m* moving under the action of a force $\boldsymbol{R} = \sum \boldsymbol{F}_k$ have a velocity $\boldsymbol{v}_0$ at time $t = 0$, and at time t_1 let its velocity be $\boldsymbol{v}_1$. Now let us multiply both sides of Eq. (5.7) by *dt* and take definite integrals. On the right side, where we integrate with respect to time, the limits of the integrals are zero and t_1; on the left side, where we integrate the velocity, the limits of the integral are the respective values of $\boldsymbol{v}_0$ and $\boldsymbol{v}_1$. As the integral of $d\,(m\boldsymbol{v})$ is $m\boldsymbol{v}$, we have

$$m\boldsymbol{v}_1 - m\boldsymbol{v}_0 = \sum \int_0^{t_1} \boldsymbol{F}_k\, dt.$$

By Eq. (5.3), the integrals on the right side are the impulses of the acting forces. Hence, we finally have

$$m\boldsymbol{v}_1 - m\boldsymbol{v}_0 = \sum \boldsymbol{S}_k. \tag{5.8}$$

Eq. (5.8) states the theorem of the change in the linear momentum of a particle: *the change in the momentum of a particle during any time interval is equal to the geometric sum of the impulses of all the forces acting on the particle during that interval of time.*

In problem solutions, projection equations are often used instead of the vector equation (5.8). Projecting both sides of Eq. (5.8) on a set of coordinate axes, we have

$$\left.\begin{aligned} mv_{1x} - mv_{0x} &= \sum S_{kx} \\ mv_{1y} - mv_{0y} &= \sum S_{ky} \\ mv_{1z} - mv_{0z} &= \sum S_{kz} \end{aligned}\right\} \tag{5.8'}$$

In the case of rectilinear motion along the *x*- axis, the theorem is stated by the first of these equations.

5.6. Theorem of the Change in Linear Momentum of the System

Consider a system of *n* particles. Writing the differential equations of motion (4.10) for this system and adding them, we obtain

$$\sum m_k \boldsymbol{a}_k = \sum \boldsymbol{F}_k^e + \sum \boldsymbol{F}_k^i.$$

From the property of internal forces the last summation is zero. Furthermore,

$$\sum m_k \boldsymbol{a}_k = \frac{d}{dt}\left(\sum m_k \boldsymbol{v}_k\right) = \frac{d\boldsymbol{Q}}{dt},$$

and we finally have

$$\frac{d\boldsymbol{Q}}{dt} = \sum \boldsymbol{F}_k^e. \tag{5.9}$$

Eq. (5.9) states the theorem of the change in the linear momentum of a system in differential form: *the derivative of the linear momentum of a system with respect to time is equal to the geometrical sum of all the external forces acting on the system.* In terms of projections on Cartesian axes we have

$$\frac{dQ_x}{dt} = \sum F_{kx}^e,\ \frac{dQ_y}{dt} = \sum F_{ky}^e,\ \frac{dQ_z}{dt} = \sum F_{kz}^e. \tag{5.10}$$

Let us develop another expression for the theorem. Let the momentum of a system be $\boldsymbol{Q}_0$ at time $t = 0$, and at time t_1 let it be $\boldsymbol{Q}_1$. Multiplying both sides of Eq. (5.9) by *dt* and integrating, we obtain

$$\boldsymbol{Q}_1 - \boldsymbol{Q}_0 = \sum \int_0^{t_1} \boldsymbol{F}_k^e\, dt,$$

or

$$\boldsymbol{Q}_1 - \boldsymbol{Q}_0 = \sum \boldsymbol{S}_k^e, \tag{5.11}$$

as the integrals to the right give the impulses of the external forces. Eq. (5.11) states the theorem of the change in the linear momentum of a system in integral form: *the change in the linear momentum of a system during any time interval is equal to the* *sum of the impulses of the external forces acting on the system during the same interval of time.* In terms of projections on Cartesian axes we have

$$\left.\begin{aligned} Q_{1x} - Q_{0x} &= \sum S_{kx}^e \\ Q_{1y} - Q_{0y} &= \sum S_{ky}^e \\ Q_{1z} - Q_{0z} &= \sum S_{kz}^e \end{aligned}\right\} \qquad (5.11')$$

Let us show the connection between this theorem and the theorem of the motion of center of mass. As $\boldsymbol{Q} = M\boldsymbol{v}_c$, by substituting this expression into Eq. (5.9) and taking into account that $\frac{d\boldsymbol{v}_c}{dt} = \boldsymbol{a}_c$ we obtain $M\boldsymbol{a}_c = \sum \boldsymbol{F}_k^e$, i.e., Eq. (5.5).

Consequently, the theorem of the motion of center of mass and the theorem of the change in the momentum of a system are, in fact, two forms of the same theorem. Whenever the motion of a rigid body (or system of bodies) is being investigated, both theorems may be used, though Eq. (5.5) is usually more convenient.

For a continuous medium (a fluid), however, the concept of center of mass of the whole system is virtually meaningless, and the theorem of the change in the momentum of a system is used in the solution of such problems.

The practical value of the theorem is that it enables us to exclude from consideration the immediately unknown internal forces (for instance, the reciprocal forces acting between the particles of a liquid).

5.7. The Law of Conservation of Linear Momentum

The following important corollaries arise from the theorem of the change in the momentum of a system:

1) Let the sum of all the external forces acting on a system be zero:

$$\sum \boldsymbol{F}_k^e = 0.$$

It follows from Eq. (5.9) that in this case $\boldsymbol{Q} = const$. Thus, *if the sum of all the external forces acting on a system is zero, the momentum vector of the system is constant in magnitude and direction.*

2) Let the external forces acting on a system be such that the sum of their projections on any axis *Ox* is zero:

$$\sum F_{kx}^e = 0.$$

It follows from Eqs. (5.10) that in this case $Q_x = const$. Thus, *if the sum of the projections on any axis of all the external forces acting on a system is zero, the projection of the momentum of that system on that axis is a constant quantity.*

These results express the law of conservation of the linear momentum of a system.

5.8. Theorem of the Change in the Angular Momentum of a Particle

Often, in analyzing the motion of a particle, it is necessary to consider the change not of the vector $m\boldsymbol{v}$ itself, but of its moment. The moment of the vector $m\boldsymbol{v}$ with respect to any center *0* or axis *z* is denoted by the symbol $\boldsymbol{m}_0(m\boldsymbol{v})$ or $m_z(m\boldsymbol{v})$

and is called the moment of momentum or angular momentum with respect to that center or axis. The moment of vector $m\boldsymbol{v}$ is calculated in the same way as the moment of a force. Vector $m\boldsymbol{v}$ is considered to be applied to the moving particle. In magnitude $|\boldsymbol{m}_0(m\boldsymbol{v})| = mvh$, where h is the perpendicular distance from 0 to the position line of the vector $m\boldsymbol{v}$ (see Fig. 29).

1. *Principle of Moments About an Axis.* Consider a particle of mass m moving under the action of a force $\boldsymbol{F}$. Let us establish the dependence between the moments of the vectors $m\boldsymbol{v}$ and $\boldsymbol{F}$ with respect to any fixed axis z.

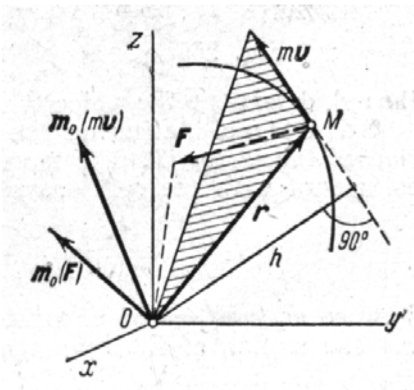

Fig.29

It is well known that

$$m_z(\boldsymbol{F}) = xF_y - yF_x. \tag{5.12}$$

Similarly, for $m_z(m\boldsymbol{v})$, and taking m out of the parentheses, we have

$$m_z(m\boldsymbol{v}) = m(xv_y - yv_x). \tag{5.13}$$

Differentiating both sides of this equation with respect to time, we obtain

$$\frac{d}{dt}[m_z(m\boldsymbol{v})] = m\left(\frac{dx}{dt}v_y - \frac{dy}{dt}v_x\right) + \left(xm\frac{dv_y}{dt} - ym\frac{dv_x}{dt}\right).$$

The expression in the first parentheses of the right side of the equation is zero, as $\frac{dx}{dt} = v_x$ and $\frac{dy}{dt} = v_y$. From Eq. (5.12), the expression in the second pair of parentheses is equal to $m_z(\boldsymbol{F})$, since, from the fundamental law of dynamics,

$$m\frac{dv_x}{dt} = F_x, m\frac{dv_y}{dt} = F_y.$$

Finally, we have

$$\frac{d}{dt}[m_z(m\boldsymbol{v})] = m_z(\boldsymbol{F}). \tag{5.14}$$

This equation states the principle of moments about an axis: *the derivative of the angular momentum of a particle about any axis with respect to time is equal to the moment of the acting force about the same axis*

2. Principle of Moments about a Center. Let us find for a particle moving under the action of a force $\boldsymbol{F}$ (Fig. 29) the relation between the moments of vectors $m\boldsymbol{v}$ and $\boldsymbol{F}$ with respect to any fixed center 0. It was shown that $\boldsymbol{m}_0(\boldsymbol{F}) = \boldsymbol{r} \times \boldsymbol{F}$. Similarly,

$$\boldsymbol{m}_0(m\boldsymbol{v}) = \boldsymbol{r} \times m\boldsymbol{v}.$$

Vector $\boldsymbol{m}_0(\boldsymbol{F})$ is normal to the plane through 0 and vector $\boldsymbol{F}$, while the vector $\boldsymbol{m}_0(m\boldsymbol{v})$ is normal to the plane through the center 0 and vector $m\boldsymbol{v}$. Differentiating the expression $\boldsymbol{m}_0(m\boldsymbol{v})$ with respect to time, we obtain

$$\frac{d}{dt}(\boldsymbol{r} \times m\boldsymbol{v}) = \left(\frac{dr}{dt} \times m\boldsymbol{v}\right) + \left(\boldsymbol{r} \times m\frac{dv}{dt}\right) = (\boldsymbol{v} \times m\boldsymbol{v}) + (\boldsymbol{r} \times m\boldsymbol{a}).$$

But $\boldsymbol{v} \times m\boldsymbol{v} = 0$, as the vector product of two parallel vectors, and $m\boldsymbol{a} = \boldsymbol{F}$. Hence,

$$\frac{d}{dt}(\boldsymbol{r} \times m\boldsymbol{v}) = \boldsymbol{r} \times \boldsymbol{F}, \text{ or } \frac{d}{dt}[\boldsymbol{m}_0(m\boldsymbol{v})] = \boldsymbol{m}_0(\boldsymbol{F}). \tag{5.15}$$

This is the principle of moments about a center: *the derivative of the angular momentum of a particle about any fixed center with respect to time is equal to the moment of the force acting on the particle about the same center.* An analogous theorem is true for the moments of vector $\boldsymbol{mv}$ and force $\boldsymbol{F}$ with respect to any axis *z,* which is evident if we project both sides of Eq. (5.15) on that axis. This was proved directly in item 1.

5.9. Total Angular Momentum of a System

The total angular momentum of a system with respect to any center 0 is defined as the quantity $\boldsymbol{K}_0$, equal to the geometrical sum of the angular momenta of all the particles of the system with respect to that center:

$$\boldsymbol{K}_0 = \sum \boldsymbol{m}_0(m_k \boldsymbol{v}_k). \tag{5.16}$$

The angular moment of a system with respect to each of three rectangular coordinate axes are found similarly:

$$K_x = \sum m_x(m_k \boldsymbol{v}_k)\,, K_y = \sum m_y(m_k \boldsymbol{v}_k), K_z = \sum m_z(m_k \boldsymbol{v}_k). \tag{5.17}$$

By the theorem proved in § 5.8, K_x, K_y, K_z are the respective projections of vector $\boldsymbol{K}_0$ on the coordinate axes.

To understand the physical meaning of $\boldsymbol{K_0}$, let us compute the angular momentum of a rotating body with respect to its axis of rotation. If a body rotates about a fixed axis *Oz* (Fig. 30), the linear velocity of any particle of the body at a distance h_k from the axis is ωh_k. Consequently, for that particle

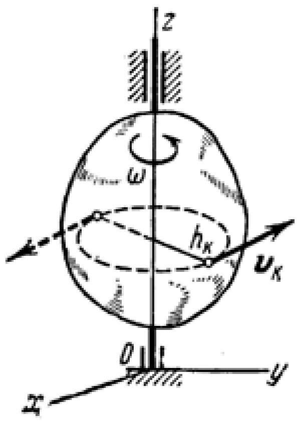

Fig. 30

$$m_z(m_k \boldsymbol{v}_k) = m_k v_k h_k = m_k \omega h_k^2.$$

Then, taking the common multiplier ω outside of the parentheses, we obtain for the whole body

$$K_z = \sum m_z\,(m_k \boldsymbol{v}_k) = (\sum m_k h_k^2)\omega.$$

The quantity in the parentheses is the moment of inertia of the body with respect to the *z*- axis (§ 4.3). We finally obtain

$$K_z = J_z \omega. \tag{5.18}$$

Thus, *the angular momentum of a rotating body with respect to the axis of rotation is equal to the product of the moment of inertia of the body and its angular velocity.*

If a system consists of several bodies rotating about the same axis, then, apparently,

$$K_z = J_{1z}\omega_1 + J_{2z}\omega_2 + \ldots + J_{nz}\omega_n.$$

The analogy between Eqs. (5.2) and (5.18) will be readily noticed: the momentum of a body is the product of its mass (the quantity characterizing the body's inertia in translational motion) and its velocity; the angular momentum of a body is equal to the product of its moment of inertia (the quantity characterizing a body's inertia in rotational motion) and its angular velocity.

Just as the momentum of a system is a characteristic of its translational motion, the total angular momentum of a system is a characteristic of its rotational motion.

5.10. Theorem of the Change in the Total Angular Momentum of a System

The principle of moments, which was proved for a single particle (§ 5.8), is valid for all the particles of a system. If, therefore, we consider a particle of mass m_k and velocity $\boldsymbol{v}_k$ belonging to a system, we have for that particle

$$\frac{d}{dt}[\boldsymbol{m}_0(m_k\boldsymbol{v}_k)] = \boldsymbol{m}_0(\boldsymbol{F}_k^e) + \boldsymbol{m}_0(\boldsymbol{F}_k^i).$$

where $\boldsymbol{F}_k^e$ and $\boldsymbol{F}_k^i$ are the resultants of all the external and internal forces acting on the particle.

