3. VIBRATION OF A PARTICLE

3.1. Free Harmonic Motion

The study of vibrations is essential for a number of physical and engineering fields. Although the vibrations studied in such different fields as mechanics, radio engineering, and acoustics are of different physical nature, the fundamental laws hold good for all of them. The study of mechanical vibrations is therefore of importance not only because they are frequently encountered in engineering but also because the results obtained in investigating mechanical vibrations can be used in studying and understanding vibration phenomena in other fields.

We shall start with examining free harmonic motion of a particle. Consider a particle *M* (Fig.11) moving rectilinearly under the action of a *restoring force* ***F*** directed towards a fixed centre *O* and proportional to the distance from that centre. The projection of ***F*** on the axis *Ox* is

$$F_x = -cx. \tag{3.1}$$

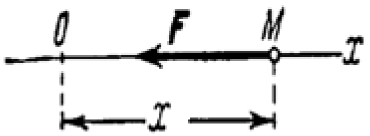

Fig. 11

We see that the force ***F*** tends to return the particle to its position of equilibrium 0, where $F = 0$, which is why it is called a "restoring" force. Let us derive the equation of motion of the particle *M*. Writing the differential equation of motion (2.1), we obtain

$$m\frac{d^2x}{dt^2} = -cx.$$

Dividing both sides of the equation by m and introducing notation

$$\frac{c}{m} = k^2, \tag{3.2}$$

we reduce the equation to the form

$$m\frac{d^2x}{dt^2} + k^2x = 0. \tag{3.3}$$

Eq. (3.3) is the *differential equation of free harmonic motion.* Referring to the theory of differential equations, as the roots of a characteristic equation of the type of Eq. (3.3) are imaginary, its general solution will be

$$x = C_1 \sin kt + C_2 \cos kt, \tag{3.4}$$

where C_1 and C_2 are constants of integration.

If we replace C_1 and C_2 by constants a and α, such that $C_1 = a\cos\alpha$ and $C_2 = a\sin\alpha$, we obtain

$$x = a(\sin kt \cos\alpha + \cos kt \sin\alpha) \text{ or}$$

$$x = a\sin(kt + \alpha). \tag{3.5}$$

This is another form of the solution of Eq. (3.3) in which the constants of integration appear as a and α, and which is more convenient for general analyses.

The velocity of a particle in this type of motion is

$$v_x = \frac{dx}{dt} = ak\sin(kt + \alpha). \tag{3.6}$$

The vibration of a particle described by Eq. (3.5) is called *simple harmonic motion.*

The quantity *a,* which is the maximum distance of *M* from the centre of vibration, is called the *amplitude of vibration* (Fig. 12). The quantity $\varphi = kt + \alpha$ is called the *phase of vibration.* Unlike the coordinate *x,* the phase φ defines both the position of the particle at any given time and the direction of its subsequent motion.

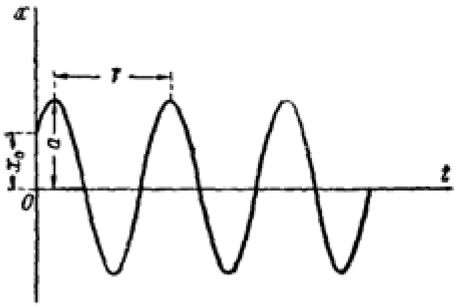

Fig. 12

The quantity *k* is called the *angular, or circular, frequency of vibration.* The time *T* in which the moving particle makes one complete oscillation is called the *period of vibration.* In one period the phase changes by 2π. Consequently, we must have *kT=2π,* whence the period

$$T = \frac{2\pi}{k}. \tag{3.7}$$

The quantity ν, which is the inverse of the period and specifies the number of oscillations per second, is called the *frequency of vibration:*

$$\nu = \frac{1}{T} = \frac{k}{2\pi}.$$

It can be seen from this that the quantity *k* differs from ν only by a constant multiplier *2π.* Usually we shall speak of the quantity *k* as of frequency.

The values of *a* and *α* are determined from the initial conditions. Assuming that, at $t=0$, $x=x_0$ and $v_x=v_0$ we obtain from Eqs. (3.5) and (3.6) $x_0 = a\sin\alpha$ and $\frac{v_0}{k} = a\cos\alpha$. By first squaring and adding these equations and then dividing them, we obtain

$$a = \sqrt{\frac{v_0^2}{k} + x_0^2},\ \tan\alpha = \frac{kx_0}{v_0}.$$

Note the following properties of free harmonic motion:

1) The amplitude and initial phase *depend* on the initial conditions;

2) The frequency *k,* and consequently the period *T, do not depend* on the initial conditions and are invariable characteristics for a given vibrating system.

It follows, in particular, that if a problem requires that only the period (or frequency) of vibration be determined, it is necessary to write a differential equation of motion in the form (3.3). Then *T* is found immediately from Eq. (3.7) without integrating.

Consider the next example: a weight is attached to end *B* of a vertical spring *AB* and released from rest (Fig.13). Determine the law of motion of the weight if the elongation of the spring in the equilibrium condition is δ_{st} (the static elongation of the spring).

