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PREFACE

The primary object of Dynamics to be gained by the student is a thorough grasp of fundamental principles. In most cases it is impossible to go beyond this object in the time available for the course. In the preparation of this textbook, the aim has been to present the fundamental principles in as clear and simple a manner as possible, and to enforce them by a sufficient number of illustrative examples.

The study of Dynamics, as presented in this manual, is founded upon a course in Statics and Kinematics. It is assumed, moreover, that the students have already become familiar with the fundamental ideas of force, energy and work through such preliminary courses of General Physics. The mathematical training required for using the book is that usually implied by an elementary knowledge of Differential and Integral Calculus.

In short, this textbook presents the subject of Dynamics in that relation to other subjects which have become established in the curricula of the technical universities. It should be emphasized, however, that the manual includes, for purposes of review, a discussion of the fundamental notions and many problems involving these notions. Attention may be called to the arrangement in the text. This arrangement is founded upon experience in teaching the subject for many years in the National Mining University of Ukraine. This manual is based on a Short Course of Theoretical Mechanics by S.M. Targ (Foreign Languages Publishing House, Moscow) [1] and prepared for foreign students and for those who study some subjects of the curricula in English.

The opinion is sometimes expressed that the needs of different classes of students require essentially different methods of treating the subject. This view, so far as it refers to the fundamental parts of an elementary course of Dynamics, is not shared by the author of this textbook. For all students, the matter of first importance is the clear understanding of fundamental general principles and the ability to apply them. That is why there are included in the text some of problems suggested for independent work on the course. These problems are taken from [2]. They were chosen with an eye to ensure a clear comprehension of the dynamical phenomena, and they embrace all the main methods of Dynamics. In order to assist the students' work the examples of the problem solutions contain the relevant instructions.

The author hopes that this manual may be useful to students of technical specialties interested in advancing their knowledge of Dynamics. If this book is in any degree successful in meeting the needs of students of engineering, it is hoped that it may be of service also to those pursuing the subject for its intrinsic scientific interest or as a preparation for the study of other engineering disciplines.

The author should be greatly obliged to those who may make use of the book if they would point out any defects or obscurities in the text or would offer suggestions for its improvement.

INTRODUCTION

This text is intended for the first course in the study of Theoretical Mechanics, and its part, Dynamics, usually taken by engineering students in the sophomore or junior year. It is assumed that the student has completed the basic courses in physics, calculus, statics, and kinematics.

The purpose of the study of Dynamics is twofold. First, students must be introduced to the basic ideas and concepts used in the area of Dynamics. This includes a thorough treatment of the basic ideas of mass, acceleration, force, energy, work, mechanical system, measures of mechanical interaction and motion, differential equations of motion, and so on. These ideas are emphasized and kept in focus throughout this text, with careful study of how their combination leads to specific theories about motion of material bodies.

Second, students need ample practice in applying these theories to practical situations. Relatively simple problems are examined in this text to analyze motion. Both of these goals require continual awareness of all the notions that are necessary parts of Dynamics, to understand and avoid situations where application of theory is unwarranted.

A traditionally difficult aspect of developing a Dynamics text has been in striking a balance between theory and the many practical applications that are important to students who will go on to use the knowledge in actual practice.

A major goal is to keep the basic ideas clearly in focus when developing theory or applying the results of the theory to actual situations. In experience, a careful separate treatment of each of the basic ideas provides an excellent framework for the study of elementary theories in Dynamics.

Section 1 presents an overview of the typical areas of application of Dynamics. The three basic ideas of Dynamics, force, mass, and acceleration, are introduced. The laws and problems of Dynamics are formulated.

The main goal of Dynamics can be stated roughly as follows. Given the loads applied to the body, what is the law of its motion? Discussions in Section 2 involve using differential equations of particle motion to determine its law of motion.

Section 3 is devoted to the study of characteristic features of motion, i.e., vibrations. The main notions and conclusions are discussed in detail. The phenomenon of resonance is considered.

Section 4 has auxiliary character for the study of System Dynamics. Some of important concepts are introduced and discussed.

So called general theorems of Dynamics for particle and mechanical system are covered in Section 5. In order to consider these theorems, many concepts are introduced. Using theorems solves the basic problem of Dynamics without integrating differential equations of motion in many practical applications.