Writing such equations for all the particles of the system and adding them, we obtain

$$\frac{d}{dt}\sum[\boldsymbol{m}_0(m_k\boldsymbol{v}_k)] = \sum \boldsymbol{m}_0(\boldsymbol{F}_k^e) + \sum \boldsymbol{m}_0(\boldsymbol{F}_k^i).$$

But from the properties of the internal forces of a system, the last summation vanishes. Hence, taking into account Eq. (5.16), we obtain finally

$$\frac{d\boldsymbol{K}_0}{dt} = \sum \boldsymbol{m}_0(\boldsymbol{F}_k^e). \tag{5.19}$$

This equation states the following principle of moments for a system: *The derivative of the total angular momentum of a system about any fixed center with respect to time is equal to the sum of the moments of all the external forces acting on that system about that center.*

Projecting both sides of Eq. (5.19) on a set of fixed axes *Oxyz* , we obtain

$$\frac{dK_x}{dt} = \sum m_x(\boldsymbol{F}_k^e), \frac{dK_y}{dt} = \sum m_y(\boldsymbol{F}_k^e), \frac{dK_z}{dt} = \sum m_z(\boldsymbol{F}_k^e). \tag{5.20}$$

5.11. The Law of Conservation of the Total Angular Momentum

The following important corollaries can be derived from the principle of moments.

1) Let the sum of the moments of all the external forces acting on a system with respect to a center *0* be zero:

$$\sum \boldsymbol{m}_0(\boldsymbol{F}_k^e) = 0.$$

It follows, then, from Eq. (5.19) that $\boldsymbol{K}_0 = const$. Thus, *if the sum of the moments of all external forces acting on a system taken with respect to any center is zero, the total angular momentum of the system with respect to that center is constant in magnitude and direction.*

2) Let the external forces acting on a system be such that the sum of their moments with respect to any fixed axis *Oz* is zero:

$$\sum m_z(\boldsymbol{F}_k^e) = 0.$$

It follows, then, from Eqs. (5.20) that $K_z = const$. Thus, *if the sum of the moments of all the external forces acting on a system with respect to any axis is zero,* *the total angular momentum of the system with respect to that axis is constant.*

These conclusions express the law of conservation of the total angular momentum of a system. It follows from them that internal forces cannot change the total angular momentum of a system.

5.12. Kinetic Energy of Particle and a System

The kinetic energy of a particle is a scalar quantity equal $\frac{1}{2}mv^2$. *The kinetic energy of a system is defined as a scalar quantity T equal to the arithmetical sum of the kinetic energies of all the particles of the system:*

$$T = \sum \frac{m_k v_k^2}{2}. \tag{5.21}$$

If a system consists of several bodies, its kinetic energy is, evidently, equal to the sum of the kinetic energies of all the bodies:

$$T = \sum T_k.$$

Let us develop the equations for computing the kinetic energy of a body in different types of motion.

Translational Motion. In this case all the points of a body have the same velocity, which is equal to the velocity of the centre of mass. Therefore, for any point $v_k = v_c$, and Eq. (5.21) gives

$$T_{trans} = \sum \frac{m_k v_k^2}{2} = \frac{1}{2}\left(\sum m_k\right) v_c^2,$$

$$T_{trans} = \frac{1}{2} M v_c^2. \tag{5.22}$$

Rotational Motion. The velocity of any point of a body rotating about an axis *Oz* is $v_k = \omega h_k$, where h_k is the distance of the point from the axis of rotation, and ω is the angular velocity of the body. Substituting this expression into Eq. (5.21) and taking the common multipliers outside the parentheses we obtain

$$T_{rotation} = \sum \frac{m_k \omega^2 h_k^2}{2} = \frac{1}{2}\left(\sum m_k h_k^2\right)\omega^2.$$

The term in the parentheses is the moment of inertia of the body with respect to the axis *z*. Thus we finally obtain

$$T_{rotaion} = \frac{1}{2} J_z \omega^2. \tag{5.23}$$

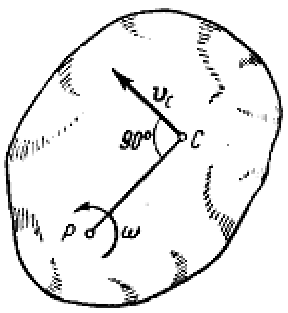

Fig. 31

Plane Motion. In plane motion, the velocities of all the points of a body are at any instant directed as if the body were rotating about an axis perpendicular to the plane of motion and passing through the instantaneous centre of zero velocity *P* (Fig. 31). Hence, by Eq. (5.23)

$$T_{plane} = \frac{1}{2} J_p \omega^2.$$

where J_p is the moment of inertia of the body with respect to the instantaneous axis of rotation.

The quantity J_p is variable, as the position of the centre P continuously changes with the motion of the body. Let us introduce instead of J_p a constant moment of inertia J_c with respect to an axis through the centre of mass C of the body. By the parallel-axis theorem, $J_p = J_c + Md^2$, where $d=PC$.

Substituting this expression for J_p and taking into account that point P is the instantaneous centre of zero velocity and therefore $\omega d = \omega\, PC = v_c$, where v_c is the velocity of the centre of mass, we obtain finally

$$T_{plane} = \frac{1}{2}Mv_c^2 + \frac{1}{2}J_c\omega^2. \qquad (5.24)$$

5.13. Work Done by a Force. Power

The concept of work is introduced as a measure of the action of a force on a body in a given displacement, specifically that action which is represented by the change in the magnitude of the velocity of a moving particle.

First let us introduce the concept of elementary work done by a force in infinitesimal displacement *ds*. The elementary work done by a force $\boldsymbol{F}$ (Fig. 32) is defined as a scalar quantity

$$dA = F_\tau\, ds, \qquad (5.25)$$

where F_τ is the projection of the force on the tangent to the path in the direction of the displacement, and ds is an infinitesimal displacement of the particle along that tangent.

This definition corresponds to the concept of work as a characteristic of that action of a force which tends to change the magnitude of velocity. For if force $\boldsymbol{F}$ is resolved into components $\boldsymbol{F}_\tau$ and $\boldsymbol{F}_n$, only the component $\boldsymbol{F}_\tau$, which imparts the particle its tangential acceleration, will change the magnitude of the velocity. Noting that $F_\tau = F\cos\alpha$, we further obtain from Eq. (5.25)

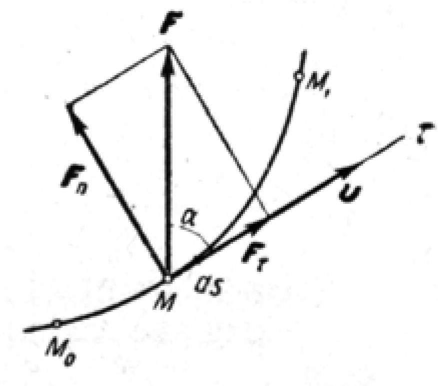

Fig. 32

$$dA = F\, ds \cos\alpha. \qquad (5.26)$$

If angle α is acute, the work is of positive sense. In particular, at $\alpha = 0$, the elementary work $dA = F\, ds$.

If angle α is obtuse, the work is of negative sense. In particular, at $\alpha = 180°$, the elementary work $dA = -F\, ds$.

If angle $\alpha = 90°$, i.e., if a force is directed perpendicular to the displacement, the elementary work done by the force is zero.

Let us now find an analytical expression for elementary work. For this we resolve force F into components F_x, F_y, F_z, parallel to the coordinate axes (Fig. 33). The infinitesimal displacement $MM'=ds$ is compounded of the displacements dx, dy, dz parallel to the coordinate axes, where x, y, z are the coordinates of point M. The work done by force F in the displacement ds can be calculated as the sum of the work done by its components F_x, F_z, F_z in the displacements *dx, dy, dz.*

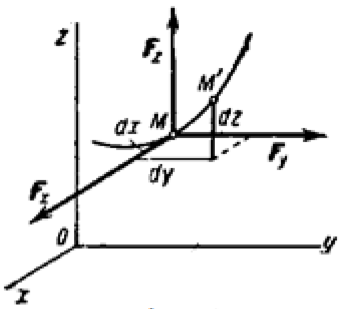

Fig. 33

But the work in the displacement dx is done only by component F_x and is equal to $F_x\, dx$. The work in the displacements dy and dz is calculated similarly. Thus, we finally obtain

$$dA = F_x\, dx + F_y\, dy + F_z\, dz. \qquad (5.27)$$

Eq. (5.27) gives the analytical expression of the elementary work done by a force.

The work done by a force in any finite displacement M_0M_1 (see Fig. 32) is calculated as the integral sum of the corresponding elementary works and is equal to

$$A_{(M_0M_1)} = \int_{(M_0)}^{(M_1)} F_\tau\, ds, \qquad (5.28)$$

$$A_{(M_0M_1)} = \int_{(M_0)}^{(M_1)} \left(F_x\, dx + F_y\, dy + F_z\, dz\right). \qquad (5.29)$$

The limits of the integral correspond to the values of the variables of integration at points M_0 and M_1, (or, more exactly, the integral is taken along the curve M_1M_0, i.e., it is curvilinear).

In order to solve the principal problem of dynamics, it is important to establish the forces whose work can be calculated immediately without knowing the equation of motion of the particle on which they are acting. It can be seen that to these forces belong only constant forces or forces which depend on the position (coordinates) of a moving particle.

Power. The term *power* is defined as the work done by a force in a unit of time (the time rate of doing work). If work is done at a constant rate, the power

$$N = \frac{A}{t_1},$$

where t_1 is the time in which the work *A* is done. In the general case,

$$N = \frac{dA}{dt} = \frac{F_\tau ds}{dt} = F_\tau v.$$

It can be seen from the equation $N = F_\tau v$ that if a motor has a given power *N*, the tractive force F_τ is inversely proportional to the velocity v. That is why, for instance, on an upgrade or poor road a motor car goes into lower gear, thereby reducing the speed and developing a greater tractive force with the same power.

5.14. Examples of Calculation of Work

The examples considered below give results which can be used immediately in solving problems.

Work Done by a Gravitational Force. Let a particle *M* subjected to a gravitational force ***P*** be moving from a point $M_0(x_0,y_0,z_0)$ to a point $M_1(x_1,y_1,z_1)$. Choose a coordinate system so that the axis *Oz* would point vertically up (Fig. 34). Then $P_x = 0$, $P_y = 0$, $P_z = -P$. Substituting these expressions into Eq. (5.29) and taking into account that the integration variable is *z,* we obtain

$$A_{(M_0M_1)} = \int_{(M_0)}^{(M_1)} (-P\,dz) = -P\int_{z_0}^{z_1} dz = P(z_0 - z_1).$$

If point M_0 is higher than M_1 then $z_0 - z_1 = h$, where h is the vertical displacement of the particle; if, on the other hand, M_0 is below M_1 then $z_0 - z_1 = -(z_1 - z_0) = -h$. Finally we have

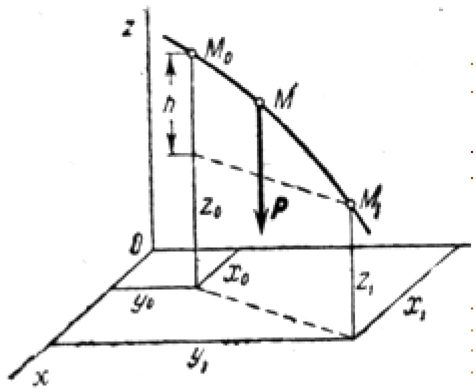

Fig. 34

$$A_{(M_0M_1)} = \pm Ph.$$

The work is positive if the initial point is higher than the final one and negative if it is lower.

It follows from this that the work done by gravity does not depend on the path along which the point of its application moves. Forces possessing this property are called conservative forces.

Work Done by an Elastic Force. Consider a weight *M* lying in a horizontal plane and attached to the free end of a spring (Fig. 35). Let point *O* on the plane represent the position of the end of the spring when it is not in tension (*AO* $=l_0$ is the length of the unextended spring) and let it be the origin of our coordinate system.

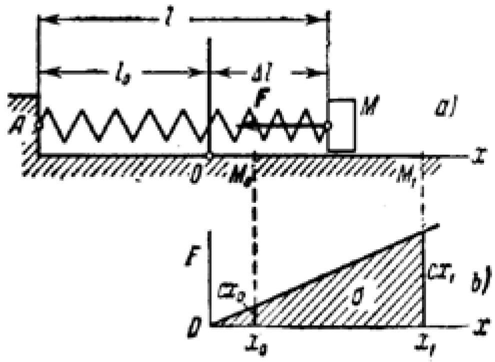

Fig. 35

Now if we draw the weight from its position of equilibrium *0*, stretching the spring to length *l*, acting on the weight will be the elastic force of the spring ***F*** directed towards *0*. According to Hooke's Law, the magnitude of this force is proportional to the extension of the spring $\Delta l = l - l_0$.

As in our case $\Delta l = x$, then in magnitude $F = c|\Delta l| = c|x|$. The factor c is called the stiffness of the spring, or the spring constant. Let us find the work done by the elastic force in the displacement of the weight from position $M_0(x_0)$ to position $M_1(x_1)$. As in this case $F_x = -F = -cx, F_y = F_z = 0$, then, substituting these expressions into Eq. (5.29), we obtain

$$A_{(M_0M_1)} = \int_{(M_0)}^{(M_1)} (-cx)\,dx = -c\int_{x_0}^{x_1} x\,dx = \frac{c}{2}(x_0^2 - x_1^2).$$

In the obtained formula x_0 is the initial extension of the spring Δl_{in}, and x_1 is the final extension Δl_{fin}. Hence

$$A_{(M_0M_1)} = \frac{c}{2}\left[(\Delta l_{in})^2 - (\Delta l_{fin})^2\right].$$

The work is positive if $|\Delta l_{in}| > |\Delta l_{fin}|$, i.e., when the end of the spring moves towards the position of equilibrium, and negative when $|\Delta l_{in}| < |\Delta l_{fin}|$ i.e., when the end of the spring moves away from the position of equilibrium.

It follows, therefore, that the work done by the force ***F*** depends only on the quantities Δl_{in} and Δl_{fin} and does not depend on the actual path traveled by *M*.