Place the origin O of the coordinate axis in the position of static equilibrium of the system and direct the axis Ox vertically down. The elastic force $F = c|\Delta l|$. In our case $\Delta l = \delta_{st} + x$, hence

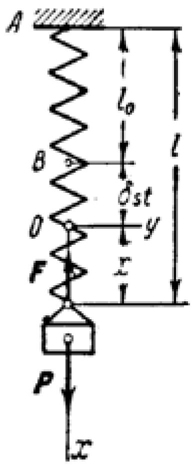

Fig. 13

$$F_x = -c(\delta_{st} + x).$$

Writing the differential equation of motion, we obtain

$$m\frac{d^2x}{dt^2} = -c(\delta_{st} + x) + P.$$

But from the conditions of the problem the gravitational force $P = mg = c\delta_{st}$ (in the position of equilibrium force P is balanced by the elastic force $c\delta_{st}$). Introducing the notation $\frac{c}{m} = \frac{g}{\delta_{st}} = k^2$, we reduce the equation to the form

$$\frac{d^2x}{dt^2} + k^2 x = 0,$$

whence immediately we find the period of vibration

$$T = \frac{2\pi}{k} = 2\pi\sqrt{\frac{\delta_{st}}{g}}.$$

Thus, the *period of vibration is proportional to the square root of the static elongation of the spring* (this holds good also for a load vibrating on an elastic beam, where δ_{st} *is* the static deflection of the beam).

The solution of the obtained differential equation is

$$x = C_1 \sin kt + C_2 \cos kt.$$

From the initial conditions, *at* $t=0$, $x=\delta_{st}$, and $v_x = 0$. *A*s $v_x = \frac{dx}{dt} = kC_1 \cos kt - kC_2 \sin kt$, substituting the initial conditions, we obtain $C_2 = -\delta_{st}, C_1 = 0$. Hence, the amplitude of vibration is δ_{st} and the motion is according to the law

$$x = -\delta_{st} \cos kt.$$

We see that the maximum elongation of the spring in this motion is $2\delta_{st}$.

This solution shows that *a constant force* ***P*** *does not change the type of motion under the action of an elastic force* ***F*** *but only shifts the center of the vibrations in the direction of the action of the force by the quantity* δ_{st}(without the force ***P*** the vibration would, evidently, be about *B*).

3.2. Damped Vibration

Let us see how the resistance of a surrounding medium affects vibrations, assuming the resisting force proportional to the first power of the velocity: $\boldsymbol{R} = -\mu\boldsymbol{v}$ (the minus indicates that force ***R*** is opposite to $\boldsymbol{v}$). Let a moving particle be acted upon by a restoring force ***F*** and a resisting force ***R***

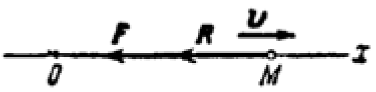

Fig. 14

(Fig. 14).

Then $F_x = -cx,\ R_x = -\mu v_x = -\mu \frac{dx}{dt}$ and the differential equation of motion is

$$m \frac{d^2x}{dt^2} = -cx - \mu \frac{dx}{dt}.$$

Dividing both sides by *m,* we obtain

$$\frac{d^2x}{dt^2} + 2b\frac{dx}{dt} + k^2 x = 0, \tag{3.8}$$

where
$$\frac{c}{m} = k^2,\ \frac{\mu}{m} = 2b. \tag{3.9}$$

It is easy to verify that *k* and *b* have the same dimension (sec^{-1}), which makes it possible to compare them.

Eq. (3.8) is called the *differential equation of damped vibration.* The solution of Eq (3.8) can be found by passing to a new variable *z* through the equality $x = ze^{-bt}$. Then

$$\frac{dx}{dt} = e^{-bt}\left(\frac{dz}{dt} - bz\right);\ \frac{d^2x}{dt^2} = e^{-bt}\left(\frac{d^2z}{dt^2} - 2b\frac{dz}{dt} + b^2 z\right).$$

Substituting these expressions and the expression of *x* into Eq. (3.8), and after the necessary computation, we obtain

$$\frac{d^2z}{dt^2} + (k^2 - b^2)z = 0. \tag{3.10}$$

Let us consider the case when *k*>*b*, i.e., when the resistance is small as compared with the restoring force. Introducing the notation

$$\tilde{k} = \sqrt{k^2 - b^2}, \tag{3.11}$$

we see that Eq. (3.10) coincides with Eq. (3.3).Consequently, $z = a\sin(\tilde{k}t + \alpha)$ or, passing to *x,*

$$x = ae^{-bt}\sin(\tilde{k}t + \alpha). \tag{3.12}$$

The expression (3.12) gives the solution of differential equation (3.8). The quantities a and α are constants of integration and are determined by the initial conditions.

Vibrations according to the law (3.12) are called *damped* because, due to the multiplier e^{-bt}, the value of *x* decreases with time and tends to zero. A graph of such vibrations is given in Fig. 15. The graph shows that the vibrations are not periodic, though they do show a certain repetition. For example, a particle oscillating about a centre O returns to that centre at certain intervals $\tilde{T}$ equal to the period of $\sin(\tilde{k}t + \alpha)$.

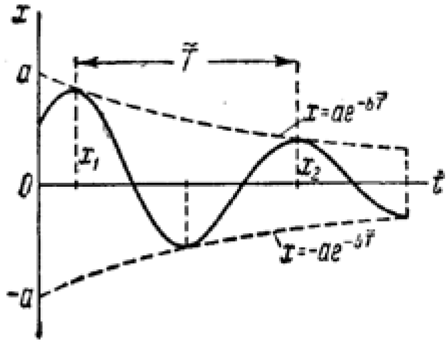

Fig. 15

Therefore, the quantity

$$\tilde{T} = \frac{2\pi}{\tilde{k}} = \frac{2\pi}{\sqrt{k^2 - b^2}} \tag{3.13}$$

is conventionally called the *period of damped vibration.* Comparing, Eqs. (3.13) and (3.7), we see that $\tilde{T} > T$, i.e., that resistance to vibration tends to increase the period of the vibration. When however, the resistance is small $(b \ll k)$ the quantity b^2 can be neglected in comparison with k^2 and we can assume $\tilde{T} \approx T$. Thus, a small resistance has no practical effect on the period of vibration.