Some elements of Analytical Mechanics are considered in Section 6. This Section studies principles of Dynamics which represent general approach to the dynamic and static problems.

Section 7 deals with Lagrangian techniques for developing differential equations of motion for mechanical system. The advantages' of Lagrange's equations are demonstrated in detail. Very likely, this Section is the most important from the practical point of view.

All Sections except Section 4 contain examples of problems solution.

There are problems for self-instruction with examples of their solution in Section 8. These, more challenging problems, are intended to assist the students in understanding and applying the basic ideas.

1. LAWS AND PROBLEMS OF DYNAMICS

1.1. Basic Concepts and Definitions

Dynamics is that section of mechanics which treats of the laws of motion of material bodies subjected to the action of forces.

The motion of bodies from a purely geometrical point of view is discussed in kinematics. Unlike kinematics, in dynamics the motion of bodies is investigated in connection with the acting forces and the inertia of the material bodies themselves.

The concept of force as a quantity characterizing the measure of mechanical interaction of material bodies is introduced in the course of statics. But in statics we treat all forces as constant without considering the possibility of their changing with time. In real systems, though, alongside of constant forces a body is often subjected to the action of variable forces whose magnitudes and directions change when the body moves. Variable forces may be both applied (active) forces and the reactions of constraints.

Experience shows that variable forces may depend in some specific ways on time, on the position of a body, or on its velocity (examples of dependence on time are furnished by the tractive force of an electric locomotive whose rheostat is gradually switched on or off, or the force causing the vibration of a foundation of a motor with a poorly centered shaft; the Newtonian force of gravitation or the elastic force of a spring depend on the position of a body; the resistance experienced by a body moving through air or water depends on the velocity). In dynamics we shall deal with such forces alongside of constant forces. The laws for the composition and resolution of variable forces are the same as for constant forces.

The concept of inertia of bodies arises when we compare the results of the action of an identical force on different material bodies. Experience shows that if the same force is applied to two different bodies initially at rest and free from any other actions, in the most general case the bodies will travel different distances and acquire different velocities in the same interval of time.

Inertia is the property of material bodies to resist a change in their velocity under the action of applied forces. If, for example, the velocity of one body changes slower than that of another body subjected to the same force, the former is said to have greater inertia, and vice-versa. The inertia of a body depends on the amount of matter it contains.

The quantitative measure of the inertia of body, which depends on the quantity of matter in the body, is called the mass of that body. In mechanics mass m is treated as a scalar quantity which is positive and constant for body. The measurement of mass will be discussed in the following article.

In the most general case the motion of a body depends not only on its aggregate mass and the applied forces, the nature of motion may also depend on the dimensions of the body and the mutual position of its particles (i.e., on the distribution of its mass).

In the initial course of dynamics, in order to neglect the influence of the dimensions and the distribution of the mass of a body, the concept of a material point, or particle, is introduced.

A particle is a material body (a body possessing mass) the size of which can be neglected in investigating its motion.

Actually any body can be treated as a particle when the distances traveled by its points are very great as compared with the size of the body itself. Furthermore, as will be shown in the dynamics of systems, a body in translational motion can always be considered as a particle of mass equal to the mass of the whole body.

Finally, the parts into which we shall mentally divide bodies in analyzing any of their dynamic characteristics can also be treated as material points.

Obviously, the investigation of the motion of a single particle should precede the investigation of systems of particles, and in particular of rigid bodies. Accordingly, the course of dynamics is conventionally subdivided into particle dynamics and the dynamics of systems of particles.

1.2. The Laws of Dynamics

The study of dynamics is based on a number of laws generalizing the results of a wide range of experiments and observations of the motions of bodies, i.e., laws that have been verified in the long course of human history.

The First Law (the Inertia Law): a particle free from any external influences continues in its state of rest, or of uniform rectilinear motion, except in so far as it is compelled to change that state by impressed forces.

The motion of a body not subjected to any force is called motion under no forces, or inertial motion.

The inertia law states one of the basic properties of matter: that of being always in motion. It establishes the equivalence, for material bodies, of the states of rest and of motion under no forces.

A frame of reference for which the inertia law is valid is called an inertial system (or, conventionally, a fixed system). Experience shows that, for our solar system, an inertial frame of reference has its origin in the center of the sun and its axes are pointed towards the so-called "fixed" stars. In solving most engineering problems a sufficient degree of accuracy is obtained by assuming any frame of reference connected with the earth to be an inertial system.