Consequently, an elastic force is also a conservative force.

Work Done by Friction. Consider a particle moving on a rough surface (Fig. 36) or a rough curve. The magnitude of the frictional force acting on the particle is *fN,* where f is the coefficient of friction and N is the normal reaction of the surface.

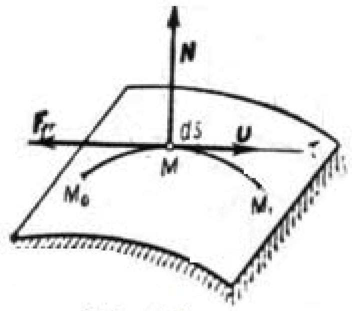

Fig. 36

Frictional force is directed opposite to the displacement of the particle, whence $F_{fr\tau} = -fN$, and from Eq. (5.28),

$$A_{(M_0M_1)} = -\int_{(M_0)}^{(M_1)} F_{fr}\, ds = -\int_{(M_0)}^{(M_1)} fN\, ds.$$

If the frictional force is constant, then $A_{(M_0M_1)} = -F_{fr}s$ where s is the length of the arc M_0M_1 along which the particle moves. Thus, the work done by kinetic friction is always negative. It depends on the length of the arc M_0M_1 and consequently it is non-conservative.

Work Done by Gravitational Forces Acting on a System. The work done by a gravitational force acting on a particle of weight p_k will be $p_k(z_{k0} - z_{k1})$, where z_{k0} and z_{k1} are the coordinates of the initial and final positions of the particle . Then the total work done by all the gravitational forces acting on a system will be

$$A = \sum p_k z_{k0} - \sum p_k z_{k1} = P(z_{c0} - z_{c1}) = \pm P h_c.$$

where P is the weight of the system, and h_c is the vertical displacement of the centre of gravity (or centre of mass) of the system.

Work Done by Forces Applied to a Rotating Body. The elemental work done by the force $\boldsymbol{F}$ applied to the body in Fig. 37 will be

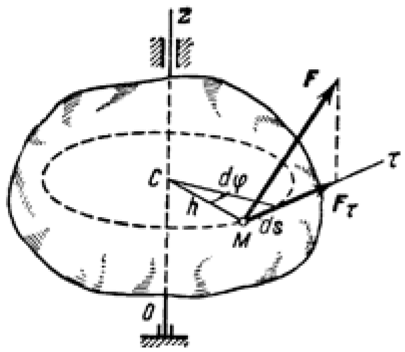

Fig. 37

$$dA = F_\tau\, ds = F_\tau h\, d\varphi$$

since $ds = hd\varphi$, where $d\varphi$ is the angle of rotation of the body.

But it is evident that $F_\tau h = m_z(\boldsymbol{F})$. We shall call the quantity $M_z = m_z(\boldsymbol{F})$ the *turning moment,* or *torque.* Thus we obtain

$$dA = M_z\, d\varphi. \tag{5.30}$$

Eq. (5.30) is valid when several forces are acting, if it is assumed that $M_z = \sum m_z(\boldsymbol{F}_k)$. The work done in a turn through a finite angle $\varphi_1$will be

$$A = \int_0^{\varphi_1} M_z\, d\varphi. \tag{5.31}$$

and, for a constant torque ($M_z = const.$),

$$A = M_z \varphi_1. \tag{5.32}$$

If acting on a body is a force couple laying in a plane normal to *Oz,* then, evidently, M_z in Eqs. (5.30)-(5.32) will denote the moment of that couple.

Let us see how power is determined in this case. From Eq. (5.30) we find

$$N = \frac{dA}{dt} = \frac{M_z d\varphi}{dt} = M_z \omega.$$

Work Done by Frictional Forces Acting on a Rolling Body. A wheel of radius R rolling without slipping on a plane (surface) is subjected to the action of a frictional force $\boldsymbol{F}_{fr}$, which prevents the slipping of the point of contact B on the surface (Fig. 38).

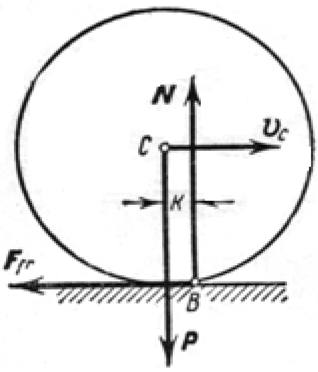

Fig. 38

The elemental work done by this force is $dA = -F_{fr}\, ds_B$. But point B is the instantaneous centre of zero velocity, and $\boldsymbol{v}_B = 0$. As $ds_B = v_B\, dt$, $ds_B = 0$, and for every elemental displacement $dA = 0$.

Thus, in rolling without slipping, the work done by the frictional force preventing slipping is zero in any displacement of the body. For the same reason, the work done by the normal reaction N is also zero. The resistance to rolling is created by the couple $(\boldsymbol{N}, \boldsymbol{P})$ of moment $M = kN$, where k is the coefficient of rolling friction. Then by Eq.(5.30), and taking into account that the angle of rotation of a rolling wheel is $d\varphi = \frac{ds_C}{R}$,

$$dA_{roll} = -kN\, d\varphi = -\frac{k}{R} N\, ds_C, \tag{5.33}$$

where ds_C, is the elemental displacement of the centre C of the wheel. If $N = const.$, then the total work done by the forces resisting rolling will be

$$A_{roll} = -kN\varphi_1 = -\frac{k}{R} N s_C. \tag{5.34}$$

5.15. Theorem of the Change in the Kinetic Energy of a Particle

Consider a particle of mass m displaced by acting forces from a position M_0 where its velocity is v_0, to position M_1 where its velocity is v_1.

To obtain the required relation, consider the equation $m\boldsymbol{a} = \sum \boldsymbol{F}_k$, which expresses the fundamental law of dynamics. Projecting both parts of this equation on the tangent $M\tau$ to the path of the particle in the direction of motion, we obtain

$$ma_\tau = \sum F_{k\tau}.$$

The tangential acceleration in the left side of the equation can be written in the form

$$a_\tau = \frac{dv}{dt} = \frac{dv}{ds}\frac{ds}{dt} = \frac{dv}{ds} v,$$

hence, we have $mv\frac{dv}{ds} = \sum F_{k\tau}$.

Multiplying both sides of the equation by ds, bring mv under the differential sign. Then, noting that $F_{k\tau}\, ds = dA_k$, where dA_k is the elementary work done by the force $\boldsymbol{F}_k$, we obtain an expression of the theorem of the change in kinetic energy in differential form:

$$d\left(\frac{mv^2}{2}\right) = \sum_k dA_k. \tag{5.35}$$

Integrating both sides of Eq. (5.35) in the limits between the corresponding values of the variables at points M_0 and M_1, we finally obtain

$$\frac{mv_1^2}{2} - \frac{mv_0^2}{2} = \sum_k A_k. \tag{5.36}$$

Eq. (5.36) states the theorem of the change in the kinetic energy of a particle in the final form: *the change in the kinetic energy of a particle in any displacement is equal to the algebraic sum of the work done by all the forces acting on the particle in the same displacement.*

The Case of Constrained Motion. If the motion of a particle is constrained, then the right side of Eq. (5.36) will include the work done by the given (active) forces $\boldsymbol{F}_k^a$ and the work done by the reaction of the constraint. Let us limit ourselves to the case of a particle moving on a fixed smooth (frictionless) surface or curve. In this case the reaction *N* is normal to the path of the particle, and $N_\tau = 0$. Then by Eq. (5.28), the work done by the reaction of a fixed smooth surface (or curve) in any displacement of a particle is zero, and from Eq. (5.36) we obtain

$$\frac{mv_1^2}{2} - \frac{mv_0^2}{2} = \sum A^a_{(M_0M_1)}. \tag{5.36'}$$

Thus, in a displacement of a particle on a fixed smooth surface (or curve) the change in the kinetic energy of the particle is equal to the sum of the work done in this displacement by the active forces applied to that particle.

If the surface, (curve) is not smooth, the work done by frictional force will be added to the work done by the active forces.

5.16. Theorem of the Change in the Kinetic Energy of a System

The theorem proved in § 5.15 is valid for any point of a system. Therefore, if we take any particle of mass m_k and velocity $\boldsymbol{v}_k$ belonging to a system, we have for this particle

$$\frac{m_k v_{k1}^2}{2} - \frac{m_k v_{k0}^2}{2} = A_k^e + A_k^i,$$

where v_{k0} and v_{k1} denote the particle's velocity at the beginning and the end of the displacement, and A_k^e and A_k^i are the sums of the work done by all the external and internal forces acting on the particle through this displacement.

If we write similar equations for all the particles of a system and add them up, we obtain

$$\sum \frac{m_k v_{k1}^2}{2} - \sum \frac{m_k v_{k0}^2}{2} = \sum A_k^e + \sum A_k^i \text{ or } T_1 - T_0 = \sum A_k^e + \sum A_k^i, \tag{5.37}$$

where T_1 and T_0 denote the kinetic energy of the system at the beginning and the end of the displacement.

This equation states the following theorem of the change in kinetic energy: *The change in the kinetic energy of a system during any displacement is equal to the sum of the work done by all the external and internal forces acting on the system in that displacement.*

For an infinitesimal displacement of a the system theorem takes the form

$$dT = dA^e - dA^i, \tag{5.38}$$

where dA^e and dA^i denote the elemental work done by all the external and internal forces acting on the system. Unlike the previously proved theorems, in Eqs. (5.37) and (5.38) the internal forces are not ignored. For, if $\boldsymbol{F}_{12}^i$ and $\boldsymbol{F}_{21}^i$ are the forces of interaction between points B_1 and B_2 of a system (see Fig. 39), then $\boldsymbol{F}_{12}^i + \boldsymbol{F}_{21}^i = 0$, but at the same time point B_1 may be moving towards B_2 and point B_2 towards B_1. The work done by each force is positive, and the total work will not be zero.

The Case of Non-Deformable Systems. A non-deformable system is defined as one in which the distance between the points of application of the internal forces does not change during the motion of the system. Special cases of such systems are a rigid body and an inextensible string. Let two points B_1 and B_2 of a non-deformable system (Fig.39) be acting on each other with forces $\boldsymbol{F}_{12}^i$ and $\boldsymbol{F}_{21}^i$ $\left(\boldsymbol{F}_{12}^i = -\boldsymbol{F}_{21}^i\right)$ and let their velocities at some instant be $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ Their displacements in a time interval *dt* will be $ds_1 = v_1\,dt$ and $ds_2 = v_2\,dt$ directed along vectors $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$. But as line B_1B_2 is non-deformable, it follows from the laws of kinematics that the projections of vectors $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ and consequently of the displacements ds_1 and ds_2 on the direction of B_1B_2 will be equal, i.e., $B_1B_1' = B_2B_2'$. Then the elemental work done by forces $\boldsymbol{F}_{12}^i$ and $\boldsymbol{F}_{21}^i$will be equal in magnitude and opposite in sense, and their sum will be zero. This holds for all internal forces in any displacement of a system.

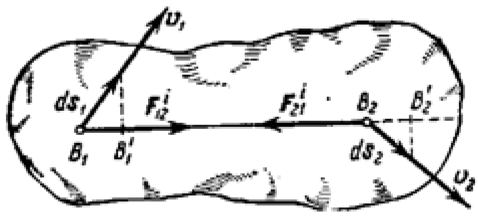

Fig. 39

We conclude from this that the sum of the work done by all the internal forces of a non-deformable system is zero, and Eq. (5.37) takes the form

$$T_1 - T_0 = \sum A_k^e. \tag{5.39}$$

Both the external and internal forces in Eqs. (5.37)-(5.39) include the reactions of constraints. If the constraints on which the bodies of a system move are smooth, then the work done by the reactions of these constraints in any displacement of the system is zero and the reactions will not enter into Eqs (5.37)-(5.39).

Thus in applying the theorem of the change in kinetic energy to frictionless systems, all the immediately unknown reactions of the constraints will be excluded from the problem. This is where its practical value lies.

5.17. Solution of Problems

Problem 11. The centre of gravity of the shaft of the motor in Fig. 40 is located at a distance $AB = a$ from the axis of rotation. The shaft is of weight $\boldsymbol{p}$, and the weight of all other parts of the motor is $\boldsymbol{P}$. Deduce the law of motion of the motor on a smooth horizontal surface if the shaft rotates with a uniform angular velocity ω. Also determine the maximum stress that will be developed in a bolt D fastening the motor to the surface.

Solution. In order to eliminate the forces rotating the shaft by making them internal, consider the motor with the shaft as a single system.

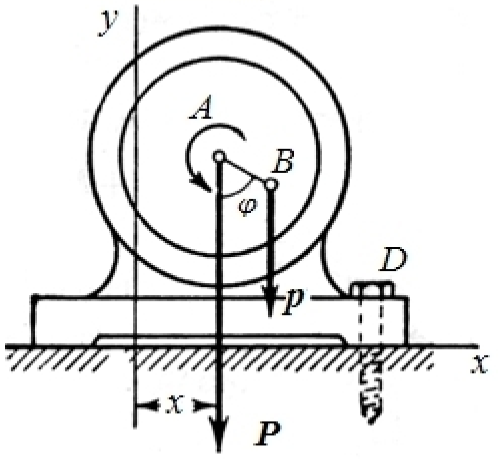

Fig. 40

1) For the motor standing freely on the plane, all the forces acting on it are vertical and the law of conservation of the motion of the centre of mass parallel to axis Ox will apply. Show the motor in an arbitrary position, assuming as initial the position in which points B and A are on the same vertical (on the y axis). Then in the arbitrary position $\xi_A = x, \xi_B = x + a\sin\varphi$. Here ξ_A and ξ_B perform projections of absolute displacements on x axis. Hence, taking into account the law of conservation of motion of the center of mass of a system ($x_c = const.$) and that $\varphi = \omega,$ we find

$$P\xi_A + p\xi_B = 0 \text{ or } Px + p(x + a\sin\omega t) = 0,$$

whence

$$x = -\frac{pa}{P+p}\sin\omega t.$$

Thus, the motor will perform simple harmonic motion with an angular frequency ω.