The time interval between two successive displacements of an oscillating particle to the right or to the left is also equal to $\tilde{T}$. Hence, if the maximum displacement x, to the right takes place at time t_1 the second displacement x_2 will be at time $t_2 = t_1 + \tilde{T}$, etc. Then, by Eq. (3.12) and taking into account that $\tilde{k}\tilde{T} = 2\pi$, we have

$$x_1 = ae^{-bt_1}\sin(\tilde{k}t_1 + \alpha),$$

$$x_2 = ae^{-b(t_1+\tilde{T})}\sin(\tilde{k}t_1 + \tilde{k}\tilde{T} + \alpha) = x_1 e^{-b\tilde{T}}.$$

Similarly, for any displacement x_{n+1} we will have $x_{n+1} = x_n e^{-b\tilde{T}}$.Thus we find that the amplitude of vibration decreases in geometric progression. The denominator of this progression $e^{-b\tilde{T}}$ is called the *damping decrement,* and the modulus of its logarithm, i.e., the quantity $b\tilde{T}$, the *logarithmic decrement.*

It follows from these results that a small resistance has practically no effect on the period of vibration, but gradually damps it by virtue of the amplitude of vibration decreasing according to a law of geometric progression.

When the resistance is large and $b > k$, the solution of Eq. (3.10) contains no trigonometric functions. The particle no longer oscillates but instead, under the influence of the restoring force, gradually approaches the position of equilibrium.

3.3. Damped Forced Vibrations. Resonance

Consider the motion of a particle on which are acting a restoring force $\boldsymbol{F}$, a damping force $\boldsymbol{R}$ proportional to the velocity (see § 3.2), and a disturbing force $\boldsymbol{Q}$, whose projection on the axis *Ox* is $Q_x = Q_0 \sin pt$. The differential equation of this motion has the form

$$m\frac{d^2x}{dt^2} = -cx - \mu\frac{dx}{dt} + Q_0 \sin pt.$$

Dividing both sides of the equation by m, assuming $\frac{Q_0}{m} = P_0$ and taking into account the expression (3.9), we obtain

$$\frac{d^2x}{dt^2} + 2b\frac{dx}{dt} + k^2x = Q_0 \sin pt. \tag{3.14}$$

Eq. (3.14) is the *differential equation of damped forced vibration of a particle.* Its general solution, as is known, has the form $x = x_1 + x_2$, where x_1 is the general solution of the equation without the right side, i.e., of Eq. (3.8) [at $k > b$ this solution is given by Eq. (3.12)], and x_2, is a particular solution of the complete equation (3.14). Let us find the solution x_2 in the form

$$x_2 = A\sin(pt - \beta),$$

where A and β are constants so chosen that Eq. (3.14) should become an identity. Differentiating, we obtain

$$\frac{dx_2}{dt} = Ap\cos(pt-\beta)\,,\ \frac{d^2x_2}{dt^2} = -Ap^2\sin(pt-\beta).$$

Substituting these expressions of the derivatives and x_2 into the left side of Eq. (3.14) and introducing for the sake of brevity the notation $pt-\beta=\psi$ (or $pt=\psi+\beta$), we obtain

$$A(-p^2+k^2)\sin\psi + 2bpA\cos\psi = P_0(\cos\beta\sin\psi+\sin\beta\cos\psi).$$

For this equation to be satisfied at any value of ψ, i.e., at any instant of time, the factors of $\sin\psi$ and $\cos\psi$ in the left and right sides should be separately equal. Hence,

$$A(k^2-p^2) = P_0\cos\beta,\quad 2bpA = P_0\sin\beta.$$

First squaring and adding these equations, and then dividing one by the other, we obtain:

$$A = \frac{P_0}{\sqrt{(k^2-p^2)^2+4b^2p^2}},\ \tan\beta = \frac{2bp}{k^2-p^2}. \tag{3.15}$$

As $x=x_1+x_2$, and the expression x_1 is given by Eq. (3.12), we have the final solution of Eq. (3.14) in the form

$$x = ae^{-bt}\sin\left(\tilde{k}t+\alpha\right) + A\sin(pt-\beta). \tag{3.16}$$

Here a and α are constants of integration determined from the initial conditions, and the expressions for A and β are given by Eqs. (3.15) and do not depend on the initial conditions. These vibrations are compounded of *natural vibration* [the first term in Eq. (3.16); Fig. 16 *a*] and *forced* vibration [the second term in Eq. (3.16); Fig. 16 *b*]. The natural vibration of the particle in such a case was discussed in § 3.2. It was established that it is transient and is damped fairly quickly, and after a certain interval of time t_t called the *transient period,* can be neglected. A curve showing the transient vibration is given in Fig. 16 *c*. For practical purposes it can thus be assumed that after a certain transient period a particle will vibrate according to the law

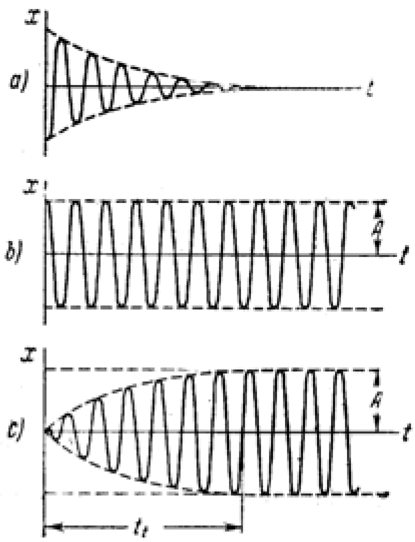

Fig. 16

$$x = A\sin(pt-\beta).$$

This is *steady-state forced vibration,* a sustained periodic motion with amplitude A defined by Eq. (3.15) and a frequency p equal to the impressed frequency. The quantity β characterizes the *phase shift* of forced vibration with respect to the disturbing force. Let us investigate the results obtained. First let us introduce the notation

$$\frac{p}{k} = \lambda,\ \frac{b}{k} = h,\ \frac{P_0}{k^2} = \frac{Q_0}{c} = \delta_0, \tag{3.18}$$

where λ is the frequency ratio, h a quantity characterizing the damping effect, δ_0 the magnitude of the static deflection of a particle under the action of force Q_0.