The Second Law (the Fundamental Law of Dynamics) establishes the mode in which the velocity of a particle changes under the action of a force. It states: *the product of the mass of a particle and the acceleration imparted to it by a force is proportional to the acting force; the acceleration takes place in the direction of the force.*

Mathematically this law is expressed by the vector equation:

$$m\boldsymbol{a} = \boldsymbol{F}. \qquad (1.1)$$

The second law of dynamics, like the first, is valid only for an inertial system. It can be immediately seen from the law that the measure of the inertia of a particle is its mass, since two different particles subjected to the action of the same force receive the same acceleration only if their masses are equal; if their masses are different, the particle with the larger mass (i.e., the more inert one) will receive a smaller acceleration, and vice-versa.

A set of forces acting on a particle can, as we know, be replaced by a single resultant $\boldsymbol{R}$ equal to the geometrical sum of those forces. In this case the equation expressing the fundamental law of dynamics acquires the form

$$m\boldsymbol{a} = \boldsymbol{R} \text{ or } m\boldsymbol{a} = \sum \boldsymbol{F}_k. \qquad (1.2)$$

Measure of mass. Eq. (1.2) makes it possible to determine the mass of a body if its acceleration in translational motion and the acting force are known. It has been established experimentally that under the action of the force of gravitation $\boldsymbol{P}$ all bodies falling to the earth (from a small height and in vacuum) possess the same acceleration g, this is known as the acceleration of gravity or of free fall. Applying Eq. (1.2) to this motion, we obtain $m\,g = P$, whence

$$m = \frac{P}{g}. \qquad (1.3)$$

Thus, the mass of a body is equal to its weight divided by the acceleration of gravity g.

The Third Law (the Law of Action and Reaction) establishes the character of mechanical interaction between material bodies. For two particles it states: *two particles exert on each other forces equal in magnitude and acting in opposite directions along the straight line connecting the two particles.*

It should be noted that the forces of interaction between free particles (or bodies) do not form a balanced system, as they act on different objects.

The third law of dynamics, which establishes the character of interaction of material particles, plays an important part in the dynamics of systems.

1.3. The Problems of Dynamics for a Free and a Constrained Particle

The problems of dynamics for a free particle are: 1) knowing the equation of motion of a particle, to determine the force acting on it (the first problem of dynamics); 2) knowing the forces acting on a particle, to determine its equation of motion (the second, or principal, problem of dynamics).

Both problems are solved with the help of Eq. (1.1) or (1.2), which expresses the fundamental law of dynamics, since they give the relation between acceleration, i.e., the quantity characterizing the motion of a particle, and the forces acting on it.

In engineering it is often necessary to investigate constrained motions of a particle, i.e., cases when constraints attached to a particle compel it to move along a given fixed surface or curve.

In such cases we shall use, as in statics, the axiom of constraints, which states that any constrained particle can be treated as a free body detached from its constraints provided the latter is represented by their reactions $\boldsymbol{N}$. Then the fundamental law of dynamics for the constrained motion of a particle takes the form

$$m\boldsymbol{a} = \sum \boldsymbol{F}_k^a + \boldsymbol{N}, \tag{1.4}$$

where $\boldsymbol{F}_k^a$ denotes the applied forces acting on the particle.

For constrained motion the first problem of dynamics will usually be: to determine the reactions of the constraints acting on a particle if the motion and applied forces are known. The second (principal) problem of dynamics for such motion will pose two questions: knowing the applied forces, to determine: a) the equation of motion of the particle and b) the reaction of its constraints.

1.4. Solution of Problems

Problem 1. A balloon of weight *P* descends with acceleration a. What weight (ballast) Q must be thrown overboard in order to give the balloon an equal upward acceleration?