2) When the motor is fastened, the horizontal reaction R_x of the bolt, by the first of Eqs. (5.6), will be

$$R_x = M\frac{d^2x_c}{dt^2}, \text{ where } x_C = \frac{Px_A + px_B}{Mg}.$$

In this case point A is fixed, and $x_A = h$ (h = const.) and $x_B = h + a\sin\omega t$. Differentiating the expression of x_C and multiplying it by $M,$ where M is the mass of the whole system, we obtain

$$R_x = M\frac{d^2x_C}{dt^2} = \frac{p}{g}\frac{d^2x_B}{dt^2} = -\frac{pa\omega^2}{g}\sin\omega t.$$

The pressure on the bolt is equal to $|R_x|$ in magnitude and opposite in direction. Its maximum value will be

$$\frac{pa\omega^2}{g}.$$

Problem 12. Crank *AB* of length *r* and weight *p* of the mechanism in Fig. 41 rotates with a constant angular velocity *ω* and actuates the slotted bar and the piston *D* connected to it. The total weight of the bar and piston is *P.* Acting on the piston during the motion is a constant force Q. Neglecting friction, determine the maximum horizontal pressure of the crank on its axle *A*.

Solution. In order to eliminate the forces rotating the crank and the pressure exerted on it by the slotted bar, consider the motion of the system as a whole.

Denoting the horizontal reaction of the axle *A* by $\boldsymbol{R_x}$ we have from the first of Eqs. (5.6)

$$M\frac{d^2x_C}{dt^2} = R_x - Q,$$

where, by Eqs. (4.4), $Mx_C = m_1x_1 + m_2x_2$.

In our case

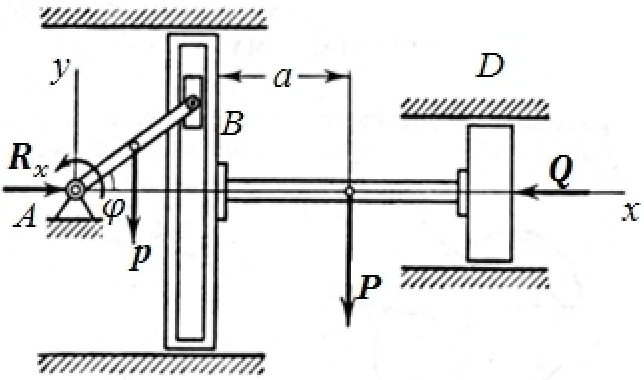

Fig. 41

$$m_1 = \frac{p}{g}, \quad x_1 = \frac{r}{2}\cos\omega t;$$

$$m_2 = \frac{P}{g}, \quad x_2 = a + r\cos\omega t,$$

as $\varphi = \omega t$. We finally obtain

$$R_x = Q + M\frac{d^2x_C}{dt^2} = Q - \frac{r\omega^2}{g}\left(\frac{p}{2} + P\right)\cos\omega t.$$

The pressure on the shaft is equal in magnitude to $|R_x|$ and oppositely directed. The maximum pressure will be at $\varphi = 180°$ and will be equal to

$$Q + \frac{r\omega^2}{g}\left(\frac{p}{2} + P\right).$$

Problem 13. A load of weight $p = 1\,N$ moves in a circle with a constant velocity $2\,m/sec$. Determine the impulse and the work done by the force acting on the load during the time the load takes to travel one quarter of the circle.

Solution. From the theorem of the change in momentum,

$$\boldsymbol{S} = m\boldsymbol{v_1} - m\boldsymbol{v_0}.$$

Constructing geometrically the difference between these momenta (Fig. 42), we find from the right-angled triangle:

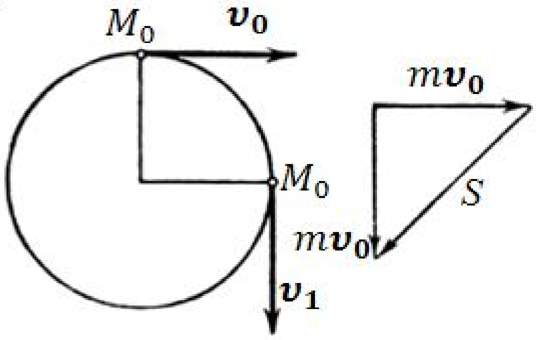

Fig. 42

$$S = m\sqrt{v_1^2 + v_0^2}.$$

But from the conditions of the problem $v_1 = v_0 = v$,**,** consequently

$$S = \frac{p}{g}v\sqrt{2} = 0.29\,N\cdot sec.$$

Problem 14. A load of mass m lying on a horizontal plane is pushed and imparted an initial velocity $\boldsymbol{v}_0$. The motion of the load is then retarded by a constant force ***F*.** Determine the time it takes the load to stop and how far it will have travelled.

Solution. Draw the load in an arbitrary position (Fig. 43), M_0 and M_1 being its initial and final positions. Acting on the load are its weight ***P*,** the reaction of the plane ***N*,** and the opposing force ***F*.** Pointing axis *Ox* in the direction of the motion, we have from Eq. (5.8'):

$$mv_{1x} - mv_{0x} = \sum S_x.$$

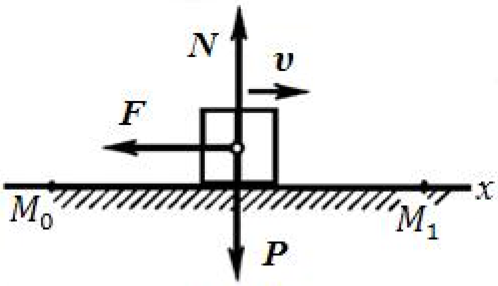

Fig. 43

In this case $v_{1x} = 0$ (v_1 is the velocity at the instant when the load stops) and $v_{0x} = v_0$. Force $\boldsymbol{F}$ is the only one projected on the x axis. As it is constant, $S_x = F_x t_1 = -Ft_1$, where t_1 is the deceleration time. Substituting these expressions into our equation, we obtain $-mv_0 = -Ft_1$, whence the required time is

$$t_1 = \frac{mv_0}{F}.$$

To determine the braking distance we use the theorem of the change in kinetic energy:

$$\frac{mv_1^2}{2} - \frac{mv_0^2}{2} = \sum A_k.$$

Here again $v_1 = 0$ and only force $\boldsymbol{F}$ does the work: $A(\boldsymbol{F}) = -Fs$, where s is the braking distance. The work done by forces $\boldsymbol{P}$ and $\boldsymbol{N}$ is zero as they are perpendicular to the displacement. Hence we obtain $-\frac{mv_0^2}{2} = -Fs_1$, and the braking distance is

$$S_1 = \frac{mv_0^2}{2F}.$$

Problem 15. The resultant $\boldsymbol{R}$ of all the forces acting on the piston in Fig. 44 changes during a certain time-interval according to the equation $R = 0.4P(1 - kt)$, where P is the weight of the piston, t is the time in seconds, and k a factor equal to 1.6 sec^{-1}. Determine the velocity of the piston at time $t_1 = 0.5\, sec$, if at time $t_0 = 0$ it was $v_0 = 0.2\, m/sec$.

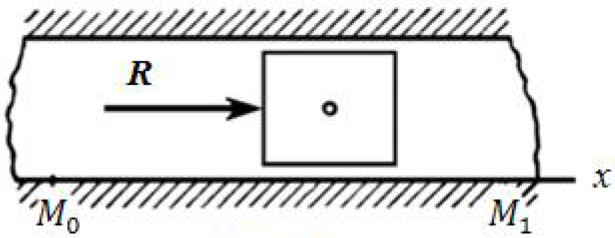

Fig. 44

Solution. As the acting force depends on the time and the given and required quantities include t_1, v_0, and v_1, we make use of Eq. (5.8'):

$$mv_{1x} - mv_{0x} = S_x.$$

In this case

$$S_x = \int_0^{t_1} R_x dt = 0.4P \int_0^{t_1} (1 - kt)dt = 0.4P\, t_1 \left(1 - \frac{k}{2} t_1\right).$$

Furthermore, $v_{0x} = 0,\ v_{1x} = v_1$ and $m = \frac{P}{g}$. Substituting these expressions, we obtain

$$v_1 = v_0 + 0.4 g t_1 \left(1 - \frac{k}{2} t_1\right) \approx 1.4\, m/sec.$$

Problem 16. A bullet of weight p fired horizontally with a velocity u hits a box of sand standing on a truck (Fig. 45). What velocity will the truck receive as a result of the impact if its weight together with the box of sand is P?

Solution. Consider the bullet and the truck as one system. This enables us to exclude the forces generated when the bullet hits the sand. The sum of the projections of the external forces on the horizontal axis x is zero. Consequently, $Q_x = const.$, or $Q_{0x} = Q_{1x}$, where Q_0 is the momentum of the system before the impact, and Q_1, after the impact. As the truck was motionless before the impact,

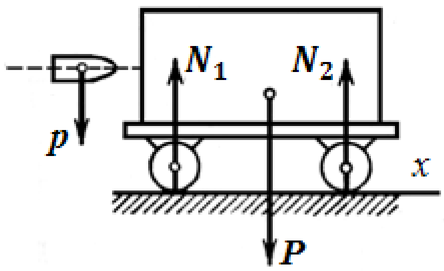

Fig. 45

$$Q_{0x} = \frac{p}{g}u.$$

After the impact the truck and bullet are moving with the same velocity v. Then

$$Q_{1x} = \frac{p+P}{g}v,$$

and equating the right sides of the expressions for Q_{0x} and Q_{1x}, we obtain

$$v = \frac{p}{p+P}u.$$

Problem 17. Determine the recoil of a gun if its barrel is horizontal, the weight of the recoiling parts is P, the weight of the shell is p, and the muzzle velocity of the shell is u.

Solution. To exclude the unknown forces developed by the pressure of the gases, consider the shell and the gun as one system.

Neglecting the resistance to the recoil during the motion of the shell in the bore, we find that the sum of the projection of the applied external forces on the axis x is zero (Fig. 46). Hence, $Q_x =$ const. and, since before the shot the system was motionless ($Q_0 = 0$), for any moment of time $Q_x = 0$.

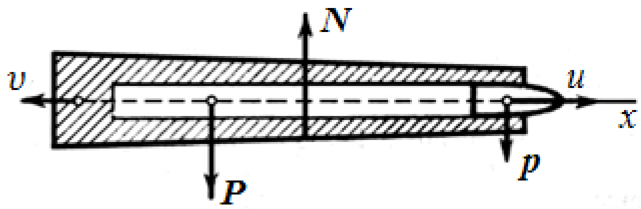

Fig. 46

If the velocity of the recoiling parts at the final instant is $\boldsymbol{v}$, then the absolute velocity of the shell at that moment is $\boldsymbol{u} + \boldsymbol{v}$. Consequently,

$$Q_x = \frac{P}{g}v_x + \frac{p}{g}(u_x + v_x) = 0,$$

whence we find

$$v_x = -\frac{p}{p+P}u_x.$$

If we knew the absolute muzzle velocity $\boldsymbol{u_a}$ of the shell, we could have substituted u_{ax} for $u_x + v_x$, whence

$$v_x = -\frac{p}{P}u_{ax}.$$

The minus sign in both cases means that $\boldsymbol{v}$ is in the opposite direction of $\boldsymbol{u}$.

Note that in calculating the total momentum of a system the absolute velocity of its parts should be considered.

Problem 18. A jet of water of diameter $d = 4\,cm$ is discharger from a nozzle with a velocity $u = 10\,m/sec$ and impinges normally against a fixed vertical wall (Fig. 47). Neglecting the compression in the jet, determine the force of water on the wall.

Solution. To exclude the internal forces of interaction of the water particles between each other at the time of impact, apply the first of Eqs. (5.11')

$$Q_{1x} - Q_{0x} = \sum S_{kx}^{e}$$

to the part of the jet filling the volume abc at the given instant. Let us calculate for this volume the difference $Q_{1x} - Q_{0x}$ for a certain time interval t_1. During this interval, the volume of water will occupy configuration $a_1b_1c_1$, and the value of Q_x will decrease by mu, where m is the mass of volume aa_1. The liquid filling volumes bb_1 and cc_1 moves normally to axis *Ox* and therefore does not increase the value of Q_x. As we have only Q_x decreasing, $Q_{1x} - Q_{0x} = -mu$. Reaction $\boldsymbol{R}$ of the wall will be the only external force acting on the given volume and giving a projection on axis *Ox*. Assuming $R = const.$, we obtain

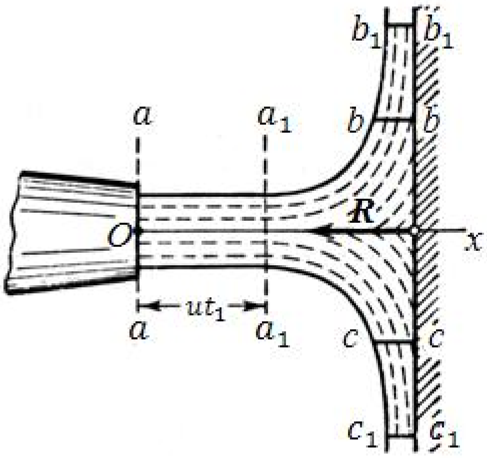

Fig. 47

$\sum S_{kx}^{e} = R_x t_1 = -Rt_1$, and $mu = Rt_1$. Now compute m. As the displacement $aa_1 = ut_1$,

$$m = \frac{\gamma}{g}\frac{\pi d^2}{4}ut_1,$$

where γ is the weight of a unit volume, and consequently, $\frac{\gamma}{g}$ is the mass of a unit volume of the liquid. Substituting this value and taking into account that for water $\gamma = 10000\,N/m^3$, we finally obtain

$$R = \frac{\gamma}{g}\frac{\pi d^2}{4}u^2 = 128\,N.$$

Problem 19. Two discs having moments of inertia J_1 and J_2 are mounted on a shaft as shown in Fig. 48. If the shaft is twisted and then released, find the dependence between the angular velocities and the angle of turn of the discs in the ensuing torsional vibrations. Neglect the mass of the shaft.