Then, dividing the numerator and denominator of Eq. (3.15) by k^2, we obtain

$$A = \frac{\delta_0}{\sqrt{(1-\lambda^2)^2 + 4h^2\lambda^2}}, \ \tan\beta = \frac{2h\lambda}{1-\lambda^2}. \tag{3.19}$$

It can be seen from Eq. (3.19) that A and β depend on two dimensionless parameters λ and h. Graphs of this relation for certain values of h are given in Fig. 17. The values of δ_0, λ and h can be computed for each specific problem from its conditions, and the values of A and p determined from the respective graphs or Eqs. (3.19).

These graphs (and equations) also show that by altering the frequency ratio λ we can induce forced vibrations of different amplitude.

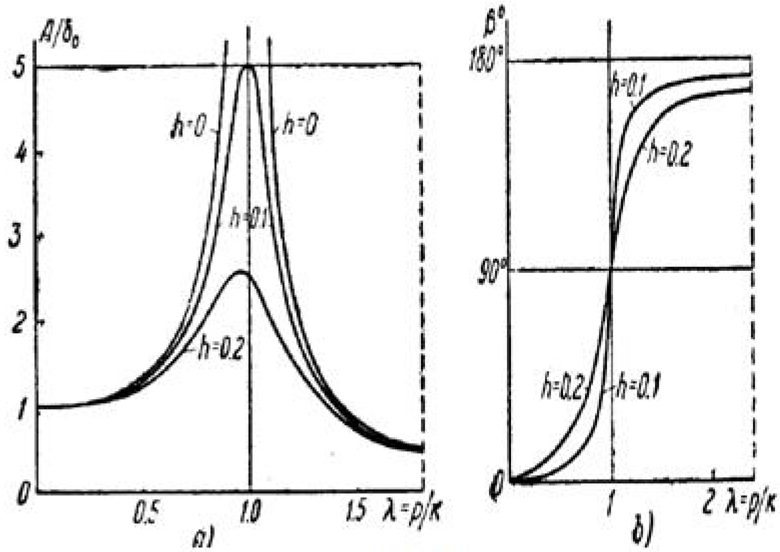

Fig. 17

When the resistance is very small (as ordinarily in the atmosphere) and λ is not close to unity, it is possible in Eqs. (3.19) assume approximately $h \approx 0$. In this case we obtain

$$A \approx \frac{\delta_0}{|1-\lambda^2|}; \ \beta \approx 0 \ (at\ \lambda < 1), \ \beta \approx 180^0 \ (at\ \lambda > 1).$$

Let us consider also the following special cases: 1) If the frequency ratio λ is very small $(p \ll k)$, then, assuming as an approximation $\lambda \approx 0$, we obtain from Eq. (3.19) $A \approx \delta_0$. The vibration in this case has an amplitude equal to the static deflection δ_0 and the phase shift is $\beta = 0$.

2) If the frequency ratio λ is very large$(p \gg k)$, A becomes very small. This case is of special interest for the absorption of vibrations in structures, instruments, etc. Assuming the resistance to be small and neglecting $2h\lambda$ and 1 as compared with λ^2 in Eq. (3.19), we obtain for computing A an approximate formula:

$$A = \frac{\delta_0}{\lambda^2} = \frac{P_0}{p^2}.$$

3) In all cases of practical interest h is very small. Then, from Eq. (3.19), if λ is almost unity the amplitude of forced vibrations becomes very large. This phenomenon is called *resonance.*

At resonance we can assume $\lambda = 1$ in Eq. (3.19), and then

$$A_r = \frac{\delta_0}{2h}, \ \beta_r = \frac{\pi}{2}. \tag{3.20}$$

We see that when h is small A_r can become very large. When the damping force, and with it h, tends to zero, the limiting value of the amplitude A_r as Eq. (3.20) shows, tends to infinity. Thus, with no damping force the vibration amplification process in resonance conditions is unlimited and the amplitude increases indefinitely.

A graph of resonance vibration is given in Fig. 18. When the damping forces are very small the picture is similar.

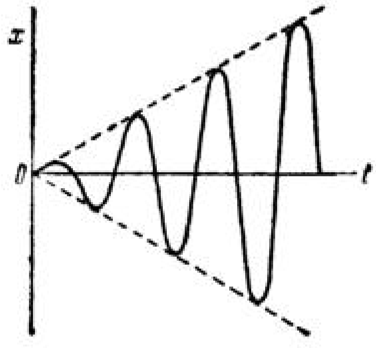

Fig. 18

General Properties of Forced Vibration. It follows from the results obtained above that forced vibration has the following important properties, which distinguish it from the natural vibration of a particle:

1) The amplitude of forced vibration does not depend on the initial conditions.

2) Forced vibration does not die out in the presence of resistance.

3) The frequency of forced vibration is equal to the frequency of the disturbing force and does not depend on the characteristics of the vibrating system (the disturbing force "impresses" its own vibration frequency on the system).