Solution. The forces acting on the falling balloon are its weight $\boldsymbol{P}$ and the buoyancy force $\boldsymbol{F}$ (Fig. 1). Hence, from Eq. (1.2)

$$\frac{P}{g}a = P - F.$$

After the ballast has been thrown out (Fig. 1), the weight of the balloon becomes $P - Q$, the buoyancy force remaining the same. Hence, taking into account that now the balloon is rising, we have

$$\frac{P-Q}{g}a = F - (P - Q).$$

Eliminating the unknown force *F* from the equations, we obtain

$$Q = \frac{2P}{1+\frac{g}{a}}.$$

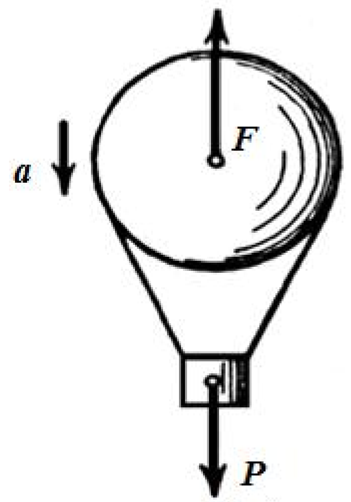

Fig.1

Problem 2. A lift of weight *P* (Fig. 2) starts ascending with acceleration a. Determine the tension in the cable.

Solution. Considering the lift as a free body, replace the action of the constraint (the cable) by its reaction $\boldsymbol{T}$. From Eq. (1.4) we obtain

$$\frac{P}{g}\, a = T - P,$$

whence

$$T = P\left(1 + \frac{a}{g}\right).$$

If the lift starts descending with the same acceleration, the tension in the cable will be

$$T = P\left(1 - \frac{a}{g}\right).$$

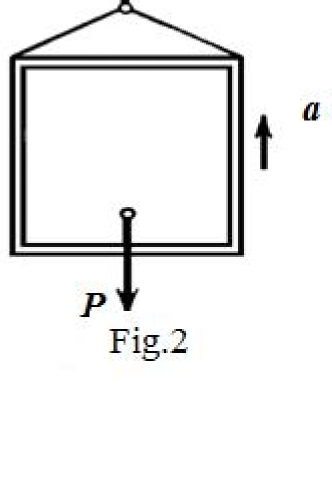

Fig.2

Problem 3. The radius of curvature of a bridge at point A is R (Fig. 3). Determine the pressure exerted on the bridge at A by a motor car of weight P moving with a velocity υ.

Solution. The normal acceleration of the car at point A is

$$a_n = \frac{\upsilon^2}{R}.$$

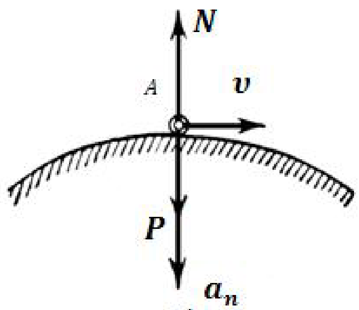

Fig.3

Acting on it are the force of gravity $\boldsymbol{P}$ and the reaction of constraint $\boldsymbol{N}$. Then, from Eq. (1.4),

$$\frac{P}{g}\frac{\upsilon^2}{R} = P - N,$$

whence

$$N = P\left(1 - \frac{\upsilon^2}{gR}\right).$$

The pressure on the bridge is equal to $\boldsymbol{N}$ in magnitude but is directed downward.

Problem 4. A crank of length l (Fig. 4) rotates with a uniform angular velocity ω and translates the slotted bar K of weight P along slides 1,1. Neglecting friction, determine the pressure exerted by the slide block A on the slotted bar.

Solution. The position of the bar is specified by its coordinate $x = l\cos\omega t$. Eq. (1.4) for the motion of the bar in terms of its projection on x axis gives $ma_x = Q_x$. But

$$a_x = \frac{d^2x}{dt^2} = -l\omega^2\cos\omega t = -\omega^2 x,$$

whence, as $Q_x = -Q$, $-\frac{P}{g}\omega^2 x = -Q$, $Q = \frac{P}{g}\omega^2 x$.

Thus, the pressure of the slide block on the slotted bar is proportional to its coordinate x.

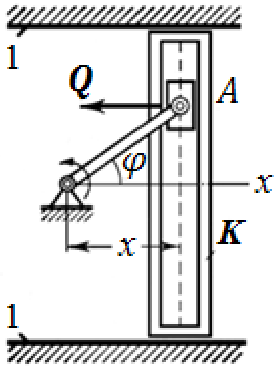

Fig. 4

2. DIFFERENTIAL EQUATIONS OF MOTION FOR A PARTICLE AND THEIR INTEGRATION

2.1. Rectilinear Motion of a Particle

We know from kinematics that in rectilinear motion the velocity and acceleration of a particle are continuously directed along the same straight line. As the direction of acceleration is coincident with the direction of force, it follows that a free particle will move in a straight line whenever the force acting on it is of constant direction and the velocity at the initial moment is either zero or is collinear with the force.