Solution. To exclude the unknown elastic forces which cause the discs to vibrate, consider both discs and the shaft as a single system. The external forces (the reactions of the bearings and the force of gravity) intersect with axis *x*, whence $\sum m_x(F_k^e) = 0$, and $K_x = const$. But since at the initial moment $K_x = 0$, during the whole of the vibration we must have $K_x = J_1\omega_1 + J_2\omega_2 = 0$ (the angular momentum of the system

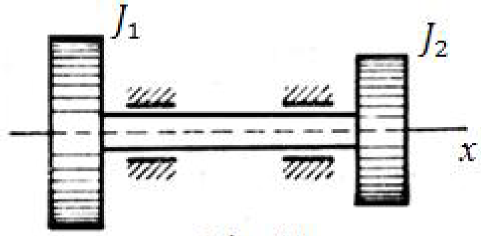

Fig. 48

with respect to the x axis equals the sum of the angular momenta of each disc with respect to the same axis). We find from this that

$$\omega_1 = \frac{J_2}{J_1}\omega_2 \quad \text{and} \quad \varphi_1 = \frac{J_2}{J_1}\varphi_2,$$

where φ_1, and φ_2 are the angles through which the discs were twisted, measured from the initial position (the latter result is obtained by integrating the first equation).

Thus, the vibrations will be in opposite directions, and the angular amplitudes will be inversely proportional to the moments of inertia of the discs.

Problem 20. A governor AB with a moment of inertia J_z consists of two symmetrically placed weights of mass m, each attached to two springs as shown in Fig. 49, and it rotates about a vertical axis Oz. At time $t_0 = 0$, the governor receives an angular velocity ω_0, and each weight starts to oscillate in damped vibration about its respective centre C at a distance l from axis Oz. Neglecting friction and considering the weights as particles, determine the dependence of the angular velocity ω of the governor on the position of the weights.

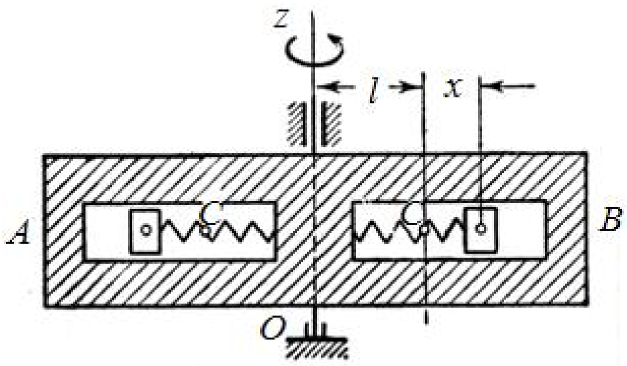

Fig. 49

Solution. To exclude the unknown elastic forces of the springs, consider the governor and the weights as one system. Then $\sum m_z(\boldsymbol{F}_k^e) = 0$, and $K_z = const$. At time $t_0 = 0$, the displacement $x = 0$ and $K_{z0} = (J_z + 2ml^2)\omega_0$. At any arbitrary instant t, $K_z = [J_z + 2m(l + x)^2]\omega$. As $K_z = K_{zo}$,

$$\omega = \frac{J_z+2ml^2}{J_z+2ml(l+x)^2}\omega_0.$$

Consequently, when $x > 0, \omega < \omega_0$, and when $x < 0, \omega > \omega_0$, i.e., the angular velocity changes about a mean value ω_0. When the vibrations of the weights dampen with time, x tends to zero, and ω to ω_0.

Problem 21. A track is laid along the circumference of a disc of radius R and weight P. Standing on the track is a toy spring-wound car of weight p. The disc rotates together with the car about a vertical axis z with an angular velocity ω_0 (Fig. 50). Determine how the angular velocity of the disc will change if at some instant the car will start moving in the direction of the rotation with a velocity u relative to the disc.

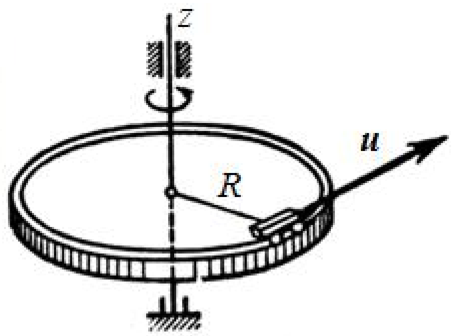

Fig. 50

Solution. To exclude the unknown frictional forces between the wheels of the car and the disc, consider both as one system. The moments of the external forces acting on the system with respect to the z axis are zero. Consequently, K_z =*const*. Considering the disc to be homogeneous ($J_z = 0.5MR^2$) and the car as a particle, we have

$$K_{z0} = \left(0.5\frac{P}{g}R^2 + \frac{p}{g}R^2\right)\omega_0.$$

When the car starts moving its absolute velocity will be $v_a = u + \omega R$, where ω is the new angular velocity of the disc. The angular momentum of the car about the z axis will be $mv_aR = m(uR + \omega R^2)$, and for the whole system we have

$$K_{z1} = 0.5\frac{P}{g}R^2\omega + \frac{p}{g}(uR + R^2\omega).$$

As $K_z = const.$, $K_{z1} = K_{z0}$, whence

$$\omega = \omega_0 - \frac{P}{0.5P + p}\frac{u}{R}.$$

The angular velocity of the disc, we see, decreases. If the car travels in the opposite direction, ω will increase.

Note that in calculating K_z the absolute velocities of all moving points of the system were taken.

Problem 22. Wound on a drum of weight P and radius r (Fig. 51) is a string carrying a load A of weight Q. Neglecting the mass of the string and friction, determine the angular acceleration of the drum when the load falls, if the radius of gyration of the drum with respect to its axis is ϱ.

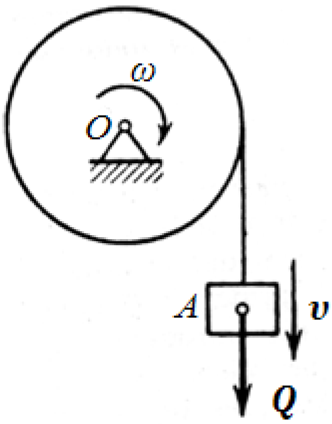

Fig. 51

Solution. Applying the theorem of moments with respect to axis O, we have

$$\frac{dK_0}{dt} = \sum m_0(\boldsymbol{F_k^e}).$$

The moving system consists of two bodies, consequently

$$K_0 = K_{drum} + K_{load}.$$

The load is in translational motion, and we consider it as a particle. Its velocity is $v = \omega r$. The drum rotates about a fixed axis, consequently,

$$K_{load} = \frac{Q}{g}vr = \frac{Q}{g}r^2\omega;\ K_{drum} = J_0\omega = \frac{P}{g}\varrho^2\omega,$$

and

$$K_0 = (Qr^2 + P\varrho^2)\frac{\omega}{g}.$$

Substituting this expression for K_0 , we obtain

$$\frac{Qr^2 + P\varrho^2}{g}\frac{d\omega}{dt} = Qr,$$

whence

$$\varepsilon = \frac{Qrg}{Qr^2 + P\varrho^2}.$$

Problem 23. A weight attached to a string of length l (Fig. 52b) is displaced from the vertical at an angle φ_0 and released from rest. Determine the velocity of the weight at the instant when the thread makes an angle φ with the vertical.

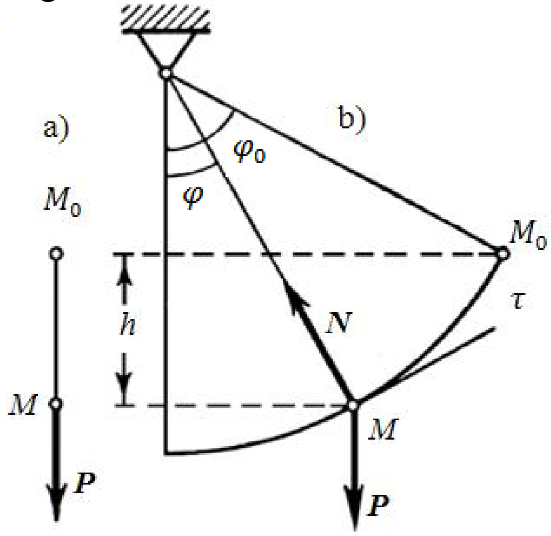

Fig. 52

Solution. As the conditions of the problem include the displacement of the weight, defined by the angle through which the thread passes, and the velocities v_0 and v_1, we make use of the theorem of the change in kinetic energy:

$$\frac{mv_1^2}{2} - \frac{mv_0^2}{2} = \sum A_k.$$

Acting on the weight is the force of gravity $\boldsymbol{P}$ and the reaction of, the thread $\boldsymbol{N}$. The work done by force $\boldsymbol{N}$ is zero, as $N_\tau = 0$. For force $\boldsymbol{P}$ we have $A(\boldsymbol{P}) = Ph = mgh$. As $v_0 = 0$, we obtain $\frac{mv_1^2}{2} = mgh$, whence

$$v_1 = \sqrt{2gh}.$$

Evidently, the result is the same for the velocity v of a freely falling weight (Fig.52a).

In our problem $h = l\cos\varphi - l\cos\varphi_0$, and finally

$$v = \sqrt{2gl(\cos\varphi - \cos\varphi_0)}.$$

Problem 24. The length l_0 of an uncompressed valve spring is $6\ cm$. When the valve is completely open it is lifted to a height $s = 0.6\ cm$ and the length of the compressed spring is $l = 4\ cm$ (Fig. 53).The stiffness of the spring is $c = 1\ N/cm$ and the weight of the valve is $p = 4\ N$. Neglecting the gravitational and resisting forces, determine the velocity of the valve at the moment of its closure.

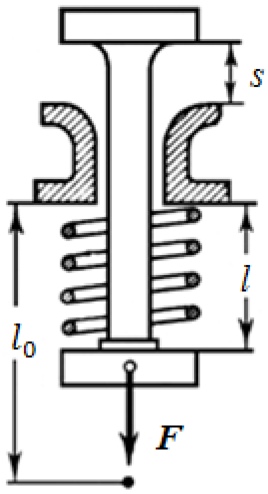

Fig. 53

Solution. The elastic force $\boldsymbol{F}$ acting on the valve depends on the displacement s of the valve, which is given. Therefore we use Eq. (5.36):

$$\frac{mv_1^2}{2} - \frac{mv_0^2}{2} = A_k.$$

According to the conditions of the problem, the only force doing work is the elastic force of the spring. Hence, we have

$$A_k = \frac{c}{2}\left[(\Delta l_{in})^2 - (\Delta l_{fin})^2\right].$$

In our case $\Delta l_{in} = l_0 - l = 2\ cm$, $\Delta l_{fin} = l_0 - l - s = 1.4\ cm$.

Furthermore, $v_0 = 0$ and $m = \frac{P}{g}$. Substituting these expression, we obtain

$$v_1 = \sqrt{\frac{cg}{p}(\Delta l_{in})^2 - (\Delta l_{fin})^2} \approx 0.22\ m/sec.$$

Problem 25. An elastic beam supporting a weight at the centre of its span (Fig. 54) deflects by an amount δ_{st} (the static deflection of the beam). Neglecting the weight of the beam, determine its maximum deflection δ_{max} if the weight is dropped on it from a height *H*.

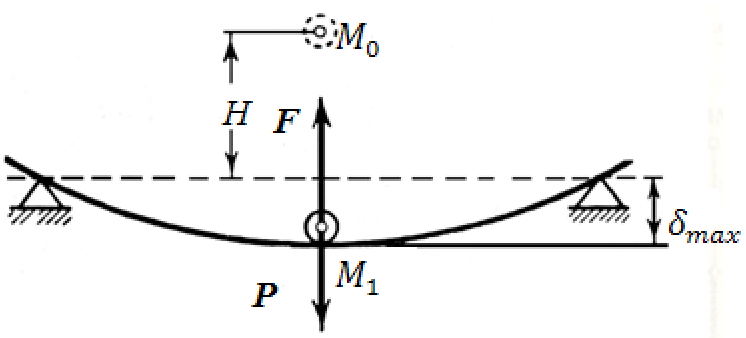

Fig. 54

Solution. As in the previous problem, we apply Eq. (5.36). The initial velocity v_0 and the final velocity v_1 (at the instant of the maximum deflection of the beam) of the weight are each zero, and Eq. (5.36) takes the from

$$\sum A_k = 0.$$

The forces doing work are the gravitational force $\boldsymbol{P}$ in the displacement M_0M_1 and the elastic force F of the beam in the deflection of the beam.

Also, $A(\boldsymbol{P}) = P(H + \delta_{max})$, $A(\boldsymbol{F}) = \frac{c}{2}\delta_{max}^2$, as for the beam $\Delta l_{in} = 0$, $\Delta l_{fin} = \delta_{max}$. Substituting these expressions, we obtain

$$P(H + \delta_{max}) - \frac{c}{2}\delta_{max}^2 = 0.$$

When the weight on the beam is in equilibrium it is balanced by elastic force. Consequently, $P = c\delta_{st}$, and the last equation can be written in the form

$$\delta_{max}^2 - 2\delta_{max}\delta_{st} - 2\delta_{st}H = 0.$$

Solving this quadratic equation, and taking into account that according to the conditions of the problem $\delta_{max} > 0$, we find

$$\delta_{max} = \delta_{st} + \sqrt{\delta_{st}^2 + 2H\delta_{st}}.$$

Problem 26. A load of weight *P* attached to a string of length l is displaced through an angle α from the vertical to a position M_0 and released from rest (Fig. 55). Determine the tension in the thread when the load is in its lowest position M_1.

Solution. Draw the load in the position for which the tension in the string has to be found, i.e., in position M_1. Acting on the load is its weight $\boldsymbol{P}$ and the reaction of the thread $\boldsymbol{T}$. Draw the inward normal M_1n and write Eq. (1.2), taking into account that $a = a_n = \frac{v^2}{\rho}$ and in the present case $\rho = l$. We have:

$$\frac{mv_1^2}{l} = T - P \text{ or } T = P + \frac{mv_1^2}{l},$$

where v_1 is the velocity of the load at position M_1. To determine v_1, we make use of Eq. (5.36'):

$$\frac{mv_1^2}{2} - \frac{mv_0^2}{2} = A^a_{(M_0M_1)}.$$

On the section M_0M_1 only force $\boldsymbol{P}$ does any work. Therefore, $A^a = Ph = Pl(1 - \cos\alpha)$.