4) Even when the disturbing force Q is small, large forced vibration can be induced if the resistance is small and the frequency p is almost equal to k (resonance).

5) Even if the disturbing force is large, forced vibration can be damped if the frequency p is much larger than k.

Forced vibration, and resonance in particular, plays an important part in many branches of physics and engineering. Lack of balance in working machines and motors, for example, usually causes forced vibration to appear in the machine or its foundation.

In radio engineering the reverse is true. Resonance is extremely useful and is used to separate the signals of one radio station from those of all others (tuning).

3.4. Solution of Problems

Problem 8. Determine the periods of vibration of a load of weight *P* attached to two springs of stiffness C_1 and C_2 as shown in Figs. 19 and 20.

Solution. a) In the first case, in the static position both springs are subjected to a tensile force *P*. Therefore, the static elongations are

$$\delta_{1st} = \frac{P}{C_1}, \delta_{2st} = \frac{P}{C_2},$$

and the total elongation is

$$\delta_{st} = \delta_{1st} + \delta_{2st} = \frac{P(C_1+C_2)}{C_1C_2}, \text{ and}$$

$$C_{eq} = \frac{C_1C_2}{C_1+C_2},$$

where C_{eq} is the equivalent spring constant of the two given springs. In particular, at $C_1 = C_2$ we have

Fig. 19

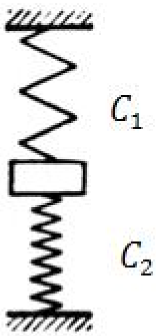

Fig. 20

$$C_{eq} = \frac{1}{2}C.$$

The period of vibration is

$$T = 2\pi\sqrt{\frac{\delta_{st}}{g}} = 2\pi\sqrt{\frac{P(C_1+C_2)}{gC_1C_2}}.$$

b) In the second case the top spring is subjected to a tensile force P_1, and the bottom spring is subjected to a compressive force P_2, such that $P_1 + P_2 = P$. For these springs we have $\delta_{1st} = \frac{P_1}{c_1}$, $\delta_{2st} = \frac{P_2}{c_2}$. But obviously $\delta_{1st} = \delta_{2st} = \delta_{st}$, and by virtue of the property of proportions

$$\delta_{st} = \frac{P_1}{c_1} = \frac{P_2}{c_2} = \frac{P_1+P_2}{c_1+c_2} \text{ or } \delta_{st} = \frac{P}{c_1+c_2}.$$

The equivalent stiffness $c_{eq} = c_1 + c_2$ and the period of vibration

$$T = 2\pi\sqrt{\frac{\delta_{st}}{g}} = 2\pi\sqrt{\frac{P}{g(c_1+c_2)}}.$$

Problem 9. The deflection caused in a beam by the weight of a motor mounted as shown in Fig. 21 is $\delta_{st} = 1\ cm$. At how many rpm of the shaft will resonance appear?

Solution. The period of natural vibration of the beam is

$$T = 2\pi\sqrt{\frac{\delta_{st}}{g}}.$$

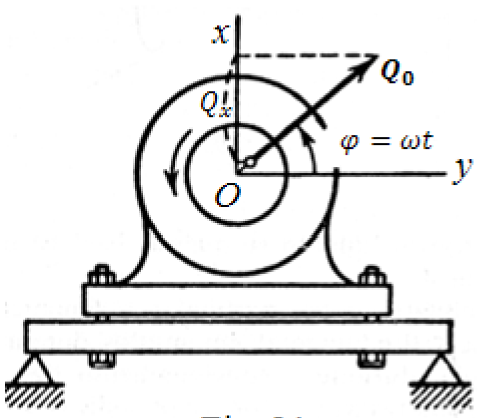

Fig. 21

If the centre of gravity of the shaft is not concentric with its axis, a centrifugal force $\boldsymbol{Q}_0$ will develop (Fig. 21). Its component $Q_x = Q_0 \sin \omega t$ (where ω is the angular velocity of the shaft) is the disturbing force acting on the beam; its frequency is $p = \omega$. Hence, the period of the forced vibration is $T_f = \frac{2\pi}{\omega}$.

Resonance will appear when $T_f = T$, i.e., at

$$\omega_{cr} = \sqrt{\frac{g}{\delta_{st}}} = 31.3\ sec^{-1}.$$

Hence, the critical speed

$$n_{cr} = \frac{30\omega_{cr}}{\pi} = 300\ rpm.$$

The working speed of the motor should be much greater than n_{cr} .

Problem 10. Analyze the forced vibration of a load attached to a spring (see example in p.3.1) if the upper end *A* of the spring oscillates vertically according to the law $\xi = a_0 \sin pt$.

Solution. Draw axis *Ox* as in Fig. 13. If we imagine the upper end of the spring displaced from point *A* downwards by a quantity ξ, the length of the spring will be $l = l_0 - \xi + \delta_{st} + x$. Then $F_x = -c\Delta l = -c(\delta_{st} + x - \xi)$, and the differential equation of motion, neglecting the resistance of the air and taking into account that $P = c\delta_{st}$, will be

$$m\frac{d^2x}{dt^2} = -c(\delta_{st} + x - \xi) + P = -cx + c\xi.$$

Introducing the notation $\frac{c}{m} = k^2$,

we obtain

$$\frac{d^2x}{dt^2} + k^2x = k^2a_0 \sin pt.$$

Consequently, the load will experience forced vibration, since, if we assume $b = 0$ and $P_0 = k^2a_0$, the equation coincides with Eq. (3.14). It can be seen from Eq. (3.18) that in the present case $\delta_0 = a_0$, and $h = 0$**.** The amplitude of forced vibration and the phase shift are determined by the Eq. (3.19).