Consider a particle moving rectilinearly under the action of an applied force $\boldsymbol{R} = \sum \boldsymbol{F}_k$. The position of the particle on its path is specified by its coordinate *x* (Fig. 5). In this case the principal problem of dynamics is: knowing $\boldsymbol{R}$, to find the equation of motion of the particle $x = f(x)$. Eq. (1.2) gives the relation between x and $\boldsymbol{R}$. Projecting both sides of the equation on axis *Ox*, we obtain

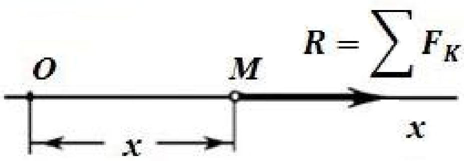

Fig. 5

$$ma_x = R_x = \sum F_{kx} \text{ or as } a_x = \frac{d^2x}{dt^2},$$

$$m\frac{d^2x}{dt^2} = \sum F_{kx}. \qquad (2.1)$$

Eq. (2.1) is called the differential equation of rectilinear motion of a particle. It is often more convenient to replace Eq. (2.1) with two differential equations containing first derivatives:

$$m\frac{dv_x}{dt} = \sum F_{kx}, \qquad (2.2)$$

$$\frac{dx}{dt} = v_x. \qquad (2.2')$$

Whenever the solution of a problem requires that the velocity be found as a function of the coordinate *x* instead of time *t* (or when the forces themselves depend on *x*), Eq. (2.2') is converted to the variables *x*. As $\frac{dv_x}{dt} = \frac{dv_x}{dx} \times \frac{dx}{dt} = \frac{dv_x}{dx} v_x$ Eq. (2.2) takes the form

$$mv_x\frac{dv_x}{dx} = \sum F_{kx}. \qquad (2.3)$$

The principal problem of dynamics is, essentially, to develop the equation of motion *x=f(t)* for a particle from the above equations, the forces being known. For this it is necessary to integrate the corresponding differential equation. In order to make clearer the nature of the mathematical problem, it should be recalled that the forces in the right side of Eq. (2.1) can depend on time *t,* on the position of the particle *x*, or on the velocity $v_x = \frac{dx}{dt}$. Consequently, in the general case Eq. (2.1) is, mathematically, a differential equation of the second order in the form

$$\frac{d^2x}{dt^2} = \Phi\left(t, x, \frac{dx}{dt}\right). \qquad (2.4)$$

The equation can be solved for every specific problem after determining the form of its right-hand member, which depends on the applied forces. When Eq. (2.4) is integrated for a given problem, the general solution will include two constants of integration C_1 and C_2 and the general form of the solution will be

$$x = f(t, C_1, C_2). \qquad (2.5)$$

To solve a concrete problem, it is necessary to determine the values of the constants C_1 and C_2. For this we introduce the so-called initial conditions.

Investigation of any motion begins from some specified instant called the initial time $t=0$, usually the moment when the motion under the action of the given forces starts. The position occupied by a particle at the initial time is called its initial displacement, and its velocity at that time is its initial velocity (a particle can have an initial velocity either because at time $t=0$ it was moving under no force or because up to time $t=0$ it was subjected to the action of some other forces). To solve the principal problem of dynamics we must know, besides the applied forces, the initial conditions, i.e., the position and velocity of the particle at the initial time.

In the case of rectilinear motion, the initial conditions are specified in the form

$$\text{at } t = 0, \ x = x_0, \ v_x = v_0. \tag{2.6}$$

From the initial conditions we can determine the meaning of the constants C_1 and C_2, and develop finally the equation of motion for the particle in the form

$$x = f(t, x_0, v_o). \tag{2.7}$$

The following simple example will explain the above. Let there be acting on a particle a force $\boldsymbol{Q}$ of constant magnitude and direction. Then Eq. (2.2) acquires the form

$$m\frac{dv_x}{dt} = Q_x.$$

As Q_x=*const.*, multiplying both members of the equation by dt and integrating, we obtain

$$v_x = \frac{Q_x}{m}t + C_1. \tag{2.8}$$

Substituting the value of v_x into Eq. (2.2'), we have

$$\frac{dx}{dt} = \frac{Q_x}{m}t + C_1.$$

Multiplying through by dt and integrating once again, we obtain

$$x = \frac{1}{2}\frac{Q_x}{m}t^2 + C_1 t + C_2. \tag{2.9}$$

This is the general solution of Eq. (2.4) for the specific problem in the form given by Eq. (2.5).