As $v_0 = 0$, substituting the expression developed for the work, we obtain $mv_1^2 = 2Pl(1 - \cos\alpha)$, and finally

$$T = P(3 - 2\cos\alpha).$$

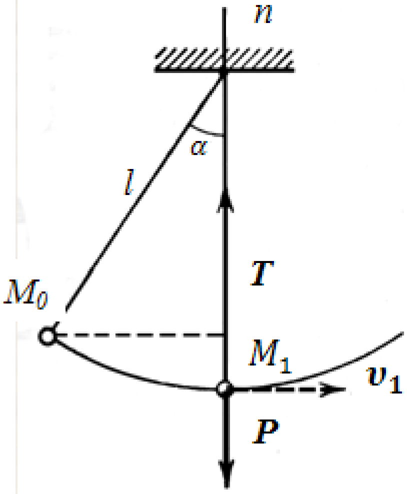

Fig. 55

In the special case, when the initial angle of deflection is 90°, the tension in the string when it is in vertical position will be $3P$, i.e., treble the weight of the load. The result shows that dynamic reactions can differ considerably from static reactions.

Problem 27. A grooved track makes two circular arcs *AB* and *BD* of radius *R* in a vertical plane; the tangent *BE* through their point of conjugation is horizontal (Fig. 56).

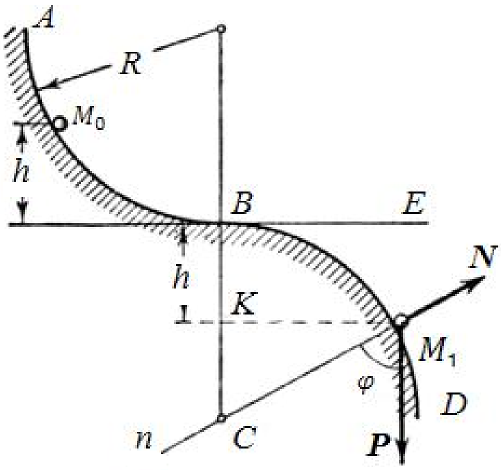

Fig. 56

Neglecting friction, determine the height *h* from *BE* at which a heavy ball should be placed on the track so that it would shoot out of the track at point M_1 lying at the same distance *h* below *BE*.

Solution. The ball will leave the track at a point M_1 such that its pressure on the track (or the reaction $\boldsymbol{N}$ of the track) is zero. Consequently, our problem is reduced to the determination of *N*. Draw the ball at M_1. Acting on it are the force of gravity $\boldsymbol{P}$ and the reaction of the track $\boldsymbol{N}$. Writing Eq. (5.36') for the projections on the inward normal M_1n, we have

$$\frac{mv_1^2}{R} = P\cos\varphi - N.$$

Since at the point of departure $N = 0$, and taking into account that $R\cos\varphi = KC = R - h$, for determining *h* we obtain equation

$$mv_1^2 = P(R - h).$$

The value of mv_1^2 can be found from the theorem of the change in kinetic energy. As $v_0 = 0$, Eq. (5.36') gives

$$\frac{mv_1^2}{2} = A^a_{(M_0M_1)}.$$

The only force that does work is $\boldsymbol{P}$, and $A(\boldsymbol{P}) = P2h$. Consequently, $mv_1^2 = 4Ph$. Substituting this expression of mv_1^2, we obtain $4h = R - h$, whence $h = 0.2R$.

Problem 28. A load M is attached to a string of length l (Fig. 57). What is the least initial velocity v_0 perpendicular to the string that should be imparted to the load for it to describe a complete circle?

Solution. The load will describe a complete circle if nowhere along its path (except, possibly, point M') will the tension in the string become zero, i.e., if the string remains taut. If on the other hand, at any point M_1 where $v_1 \neq 0$ the tension becomes zero, the string will no longer constrain the load, which will continue to move as a free body.

To solve the problem we must determine the tension T in the string at any point M defined by angle φ and then require that $T > 0$ for any angle $\varphi \neq 180^0$.

Acting on the load at point M are its weight $\boldsymbol{P}$ and the tension of the thread $\boldsymbol{T}$. Writing Eq. (1.2) for the projections on the inward normal Mn, we obtain

$$\frac{mv^2}{l} = T - P\cos\varphi \ ,$$

where v is the velocity of the load at point M. To determine v we apply the theorem of the change in kinetic energy:

$$\frac{mv^2}{2} - \frac{mv_0^2}{2} = A^a_{(M_0M_1)}.$$

In our case $A^a_{(M_0M_1)} = -Ph = -Pl(1-\cos\varphi)$, and consequently $mv^2 = mv_0^2 - 2Pl(1-\cos\varphi)$.

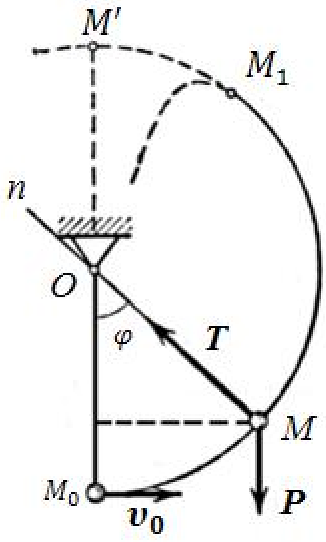

Fig. 57

Substituting this expression of mv^2 and solving for T, we obtain

$$T = P\left(\frac{v_0^2}{gl} - 2 + 3\cos\varphi\right).$$

The least value of T is at $\varphi = \ 180°$:

$$T_{\min} = P\left(\frac{v_0^2}{gl} - 5\right).$$

The condition for T never to become zero (except, possibly, at point M') is that $T_{\min} \geq 0$. Hence,

$$\frac{v_0^2}{gl} \geq 5 \quad \text{or} \quad v_0 \geq \sqrt{5gl}.$$

Thus, the least initial velocity at which the load will describe a complete circle is given by the equation

$$v_{0\,min} = \sqrt{5gl}.$$

Let us assume that the load is attached not to a thread but to a rigid light (weightless) rod of length l. In this case (since, unlike a thread, a rod can work both in tension and in compression) the load will describe a complete circle if the velocity does not become zero anywhere (except, possibly, at point M'). Applying Eq. (5.36') for the displacement M_0M' and assuming $v = 0$ at point M', we obtain

$$\frac{-mv_0^2}{2} = -mg2l\,.$$

Hence $v_{0\,min} = \sqrt{4gl}$.

Problem 29. Find the kinetic energy of a uniform cylindrical wheel of mass M rolling without slipping, if the velocity of its centre is v_C.

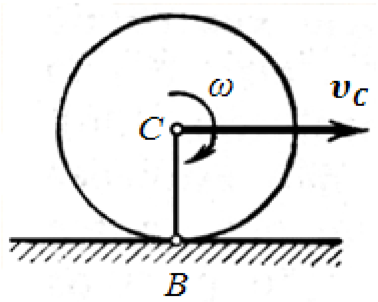

Fig. 58

Solution. The wheel is in plane motion. By Eq. (5.24),

$$T = \frac{1}{2}Mv_C^2 + \frac{1}{2}J_C\omega^2.$$

As the body is an uniform cylinder, we have $J_c = 0.5\,MR^2$, where R is the radius of the wheel. On the other hand, since point B is the instantaneous centre of zero velocity of the wheel (Fig. 58), $v_C = \omega BC = \omega R$, whence $\omega = v_C/R$.

Substituting these expressions, we find

$$T = \frac{1}{2}Mv_C^2 + \frac{1}{4}MR^2\frac{v_C^2}{R^2} = \frac{3}{4}Mv_C^2.$$

Problem 30. When body A in Fig. 59 translates with a velocity $\boldsymbol{u}$, body B moves in the slots of body A with a velocity $\boldsymbol{v}$. If angle φ is known, determine the kinetic energy of body B.

Solution. The absolute motion of body B is a translation with a velocity $\boldsymbol{v_a} = \boldsymbol{u} + \boldsymbol{v}$. Then

$$T = \tfrac{1}{2}Mv_a^2 = \tfrac{1}{2}M(u^2 + v^2 + 2uv\,\cos\alpha).$$

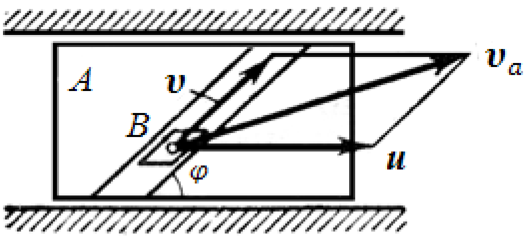

Fig. 59

A characteristic mistake in problems of this type is to regard the kinetic energy of the body as the sum of the energies of the relative and transport motions:

$$T = T_{rel} + T_{tr} = \tfrac{1}{2}Mv^2 + \tfrac{1}{2}Mu^2,$$

and thus, it will be noticed, the component $Muv\cos\alpha$ drops out of the solution.

Thus, in the most general case of relative motion, the total kinetic energy of a body does not equal the sum of the kinetic energies of its relative and transport motions.

Problem 31. A mechanism consists of a part which is translated with a velocity $\boldsymbol{u}$, and a rod AB of length l and mass M hinged at A (Fig. 60). The rod rotates about axis A with an angular velocity ω. Determine the kinetic energy of the rod if angle φ is given.

Solution. The rod performs plane motion, and by Eq. (5.24)

$$T = \frac{1}{2}Mv_C^2 + \frac{1}{2}J_C\omega^2.$$

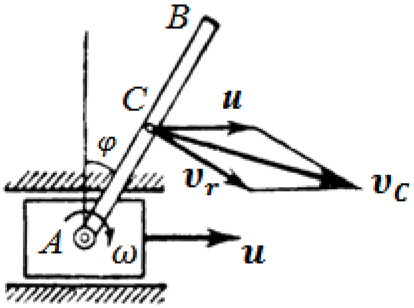

Fig. 60

The velocity of point C is compounded of the velocities $\boldsymbol{u}$ and $\boldsymbol{v}_r$, where in magnitude $v_r = \omega\frac{l}{2}$. Consequently, $v_C^2 = u^2 + v_r^2 + 2uv_r\cos\varphi$.

The angular velocity of the rod about C is the same as about A, as ω does not depend on the location of the pole. Taking into account that $J_C = \frac{Ml^2}{12}$ and substituting all these expressions, we obtain

$$T = \frac{1}{2}M\left(u^2 + \omega^2\frac{l^2}{4} + u\omega l\cos\varphi\right) + \frac{1}{24}Ml^2\omega^2 =$$
$$= \frac{1}{2}Mu^2 + \frac{1}{6}Ml^2\omega^2 + \frac{1}{2}Ml\omega u\cos\varphi.$$

Problem 32. A rod AB of length l is hinged as shown at point A (Fig. 61**).** Neglecting friction, determine the minimum velocity ω_0 that must be imparted to the rod so that it would swing into a horizontal configuration.

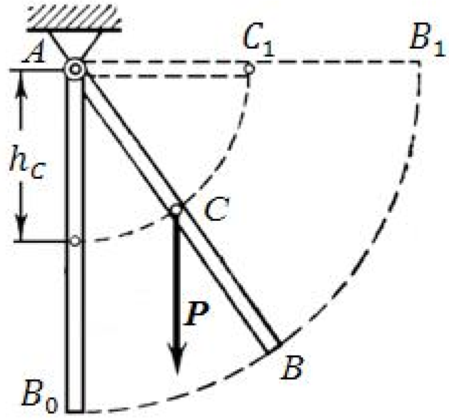

Fig. 61

Solution. The given and required quantities include ω_0, $\omega_1 = 0$, and the displacement of the system as defined by angle B_0AB_1.Therefore, the problem is best solved by applying Eq. (5.39):

$$T_1 - T_0 = \sum A_k^e.$$

Denoting the weight of the rod by $\boldsymbol{P}$, compute all the quantities in equation. From Eqs. (5.23) and taking into account that $J_A = \frac{Ml^2}{3}$, we find

$$T_0 = \frac{1}{2}J_A\omega_0^2 = \frac{1}{6}\frac{P}{g}l^2\omega_0^2.$$

Since in the final configuration the velocity of the rod is zero, $T_1 = 0$. The work is done only by force $\boldsymbol{P}$, and $A^e = -Ph_C = -P\frac{l}{2}$. Substituting these values, we obtain

$$-\frac{1}{6}\frac{P}{g}l^2\omega_0^2 = -P\frac{l}{2},$$

whence $\omega_0 = \sqrt{\frac{3g}{l}}$.

Problem 33. Two pulleys A and B are connected by a belt (Fig. 62). When the motor is switched off, pulley A of radius R, has an angular velocity ω_0. The total weight of the pulleys is P, and of the belt p. A brake shoe is applied to pulley A with a force Q to stop the rotation; the coefficient of friction of the shoe on the pulley is f.

Neglecting friction in the axles and considering the pulleys to be homogeneous discs, determine how many revolutions pulley A will make before stopping.

Solution. We shall use Eq. (5.39) to determine the required number of revolutions N:

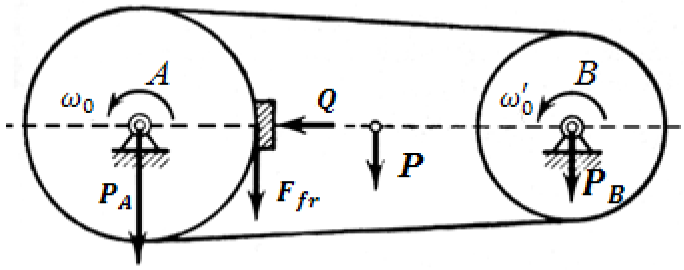

Fig. 62

$$T_1 - T_0 = \sum A_k^e.$$

From the conditions of the problem $T_1 = 0$, and $T_0 = T_A + T_B + T_b$. Taking into account that the initial velocities of all the points of the belt $v_{b0} = \omega_0 R = \omega_0' r$, where ω_0' and r are the initial angular velocity and the radius of pulley B, we find:

$$T_A = \frac{1}{2}\left(\frac{P_A}{2g}R^2\right)\omega_0^2, \qquad T_B = \frac{1}{2}\left(\frac{P_B}{2g}r^2\right)\omega_0'^2 = \frac{1}{4}\frac{P_B}{g}R^2\omega_0^2,$$

$$T_b = \frac{1}{2}\frac{p}{g}v_{b0}^2 = \frac{1}{2}\frac{p}{g}R^2\omega_0^2.$$

The last equation follows from the fact that all the points of the belt move with the same speed. Finally, as $P_A + P_B = P$, we obtain

$$T_0 = \frac{P + 2p}{4g}R^2\omega_0^2.$$

Now compute the work done by the forces. In this case, the work done by gravity is zero, as the centers of gravity of the pulleys and the belt do not change their position during the motion. The force of friction $F_{fr} = fQ$, and the work done by it is found from Eq. (5.32):

$$A_{fr} = -(fQR)\varphi_1 = -fQR \cdot 2\pi N.$$

Substituting all the found values, we obtain finally

$$N = \frac{(P + 2p)R\omega_0^2}{8\pi g f Q}.$$

Problem 34. A cart is drawn with a constant force $Q = 160\,N$ up an inclined plane making an angle $\alpha = 30°$ with the horizontal (Fig. 63). The platform of the cart weighs $P = 180\,N$, and each of its uniform wheels weighs $p = 20\,N$. Determine: 1) the linear velocity v_1 of the cart, when it has travelled a distance $l = 4\,m$, if $v_0 = 0$; 2) the acceleration of the cart. The wheels roll without slipping. Neglect the rolling friction.