If $p \ll k$ (the top end of the spring oscillates very slowly), then $\lambda \approx 0$ and $A \approx a_0$ and the phase shift $\beta = 0$. The load will oscillate as if the spring were a rigid rod, which physically corresponds to the condition $k \gg p$**.** *At p=k* resonance appears and the amplitude increases sharply. If the frequency *p* becomes larger than *k* **(**$\lambda > 1$**)** the load will vibrate in such a way that it will move down when the end of the spring moves up and vice versa (a phase shift of $\beta = 180°$), and the larger the value of *p* the smaller the amplitude. Finally, when *p* is much greater than *k* ($\lambda \gg 1$), the amplitude $A \approx 0$**.** The load will remain in the position of static equilibrium (point *O*) even though the top end of the spring will oscillate with amplitude a_0 (the frequency of this vibration is so large that the load, as it were, is unable to keep up with it).

4. INTRODUCTION TO THE DYNAMICS OF A SYSTEM

4.1. Mechanical Systems. External and Internal Forces

A mechanical system is defined as such a collection of material points (particles) or bodies in which the position or motion of each particle or body of the system depends on the position and motion of all the other particles or bodies. We shall thus regard a material body as a system of its particles.

A classical example of a mechanical system is the solar system, all the component bodies of which are connected by the forces of their mutual attraction.

A collection of bodies not connected by interacting forces does not comprise a mechanical system. In this summary we shall consider only mechanical systems, calling them just "systems" for short.

The forces acting on the particles or bodies of a system can be subdivided into external and internal forces.

External forces are defined as the forces exerted on the members of a system by particles or bodies not belonging to the given system. Internal forces are defined as the forces of interaction between the members of the same system. We shall denote external forces by the symbol $\boldsymbol{F}^e$, and internal forces by the symbol $\boldsymbol{F}^i$. Both external and internal forces can be either active forces or the reactions of constraints. The division of forces into external and internal is purely relative, and it depends on the extent of the system whose motion is being investigated. In considering the motion of the solar system as a whole, for example, the gravitational attraction of the sun acting on the earth is an internal force; in investigating the earth's motion about the sun, the same force is external.

Internal forces possess the following properties:

1. The geometrical sum (the principal vector) of all the internal forces of a system is zero. This follows from the third law of dynamics, which states that any two particles of a system (Fig. 22) act on each other with equal and oppositely directed forces $\boldsymbol{F}_{12}^{i}$ and $\boldsymbol{F}_{21}^{i}$, the sum of which is zero. Since the same is true for any pair of particles of a system, then

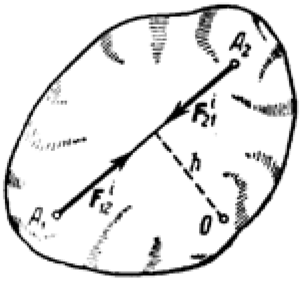

Fig. 22

$$\boldsymbol{F}_{k}^{i} = 0. \tag{4.1}$$

2. The sum of the moments (the principal moment) of all the internal forces of a system with respect to any centre or axis is zero. For if we take an arbitrary centre 0, it is apparent from Fig. 22 that $\boldsymbol{m}_0(\boldsymbol{F}_{12}^{i}) + \boldsymbol{m}_0(\boldsymbol{F}_{21}^{i}) = 0$. The same result holds good for the moments about any axis. Hence, for the system as a whole we have

$$\sum \boldsymbol{m}_0(\boldsymbol{F}_{k}^{i}) = 0 \text{ or } \sum m_x(\boldsymbol{F}_{k}^{i}) = 0. \tag{4.2}$$

It does not follow from the above, however, that the internal forces are mutually balanced and do not affect the motion of the system, for they are applied to different particles or bodies and may cause their mutual displacement. The internal forces will be balanced only when a given system is a rigid body.

4.2. Mass of a System. Centre of Mass

The motion of a system depends, besides the acting forces, on its total mass and the distribution of this mass. The *mass of a system* is equal to the arithmetical sum of the masses of all the particles or bodies comprising it:

$$M = \sum m_k. \tag{4.3}$$

The distribution of mass is characterized primarily by the location of a point called the centre of mass. *The centre of mass or centre of inertia, of a system is defined as a geometrical point C whose coordinates are given by the equations:*

$$x_c = \frac{\sum m_k x_k}{M},\ y_c = \frac{\sum m_k y_k}{M},\ z_c = \frac{\sum m_k z_k}{M}, \tag{4.4}$$

where m_k is the mass of a particle of the system, and x_k, y_k, z_k are its coordinates.

If the position of a centre of mass is defined by its radius vector $\boldsymbol{r}_c$, we can obtain from Eqs. (4.4) the following expression

$$\boldsymbol{r}_c = \frac{\sum m_k \boldsymbol{r}_k}{M}, \tag{4.5}$$

where $\boldsymbol{r}_k$ is the radius vector of a particle of the system.

For a body in a uniform gravitational field, the centre of mass coincides with the centre of gravity. The concepts of centre of gravity and centre of mass, however, are not identical. The concept of centre of gravity, as the point through which the resultant of the forces of gravity passes, has meaning only for a rigid body in a uniform field of gravity. The concept of centre of mass, as a characteristic of the distribution of mass in a system, on the other hand, has meaning for any system of particles or bodies, regardless of whether a given system is subjected to the action of forces or not.