Now let us determine the integration constants C_1 and C_2, assuming for the specific problem the initial conditions given by (2.6). Solutions (2.8) and (2.9) must satisfy any moment of time, including $t=0$. Therefore, substituting zero for t in Eqs. (2.8) and (2.9), we should obtain v_0 and x_o, instead of v_x and x, i.e., we should have $v_0=C_1$, $x_o = C_2$.

These equations give the values of the constants C_1 and C_2, which satisfy the initial conditions of a given problem. Substituting these values into Eq. (2.9), we obtain finally the relevant equation of motion in the form expressed by Eq. (2.7):

$$x = \frac{1}{2}\frac{Q_x}{m}t^2 + v_0 t + x_0. \tag{2.10}$$

We see from Eq. (2.10) that a particle subjected to a constant force performs uniformly variable motion. This could have been foreseen; for, if $\boldsymbol{Q} = const.$, $\boldsymbol{a} = const.$, too. An example of this type of motion is the motion of a particle under the force of gravity, in which case in Eq. (2.10) $\frac{Q_x}{m} = g$ and axis Ox is directed vertically down.

2.2. Curvilinear Motion of a Particle

Consider a free particle moving under the action of forces $\boldsymbol{F_1}, \boldsymbol{F_2}, \dots, \boldsymbol{F_n}$. Let us draw a fixed set of axes *Oxyz* (Fig. 6). Projecting both members of the equation (1.2) on these axes, and taking into account that $a_x = \frac{d^2x}{dt^2}, a_y = \frac{d^2y}{dt^2}, a_z = \frac{d^2z}{dt^2}$ we obtain the differential equations of curvilinear motion of a body in terms of the projections on rectangular Cartesian axes:

$$m\frac{d^2x}{dt^2} = \sum F_{kx}\,,\ m\frac{d^2y}{dt^2} = \sum F_{ky}\,,\ m\frac{d^2z}{dt^2} = \sum F_{kz}. \quad (2.11)$$

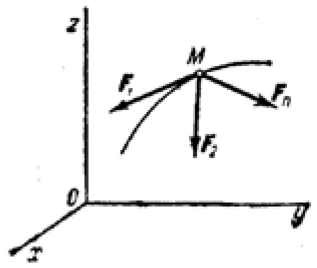

Fig.6

As the forces acting on the particle may depend on time, the displacement or the velocity of the particle, then by analogy with Eq. (2.4), the right-hand members of Eqs. (2.11) may contain the time *t,* the coordinates *x, y, z* of the particle, and the projections of its velocity $\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}$. Furthermore, the right side of each equation may include all these variables.

Eq. (2.11) can be used to solve both the first and the second (the principal) problems of dynamics. To solve the principal problem of dynamics we must know, besides the acting forces, the initial conditions, i.e., the position and velocity of the particle at the initial time. The initial conditions for a set of coordinate axes *Oxyz* are specified in the form: *at t=0,*

$$x = x_0, y = y_0, z = z_0,$$
$$v_x = v_{x0},\ v_y = v_{y0}, v_z = v_{z0}. \quad (2.12)$$

Knowing the acting forces, by integrating Eq. (2.11), we find the coordinates *x, y, z* of the moving particle as functions of time *t,* i.e., the equation of motion for the particle. The solutions will contain six constants of integration $C_1, C_2, \dots, C_6$, the values of which must be found from the initial conditions (2.12). An example of integrating of Eqs.(2.11) is given in §2.3.

2.3. Motion of a Particle Thrown at an Angle to the Horizon in a Uniform Gravitational Field

Let us investigate the motion of a projectile thrown with an initial velocity $\boldsymbol{v_0}$ at an angle α to the horizon, considering it as a material particle of mass m, neglecting the resistance of the atmosphere, assuming that the horizontal range is small as compared with the radius of the earth and considering the gravitational field to be uniform ($P = const.$).