Solution. 1) Let us use Eq. (5.39) to determine v_1:

$$T_1 - T_0 = \sum A_k^e.$$

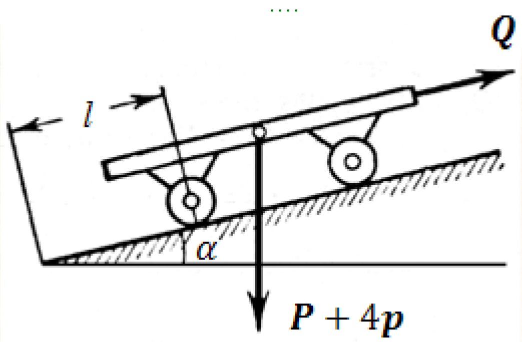

Fig. 63

In our case $T_0 = 0$ and $T_0 = T_{platform} + 4T_{wheel}$. The cart is in translatory motion, and the kinetic energy of a uniform rolling wheel was calculated in Problem 29. Thus,

$$T_1 = \frac{1}{2}\frac{P}{g}v_1^2 + 4\left(\frac{3}{4}\frac{p}{g}v_1^2\right) = \frac{1}{2g}(P + 6p)v_1^2.$$

Work is done by force Q and the force of gravity $P_1 = P + 4p$. The work done by the frictional forces preventing slippage and by the normal reactions is zero. Making the necessary computations, we find

$$A(\boldsymbol{Q}) = Ql; \ \ A(\boldsymbol{P}_1) = -(P + 4p)h_C = -(P + 4p)l \sin\alpha.$$

Substituting these expressions, we obtain

$$\frac{1}{2g}(P + 6p)v_1^2 = [Q - (P + 4p)\sin\alpha]l,$$

whence

$$v_1 = \sqrt{\frac{2gl[Q - (P + 4p)\sin\alpha]}{P + 6p}} = 2.8\, m/sec.$$

2) To determine the acceleration a, let us consider the quantities v_1 and l as variables. Then, differentiating through with respect to time, we have

$$\frac{1}{g}(P + 6p)v\frac{dv}{dt} = [Q - (P + 4p)\sin\alpha]\frac{dl}{dt}.$$

But $\frac{dl}{dt} = v$, and $\frac{dv}{dt} = a$ and, eliminating v, we have

$$a = \frac{Q - (P + 4p)\sin\alpha}{P + 6p}g = 0.98\, m/sec^2.$$

Problem 35. One end of a string passing over a pulley *O* (Fig. 64) is wound on a cylinder of radius *R* and weight *P;* attached to the other end is a load *D* of weight Q. If $v_{C0} = 0$, determine the velocity v_C of the centre *C* of the cylinder after it has travelled a distance *s*, and the acceleration a_C of the centre. The coefficient of rolling friction of the cylinder is *k,* the radius of gyration of the cylinder with respect to its axis is ϱ. Neglect the mass of the string and the pulley

Solution. 1) We use Eq. (5.39) to determine the velocity v_C:

$$T - T_0 = \sum A_k^e.$$

In our case $T_0 = 0$ and $T = T_{cyl} + T_D$. From Eqs. (4.7), (5.22), and (5.24),

$$T_D = \frac{1}{2}\frac{Q}{g}v_D^2,\ \ T_{cyl} = \frac{1}{2}\frac{P}{g}v_C^2 + \frac{1}{2}\left(\frac{P}{g}\varrho^2\right)\omega^2.$$

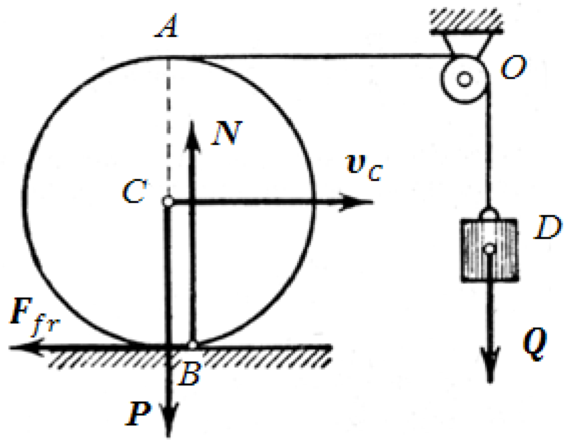

Fig. 64

As point B is the instantaneous centre of zero velocity,

$$\omega = \frac{v_C}{R}, \qquad \text{and} \qquad v_D = v_A = 2v_C.$$

Consequently,

$$T = \frac{1}{2g}\left[4Q + P\left(1 + \frac{\varrho^2}{R^2}\right)\right]v_C^2.$$

The forces doing work are $\boldsymbol{Q}$ and the couple $(\boldsymbol{N}, \boldsymbol{P})$. As $v_D = 2v_C$, the displacement of load D is $h = 2s$, and $A(\boldsymbol{Q}) = Q \cdot 2s$. The work done by the forces opposing the rolling can be found from Eq. (5.34), as $N = P = const$. Then,

$$\sum A_k^e = 2Qs - \frac{k}{R}Ps.$$

Substituting the found expressions, we obtain

$$\frac{1}{2g}\left[4Q + P\left(1 + \frac{\varrho^2}{R^2}\right)\right]v_C^2 = \left(2Q - \frac{k}{R}P\right)s,$$

whence

$$v_C = \sqrt{\frac{2g(2QR - kP)Rs}{4QR^2 + P(R^2 + \varrho^2)}}.$$

2) As in the preceding problem, to determine a_C differentiate both sides of the last equation with respect to time. Taking into account that $\frac{ds}{dt} = v_C$, we finally obtain

$$a_C = \frac{(2QR - kP)R}{4QR^2 + P(R^2 + \varrho^2)}g.$$

Problem 36. Referring to Fig. 65, a spiral spring is attached to gear *1* of radius r and weight P and to crank OC of length l and weight Q on which the gear is mounted. Gear *1* rolls on a fixed gear *2* of radius $R = l - r$. The moment of the spring $M_{sp} = c\alpha$, where α is the angle of rotation of gear *1* with respect to the crank. Neglecting friction, determine the period of vibration of the crank if it is disturbed from its position of equilibrium. The mechanism works in a horizontal plane.

Solution. We shall define the position of the crank by the angle φ measured from its equilibrium position. To exclude the unknown reaction of axis C from the computation, consider gear *1* and the crank as a single system and develop the differential equation of its motion from Eq. (5.38).

First, compute the kinetic energy T of the system in terms of the angular velocity ω_{cr} of the crank (as we are developing the law of motion of the crank). We have:

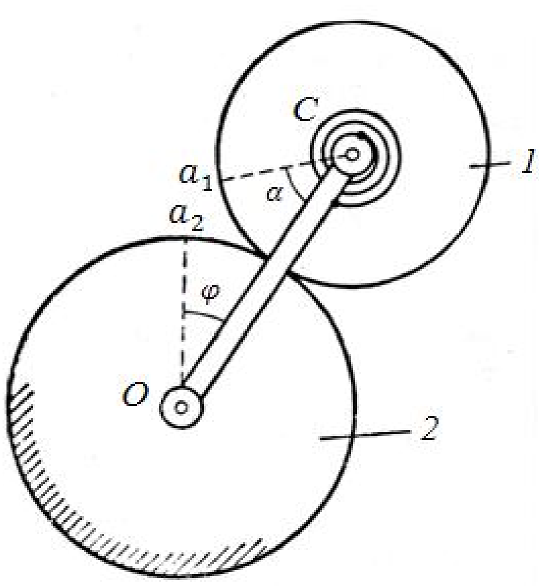

Fig. 65

$$T = T_{cr} + T_{gear} = \frac{1}{2}J_{0cr}\omega_{cr}^2 + \frac{1}{2}\frac{P}{g}v_C^2 + \frac{1}{2}J_{C_{gear}}\omega_{gear}^2.$$

Considering the crank as a homogeneous rod and the gear as a uniform disc, and taking into account that the point of contact is the instantaneous centre of zero velocity of gear *1,* we have

$$J_{0cr} = \frac{1}{3}\frac{Q}{g}l^2, \quad J_{C_{gear}} = \frac{1}{2}\frac{P}{g}r^2,$$

$$v_C = \omega_{cr}l, \quad \omega_{gear} = \frac{v_C}{r} = \frac{l}{r}\omega_{cr}.$$

Note again that Eq. (5.24) which is used to compute T_{gear}, contains the absolute angular velocity of the gear, not its relative velocity of rotation with respect to the crank. Substituting all the determined quantities, we finally obtain

$$T = \frac{1}{12g}(2Q + 9P)l^2\omega_{cr}^2.$$

Now let us compute the elemental work. In this case no external forces do any work, therefore, $dA^e = 0$. The elemental work done by the elastic force of the spring (an internal force) in turning the gear through an angle α about the crank is $dA^i = -M_{sp}d\alpha = -c\alpha d\alpha$ (the minus sign indicates that the moment is directed opposite the direction through which the gear is turned). As we are seeking the law of motion of the crank, let us express angle α in terms of φ. As $R\varphi = r\alpha$,

$$\frac{\alpha}{R} = \frac{\varphi}{r}, \ \alpha = \frac{l-r}{r}\varphi \text{ and } dA^i = -c\frac{(l-r)^2}{r^2}\varphi d\varphi.$$

Writing now the equation $dT = dA^i$, we have

$$\frac{1}{6g}(2Q + 9P)l^2\omega_{cr} \cdot d\omega_{cr} = -c\frac{(l-r)^2}{r^2}\varphi d\varphi.$$

Dividing through by dt and taking into account that $\frac{d\varphi}{dt} = \omega_{\text{cr}}$ and $\frac{d\omega_{cr}}{dt} = \frac{d^2\varphi}{dt^2}$, we finally obtain the differential equation of motion of the system in the form

$$\frac{d^2\varphi}{dt^2} + k^2\varphi = 0,$$

where

$$k^2 = \frac{6gc(l-r)^2}{(2Q+9P)l^2r^2}.$$

This equation is a differential equation of harmonic motion. When moved from its equilibrium position the crank will perform simple harmonic motion the period of which will be

$$T = \frac{2\pi}{k} = 2\pi\frac{lr}{l-r}\sqrt{\frac{2Q+9P}{6gc}}.$$

Problem 37. A wheel of radius R and weight P rotates on its axis O with an angular velocity ω_0 (Fig. 66). A brake shoe is applied to the wheel at some instant with a force $\boldsymbol{Q}$. The coefficient of friction of the shoe on the wheel is f. Neglecting friction in the axle and the weight of the spokes, determine in how many seconds the wheel will stop.

Solution. Taking into account Eqs. (5.18), (5.20) and considering the moment positive in the direction of the rotation, we have

$$J_O\frac{d\omega}{dt} = -fQr,$$

as the force of friction $F_{fr} = fQ$. From this, integrating, we obtain

$$J_O\omega = -fQrt + C_1.$$

According to the initial conditions, at $t = 0$, $\omega = \omega_0$, consequently $C_1 = J_O\omega_0$, and finally

$$\omega = \omega_0 - \frac{fQr}{J_O}t.$$

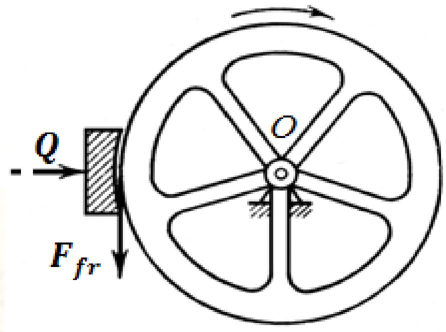

Fig. 66

At $t = t_1$, when the wheel stops, $\omega = 0$. Substituting this expression and taking into account that for the rim (a ring) $J_O = \frac{P}{g}r^2$, we obtain

$$t_1 = \frac{J_O\omega_0}{fQr} = \frac{Pr\omega_0}{fQg}.$$

If we want to determine the number of revolutions of the wheel until it stops, it is more convenient to apply the theorem of the change in kinetic energy.

Problem 38. A vertical cylindrical rotor whose moment of inertia with respect to its axis is J_z (Fig. 67) is made to revolve by an applied torque M_t. Determine how the angular velocity ω of the rotor will change during the motion if $\omega_0 = 0$ and the moment of the resisting force of the air is proportional to ω, i.e., $M_{res} = \mu\omega$.

Solution. The differential equation of the rotation of the rotor has the form (assuming the moments in the direction of rotation to be positive)

$$J_z\frac{d\omega}{dt} = M_t - \mu\omega.$$

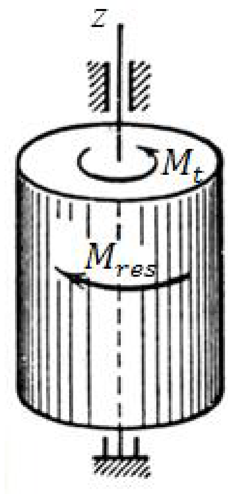

Fig. 67

Separating the variables and assuming $\frac{\mu}{J_z} = n$, we have

$$-\mu\frac{d\omega}{M_t - \mu\omega} = -ndt,$$

whence, integrating, we find

$$\ln(M_t - \mu\omega) = -nt + \ln C.$$

As, at $t = 0$, $\omega = 0$, then $C = M_t$, and

$$\ln\frac{M_t - \mu\omega}{M_t} = -nt,$$

or

$$\frac{M_t-\mu\omega}{M_t} = e^{-nt},$$

and finally we obtain

$$\omega = \frac{M_t}{\mu}(1 - e^{-nt}).$$

The angular velocity of the rotor increases with time and tends towards the limiting value $a_{lim} = \frac{M_t}{\mu}$.

Problem 39. A uniform circular cylinder rolls down an inclined plane without slipping (Fig. 68). Neglecting rolling friction, determine the acceleration of the centre of the cylinder and the limiting impending slip.