4.3. Moment of Inertia of a Body about an Axis. Radius of Gyration

The position of centre of mass does not characterize completely the distribution of mass in a system. For if in the system in Fig. 23 the distance h of each of two identical spheres A and B from the axis Oz is increased by the same quantity, the location of the centre of mass will not change, though the distribution of mass will change and influence the motion of the system (all other conditions remaining the same, the rotation about axis Oz will be slower).

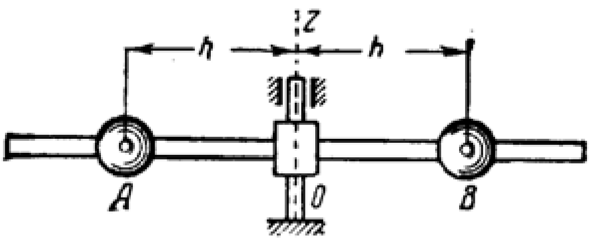

Fig. 23

Accordingly, another characteristic of the distribution of mass, called the moment of inertia, is introduced in mechanics. *The moment of inertia of a body with respect to a given axis Oz is defined as a scalar quantity equal to the sum of the masses of the particles of the body, each multiplied by the square of its perpendicular distance from the axis*

$$J_z = \sum m_k h_k^2. \tag{4.6}$$

It will be shown further on that moment of inertia plays the same part in the rotational motion of a body as mass does in translational motion, i.e., *moment of inertia is a measure of a body's inertia in rotational motion.*

By Eq. (4.6), the moment of inertia of a body is equal to the sum of the moments of inertia of all its parts with respect to the same axis. For a material point located at a distance h from an axis, $J_z = mh^2$. The dimension of moment of inertia in the international system of units is $[J] = kg \cdot m^2$.

The concept of *radius of gyration* is often employed in calculations. The radius of gyration of a body with respect to an axis Oz is a linear quantity ρ defined by the equation

$$J_z = M\rho^2, \qquad 4.7)$$

where M is the mass of the body.

It follows from the definition that geometrically the radius of gyration is equal to the distance from the axis Oz to a point, such that if the mass of the whole body were concentrated in it the moment of inertia of the point would be equal to the moment of inertia of the whole body. Knowing the radius of gyration, we can obtain the moment of inertia of a body from Eq. (4.7) and vice versa.

4.4. Moments of Inertia of Some Homogeneous Bodies

If we divide a body into elements, in the limit the sum in Eq. (4.6) will become an integral and we obtain

$$J_z = \iiint h^2\, dm, \quad (4.8)$$

where the integration is over the whole volume of the body and depends on the coordinates of the points of the body. Eq. (4.8) is convenient in computing the moments of inertia of homogeneous bodies. Let us examine some examples.

1. Thin Homogeneous Rod of Length l and Mass M. Let us find its moment of inertia with respect to an axis Az perpendicular to the rod (Fig. 24). If we lay off an axis Ax along AB, for any line element of length dx we have $h = x$ and its mass $dm = \rho_1\, dx$, where $\rho_1 = M/l$ is the mass of a unit length of the rod, and Eq. (4.8) gives:

$$J_A = \int_0^l x^2\, dm = \rho_1 \int_0^l x^2\, dx = \rho_1 \frac{l^3}{3}.$$

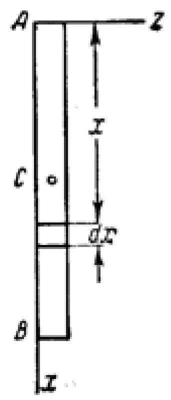

Fig. 24

Substituting the expression for ρ_1, we obtain finally

$$J_A = \frac{1}{3} M l^3.$$

2. Thin Circular Homogeneous Ring of Radius R and Mass M. Let us find its moment of inertia with respect to an axis Cz perpendicular to the plane of the ring through its centre (Fig. 25). As all the points of the ring are at a distance $h_k = R$ from axis Cz, Eq. (4.6) gives

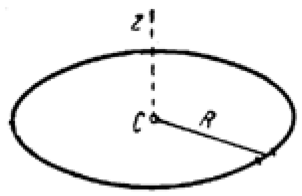

Fig. 25

$$J_c = \sum m_k R^2 = \left(\sum m_k\right) R^2 = MR^2.$$

Hence, for the ring

$$J_c = MR^2.$$

It is evident that the same result is obtained for the moment of inertia of a cylindrical shell of mass M and radius R with respect to its axis.

3. Circular Homogeneous Disc or Cylinder of Radius R and Mass M. Let us compute the moment of inertia of a circular disc with respect to an axis Cz perpendicular to it through its centre (Fig. 26a). Consider an elemental ring of radius r and width dr. Its area is $2\pi r\, dr$, and its mass $dm = \rho_2 2\pi r\, dr$, where $\rho_2 = \frac{M}{\pi R^2}$ is the mass of a unit area of the disc. From Eq.(4.8) we have for the elemental ring

$$dJ_c = r^2\, dm = 2\pi \rho_2 r^3\, dr$$

and for the whole disc

$$J_c = 2\pi\rho_2 \int_0^R r^3\, dr = \frac{1}{2}\pi\rho_2 R^4.$$

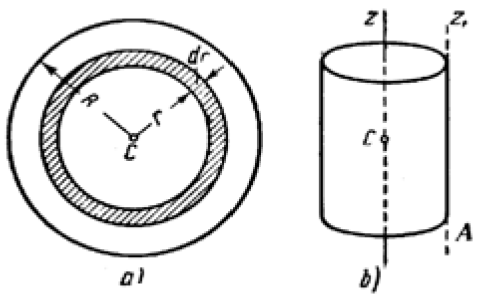

Fig. 26

Substituting the expression for ρ_2 we obtain finally

$$J_c = \frac{1}{2} MR^2.$$

It is evident that the same formula is obtained for the moment of inertia J_z of a homogeneous circular cylinder of mass M and radius R with respect to its axis Cz (Fig. 26b).