Place the origin of the coordinate axes O at the initial position of the particle, direct the y-axis vertically up, the x-axis in the plane through Oy and vector $\boldsymbol{v_0}$, and the z-axis perpendicular to the first two (Fig. 7). The angle between vector $\boldsymbol{v_0}$ and the x-axis will be α.

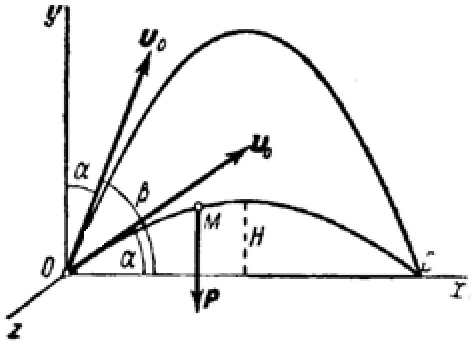

Fig. 7

Draw now moving particle M anywhere on its path. Acting on the particle is only the force of gravity $\boldsymbol{P}$, the projections of which on the coordinate axes are $P_x = 0$, $P_y = -P = -mg$, $P_z = 0$.

Substituting these values into Eq.(2.11) and noting that $\frac{d^2x}{dt^2} = \frac{dv_x}{dt}$, etc., after eliminating m, we obtain:

$$\frac{dv_x}{dt} = 0, \qquad \frac{dv_y}{dt} = -g, \quad \frac{dv_z}{dt} = 0.$$

Multiplying these equations by dt and integrating, we find $v_x = C_1$, $v_y = -gt + C_2$, $v_z = C_3$.

The initial conditions of our problem have the form

$$\text{at } x = 0, y = 0, z = 0;$$

$$v_x = v_0 \cos\alpha, \qquad v_y = v_0 \sin\alpha, \ v_z = 0.$$

Satisfying the initial conditions, we have

$$C_1 = v_0 \cos\alpha, \ C_2 = v_0 \sin\alpha = 0, C_3 = 0.$$

Substituting these values of C_1, C_2 and C_3 in the solutions above and replacing v_x, v_y, v_z by $\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}$, we arrive at the equations

$$\frac{dx}{dt} = v_0 \cos\alpha, \ \frac{dy}{dt} = v_0 \sin\alpha - gt, \frac{dz}{dt} = 0.$$

Integrating, we obtain $x = v_0\, t\cos\alpha + C_4$, $y = v_0 t \sin\alpha - \frac{gt^2}{2} + C_5$, $z = C_6$.

Substituting the initial conditions, we have $C_4 = C_5 = C_6 = 0$. And finally we obtain the equations of motion of particle M in the form

$$x = v_0\, t\cos\alpha, \ y = v_0 t \sin\alpha - \frac{gt^2}{2}, \ z = 0. \qquad (2.13)$$

From the last equation it follows that the motion takes place in the plane Oxy.

Knowing the equations of motion of a particle it is possible to determine all the characteristics of the given motion by the methods of Kinematics.

1. Path. Eliminating the time t between the first two of Eqs. (2.13), we obtain the equation of the path of the particle:

$$y = x \tan\alpha - \frac{gx^2}{2v_0^2 \cos^2\alpha}. \qquad (2.14)$$

This is an equation of a parabola the axis of which is parallel to the y-axis. Thus, a heavy particle thrown at an angle to the horizon in vacuum follows a parabolic path.

2. Horizontal Range. The horizontal range is the distance $OC=X$ along the x-axis. Assuming in Eq. (2.14) $y=0$, we obtain the points of intersection of the path with the x-axis. From the equation

$$x\left(\tan\alpha - \frac{gx}{2v_0^2\cos^2\alpha}\right) = 0$$

we obtain

$$x_1 = 0,\ x_2 = \frac{2v_0^2\cos^2\alpha\tan\alpha}{g}.$$

The first solution gives point *0*, the second point *C*. Consequently $X = x_2$ and finally

$$X = \frac{v_0^2}{g}\sin 2\alpha. \qquad (2.15)$$

From Eq. (2.15) we see that the horizontal range X is the same for angle β, where $2\beta=180°—2\alpha$, i.e., if $\beta=90°— \alpha$. Consequently, a particle thrown with a given initial velocity v_0 can reach the same point C by two paths: flat (low) $(\alpha<45°)$ or curved (high) $(\beta=90°— \alpha>45°)$. With a given initial velocity v_0 the maximum horizontal range in vacuum is obtained when $\sin 2\alpha = 1$, i.e., when angle $\alpha = 45°$.