Solution. Let us introduce the following notations: α for the angle of inclination of the surface, $\boldsymbol{P}$ for the weight of the cylinder, R for its radius, and $\boldsymbol{F}$ for the limiting friction of impending slip. Let us also direct the x axis along the inclined plane and the y axis perpendicular to it.

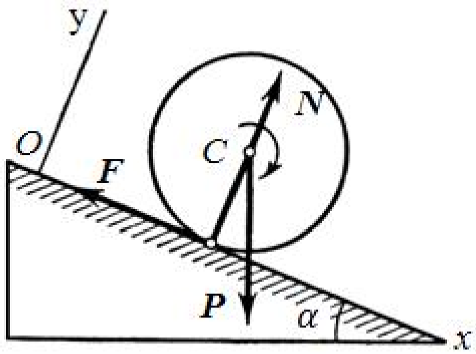

Fig. 68

As the centre of mass of the cylinder does not move parallel to the y axis, $a_{Cy} = 0$, and the sum of the projections of all the forces on the y axis is also zero. Thus, $N = P\cos\alpha$.

Take into account that $a_{Cx} = a_C$. Neglecting rolling friction and taking the direction of rotation of the cylinder as the positive direction of the moment of force, we find

$$Ma_C = P\sin\alpha - F, \quad J_c\,\varepsilon = FR.$$

These equations contain three unknown quantities J_c, ε and F (we cannot consider $F = fN$ here, because this equality is valid only when the point of contact slides on the surface; when there is no sliding it is possible for $F \le fN$. We obtain an additional relationship between the unknown quantities if we take into account that in pure rolling $v_c = \omega R$, whence, differentiating, we have $a_c = \varepsilon R$. For uniform cylinder $J_c = 0{,}5MR^2$, then the second of equations takes the form

$$\frac{1}{2}Ma_C = F.$$

Substituting this expression of F into the first of equations, we obtain $a_c = \frac{2}{3}g\sin\alpha$.

Now we find $F = \frac{1}{3}P\sin\alpha$.

This is the friction force that must act on the rolling cylinder if it is not to slip. It was pointed out before that $F = fN$. Cons1equently, pure rolling takes place when

$$\frac{1}{3}P\sin\alpha \le fP\cos\alpha \quad \text{or} \quad f \ge \frac{1}{3}\tan\alpha.$$

If the coefficient of friction is less than this, force F cannot attain the obtained value, and the cylinder will slip. In this case v_c and ω are not related by the equality $v_c = \omega R$ (the point of contact is not the instantaneous centre of zero velocity), but now F has a limiting value, i.e., $fN = fP\cos\alpha$, and equations take the form

$$\frac{P}{g}a_c = P(\sin\alpha - f\cos\alpha), \quad \frac{1}{2}\frac{P}{g}R^2\varepsilon = fPR\cos\alpha,$$

whence

$$a_c = g(\sin\alpha - f\cos\alpha), \quad \varepsilon = \frac{2gf}{R}\cos\alpha.$$

In this case the centre of the cylinder moves with acceleration a_c, while the cylinder itself rotates with an angular acceleration ε.

Problem 40. Solve the previous problem taking into account the resistance to rolling, assuming the coefficient of rolling friction to be k.

Solution. In order to give an example of another method of computation, let us find a_c with the help of the theorem of the change in kinetic energy, i.e., the equation

$$dT = dA^e.$$

In our case,

$$T = \frac{3}{4}Mv_C^2.$$

Only the force $\boldsymbol{P}$ and the resisting moment perform any work. The work done by forces $\boldsymbol{F}$ and $\boldsymbol{N}$ in rolling is zero. Then, taking into account Eq. (5.33), we obtain (see Fig. 68, but now with force $\boldsymbol{N}$ shifted by the value k in the direction of the motion):

$$dA^e = P\sin\alpha \cdot ds_C - \frac{k}{R}Nds_C = P\left(\sin\alpha - \frac{k}{R}\cos\alpha\right)dS_C.$$

Substituting the determined quantities and dividing through by dt, we have

$$\frac{3}{2}\frac{P}{g}v_C\frac{dv_C}{dt} = P\left(\sin\alpha - \frac{k}{R}\cos\alpha\right)\frac{ds_C}{dt}.$$

The last multiplier is equal to v_C, and we finally obtain

$$a_C = \frac{2}{3}g\left(\sin\alpha - \frac{k}{R}\cos\alpha\right).$$

At $k = 0$ this formula gives the result of the previous problem.

The frictional force can now be found from the equation $Ma_C = P\sin\alpha - F$, which does not change its form.

Problem 41. A uniform cylinder of weight *P* and radius *r* starts rolling from rest without slipping from a point on a cylindrical surface of radius *R* defined by angle φ_0 (Fig.69).

Determine: 1) The pressure of the cylinder on the surface for any angle φ; 2) The law of motion of the cylinder when angle φ_0 is small. Neglect rolling friction.

Solution. 1) Acting on the cylinder in any position is force ***P****,* the normal reaction $\boldsymbol{N}$, and the frictional force ***F****,* without which rolling is impossible. The path of the centre *C* is known: a circle of radius $R - r$. To determine *N* we make use of equations (5.5). Directing the normal *Cn* inwards to the path and projecting all forces on this normal, we obtain

$$M\frac{v_C^2}{R-r} = N - P\cos\varphi.$$

The quantity υ_C in this equation can be found from the theorem of the change in kinetic energy:

$$T - T_0 = \sum A_k^e.$$

In our case $T_0 = 0$ and $T = \frac{3}{4} M\upsilon_C^2$ (see Problem 29). Only force $\boldsymbol{P}$ does any work, consequently,

$$\sum A_k^e = Ph = P(R - r)(\cos\varphi - \cos\varphi_0),$$

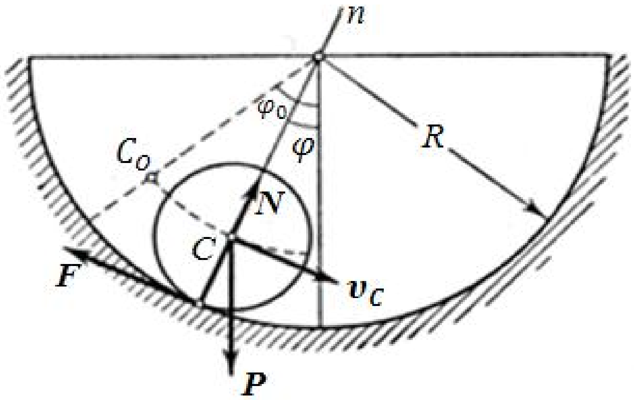

Fig. 69

and equation takes the form

$$\frac{3}{4} M\upsilon_C^2 = P(R - r)(\cos\varphi - \cos\varphi_0).$$

Computing from here $M\upsilon_C^2$, we obtain finally

$$N = \frac{P}{3}(7\cos\varphi - 4\cos\varphi_0).$$

If, for example, $\varphi_0 = 60°$ and $\varphi = 0°$, then $N = \frac{5}{3} P$.

2) To determine the law of motion of point *C*, differentiate equation with respect to time. We obtain

$$\frac{3}{2}\frac{P}{g}\upsilon_C \frac{d\upsilon_C}{dt} = -P(R - r)\sin\varphi \frac{d\varphi}{dt}.$$

In our case angle φ decreases when the cylinder moves, and $\frac{d\varphi}{dt} < 0$. Then

$$\upsilon_C = (R - r)\left|\frac{d\varphi}{dt}\right| = -(R - r)\frac{d\varphi}{dt};\ \frac{d\upsilon_C}{dt} = -(R - r)\frac{d^2\varphi}{dt^2}.$$

Substituting these expressions into the previous equation, we obtain finally

$$\frac{d^2\varphi}{dt^2} + \frac{2}{3}\frac{g}{R - r}\sin\varphi = 0.$$

If angle φ_0 is small, then, as $\varphi \leq \varphi_0$, we can assume that $\sin\varphi \approx \varphi$, and the equation takes the form

$$\frac{d^2\varphi}{dt^2} + k^2\varphi = 0,$$

where

$$k^2 = \frac{2}{3}\frac{g}{R - r}.$$

Consequently (see §3.1), the centre of the cylinder performs simple harmonic motion, its period being

$$T = \frac{2\pi}{k} = 2\pi\sqrt{\frac{3(R - r)}{2g}}.$$

Problem 42. The body in Fig. 70 rests at B on a piezoelectric sensor of an instrument for measuring pressure, and at A it is attached to a string AD.

When the system is in equilibrium, AC is horizontal and the pressure at B is $Q = Q_0$. Determine the moment of inertia J_C of the body with respect to an axis through the centre of mass C, if at the instant when the string is severed the pressure at B becomes Q_1, and the distance l is known.

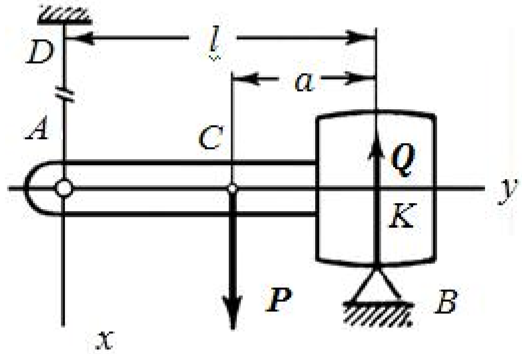

Fig. 70

Solution. 1) In the equilibrium position $Q_0 l = P(l - a)$, whence

$$a = \frac{P - Q_0}{P} l.$$

2) When the string is severed, the body begins plane motion. Its displacement in the initial time increment can be neglected. Then Eqs. (5.6) and (5.19), which are valid only for this initial time interval, will take the form

$$Ma_{C_x} = P - Q_1,\ a_{C_y} = 0,\ J_C\varepsilon = Q_1 a.$$

As $a_{C_y} = 0$, point C starts moving vertically down and point B slides horizontally (assuming the friction in the support to be very small). Erecting perpendiculars to the directions of these displacements, we find that the instantaneous centre of zero velocity is at point K. Consequently, $v_C = \omega a$. Assuming $a = const.$ for the elementary time interval, we obtain, after differentiating, $a_C = a\varepsilon$. Then the first of equations gives

$$\frac{P}{g} a\varepsilon = P - Q_1.$$

Determining ε from here, we obtain finally

$$J_C = \frac{Q_1 a}{\varepsilon} = \frac{P}{g}\frac{Q_1}{P - Q_1} a^2.$$

Problem 43. The weight of a motor car together with its wheels is P, the weight of each wheel is p, and their radii are r (Fig. 71). Acting on the rear (driving) wheels is a turning moment M_t. The car starts from rest and is subjected to the resistance of the air, which is proportional to the square of the translational velocity: $R = \mu v^2$. The frictional moment acting on the axle of each wheel is M_{fr}. Neglecting rolling friction, determine 1) the maximum velocity of the car; 2) the sliding friction acting on the driving and driven wheels during motion.

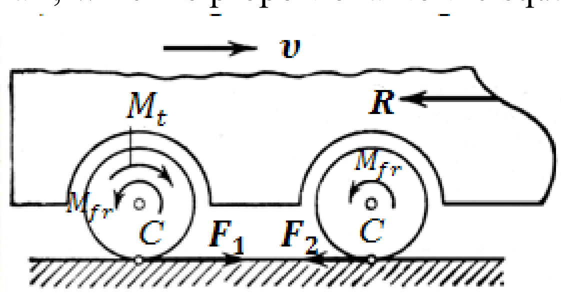

Fig. 71

Solution. 1) To determine the maximum velocity, write the equation of motion from Eq. (5.38):

$$dT = dA^e + dA^i.$$

The kinetic energy of the car is equal to the energy of the body plus the energy of the wheels. Taking into account that P is the weight of the whole car and $v_C = \omega r,$ and denoting the radius of gyration of each wheel by the symbol ϱ we obtain

$$T = \frac{1}{2}\frac{P}{g}v_C^2 + 4\left(\frac{1}{2}J_C\omega^2\right) = \frac{1}{2g}\left(P + 4p\frac{\varrho^2}{r^2}\right)v_C^2.$$

Of all the external forces, only the resistance of the air does work, as we have neglected rolling resistance, and in this case the work done by the frictional forces F_1 and F_2 of the wheels on the road is zero. Therefore,

$$dA^e = -\mu v_C^2 ds_C.$$

The work done by the internal forces (the torque and the friction in the axes) is

$$dA^i = \left(M_t - 4M_{fr}\right)d\varphi = \left(M_t - 4M_{fr}\right)\frac{ds_C}{r}.$$

Substituting all these expressions and dividing through by dt we obtain

$$\frac{1}{g}\left(p + 4p\frac{\varrho^2}{r^2}\right)v_C\frac{dv_C}{dt} = \frac{1}{r}\left(M_t - 4M_{fr} - \mu r v_C^2\right)\frac{ds_C}{dt},$$

from which, cancelling out $v_C = \frac{ds_C}{dt}$, we find

$$\left(P + 4p\frac{\varrho^2}{r^2}\right)a_C = \frac{g}{r}\left(M_t - 4M_{fr} - \mu r v_C^2\right).$$

When the velocity reaches its limiting value, the acceleration a_C becomes zero. Therefore v_C^{lim} can be found from the equation

$$M_t - 4M_{fr} - \mu r v_C^2 = 0,$$

whence

$$v_C^{lim} = \sqrt{\frac{M_t - 4M_{fr}}{\mu r}}.$$

2) To determine the frictional forces acting on each wheel, we deduce the equations of the rotation of the wheels about their axes. For the driving wheels, taking into account that the frictional force $\boldsymbol{F_1}$ acting on each of them is directed forward, we obtain

$$2\frac{p}{g}\varrho^2\varepsilon = M_t - 2M_{fr} - 2F_1 r.$$

Since in rolling $a_C = \varepsilon r,$ we obtain finally

$$F_1 = \frac{0.5M_t - M_{fr}}{r} - \frac{\varrho^2}{r^2}\frac{p}{g}a_C.$$

The frictional force $\boldsymbol{F_2}$ acting on each of the driven wheels is directed backwards. Therefore, for the driven wheels we have

$$\frac{p}{g}\varrho^2\varepsilon = F_2 r - M_{fr},$$

whence

$$F_2 = \frac{M_{fr}}{r} + \frac{\varrho^2}{r^2}\frac{p}{g}a_C.$$