The moments of inertia of non-homogeneous and composite bodies can be determined experimentally with the help of appropriate instruments.

4.5. Moments of Inertia of a Body about Parallel Axes. The Parallel-Axis (Huygens') Theorem

In the most general case, the moments of inertia of the same body with respect to different axes are different. Let us see how to determine the moment of inertia of a body with respect to any axis if its moment of inertia with respect to a parallel axis through the body is known.

Draw an axis *Cz* through the centre of mass *C* of a body, and an axis Oz_1 parallel to it (Fig.27), denoting the distance between the two axes by the symbol *d.* By definition we have

$$J_{Oz_1} = \sum m_k h_k^2,\ J_{Cz} = \sum m_k\, h_k'^{\,2},$$

where h_k is the distance of an arbitrary point *B* of the body from axis Oz_1, and h_k' is the distance of the same point from axis *Cz,* It follows from ΔBae that

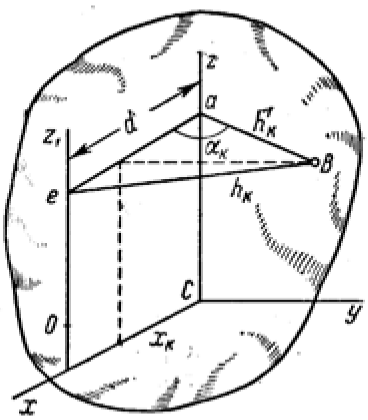

Fig. 27

$$h_k^2 = h_k'^{\,2} + d^2 - 2dh_k' \cos\alpha_k.$$

Let us draw from point *C*, as the origin of a coordinate system, axes *x* and *y* perpendicular to *Cz,* such that *x* intersects with axis Oz_1. It is evident that $Cx \parallel ae$. Denoting the coordinates of point *B* as x_k, y_k, z_k, we obtain

$$h_k' \cos\alpha_k = x_k \text{ and } h_k^2 = h_k'^2 + d^2 - 2dx_k.$$

Substituting this expression of h_k^2 into the expression for J_{Oz_1} and taking the common factors d^2 and *2d* outside the summation signs, we have

$$J_{Oz_1} = \sum m_k\, h_k'^{\,2} + \left(\sum m_k\right)d^2 - 2d\sum m_k\, x_k.$$

The first summation in the right side of the equation is equal to J_{Cz} and the second to the mass *M* of the body. Let us find the value of the third summation. From Eq. (4.4) we know that, for the coordinates of the centre of mass, $\sum m_k x_k = Mx_C$. But since in our case point *C* is the origin, $x_c = 0$, and consequently $\sum m_k\, x_k = 0$. We finally obtain

$$J_{Oz_1} = J_{cz} + Md^2. \tag{4.9}$$

Eq. (4.9) expresses the parallel-axis theorem enunciated by Huygens:

the moment of inertia of a body with respect to any axis is equal to the moment of inertia of the body with respect to a parallel axis through the centre of mass of the body plus the product of the mass of the body and the square of the distance between the two axes.

It follows from Eq. (4.9) that $J_{Oz_1} > J_{Cz}$. Consequently, of all the axes of same direction, the moment of inertia is least with respect to the one through the centre of mass.

4.6. The Differential Equations of Motion of a System

Suppose we have a system of n particles. Choosing any particle of mass m_k belonging to the system, let us denote the resultant of all the external forces acting on the particle (both active forces and the forces of reaction) by the symbol $\boldsymbol{F}_k^e$ and the resultant of all the internal forces by $\boldsymbol{F}_k^i$. If the particle has an acceleration $\boldsymbol{a}_k$, then, by the fundamental law of dynamics,

$$m_k \boldsymbol{a}_k = \boldsymbol{F}_k^e + \boldsymbol{F}_k^i.$$

Similar results are obtained for any other particle, whence, for the whole system, we have

$$\begin{cases} m_1 \boldsymbol{a}_1 = \boldsymbol{F}_1^e + \boldsymbol{F}_1^i \\ m_2 \boldsymbol{a}_2 = \boldsymbol{F}_2^e + \boldsymbol{F}_2^i \\ \dots\dots\dots\dots\dots\dots\dots \\ m_n \boldsymbol{a}_n = \boldsymbol{F}_n^e + \boldsymbol{F}_n^i \end{cases} \tag{4.10}$$

These equations, from which we can develop the law of motion of any particle of the system, are called *the differential equations of motion of a system* in vector form. Eqs. (4.10) are differential because $\boldsymbol{a}_k = \frac{d\boldsymbol{v}_k}{dt} = \frac{d^2\boldsymbol{r}_k}{dt^2}$. In the most general case the forces in the right side of the equations depend on the time, the coordinates of the particles of the system, and their velocities.

By projecting Eqs. (4.10) on coordinate axes, we can obtain the differential equations of motion of a given system in terms of the projections on these axes.

The complete solution of the principal problem of dynamics for a system would be to develop the equation of motion for each particle of the system from the given forces by integrating the corresponding differential equations. For two reasons, however, this solution is not usually employed.

Firstly, the solution is too involved and will almost inevitably lead into insurmountable mathematical difficulties.

Secondly, in solving problems of mechanics it is usually sufficient to know certain overall characteristics of the motion of a system without investigating the motion of each particle. These overall characteristics can be found with the help of the *general theorems* of systems dynamics, which we shall now study.

The main application of Eqs.(4.10) or their corollaries will be to develop the respective general theorems.