3. Height of path. If in Eq. (2.14) we assume $x = \frac{1}{2}X = \frac{v_0^2}{g}\sin\alpha\cos\alpha$, we obtain the height H of the path:

$$H = \frac{v_0^2}{2g}\sin^2\alpha. \qquad (2.16)$$

4. Time of flight. It follows from Eq. (2.13) that the total time of flight is defined by the equation $X = v_0 T\cos\alpha$. Substituting the expression for X, we obtain

$$T = \frac{2v_0}{g}\sin\alpha. \qquad (2.17)$$

At the maximum range angle $\alpha^*=45°$, all the quantities become respectively

$$X^* = \frac{v_0^2}{g},\ T^* = \frac{\sqrt{2}v_0}{g},\ H^* = \frac{v_0^2}{4g}X.$$

2.4. Solution of Problems

Problem 5. A load of weight P starts moving from rest along a smooth horizontal plane under the action of a force $\boldsymbol{R}$ the magnitude of which increases in proportion to the time, the relation being $R=kt$. Develop the equation of motion for the load.

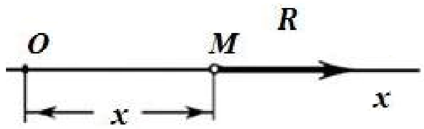

Fig. 8

Solution. Place the origin O in the initial position of the load and direct the axis Ox in the direction of motion (see Fig. 8). Then the initial conditions are: at t=0, x=0 and $v_x = 0$. Draw the load in an arbitrary position and the forces acting on it. We have $R_x = R = kt$, and Eq. (2.2) takes the form

$$\frac{P}{g}\frac{dv}{dt} = kt.$$

Multiplying through by dt, we immediately separate the variables and obtain

$$v_x = \frac{kg}{P}\frac{t^2}{2} + C_1.$$

Substituting the initial values into this equation, we find that $C_1 = 0$. Then, substituting $\frac{dx}{dt}$ for v_x, we have

$$\frac{dx}{dt} = \frac{kg}{2P}t^2.$$

Multiplying through by dt we again separate the variables and, integrating, we find

$$x = \frac{kg}{2P}\frac{t^3}{3} + C_2.$$

Substitution of the initial values gives $C_2 = 0$, and we obtain the equation of motion for the load in the form

$$x = \frac{kg}{6P}t^3.$$

Problem 6. Neglecting the resistance of the air, determine the time it would take a body to travel from end to end of a tunnel AB dug through the earth along a chord (Fig. 9). Assume the earth's radius to be $R = 6{,}370\ km$.

Note. The theory of gravitation states that a body inside the earth is attracted towards the centre of the earth with a force F directly proportional to the distance r from the centre. Taking into account that, at $r = R$ (i.e., at the surface of the earth), force F is equal to the weight of the body $(F = mg)$, we find that inside the earth $F = \frac{mg}{R}r$, where $r = MC$ is the distance of point *M* from the centre of the earth.

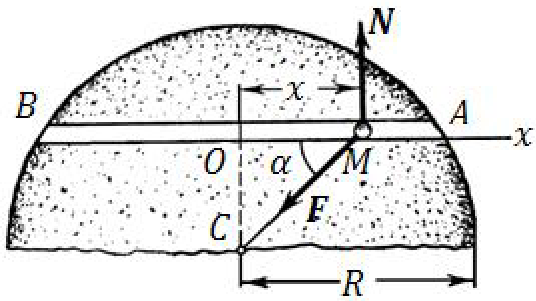

Fig. 9

Solution. Place the origin O in the middle of the chord AB (where a body in the tunnel would be in equilibrium) and direct the axis Ox along OA. If we assume the chord to be of length $2a$, initial conditions will be: at $t = 0, x = a$ and $v_x = 0$.

The forces acting on the body in an arbitrary position are $\boldsymbol{F}$ and $\boldsymbol{N}$. Consequently,

$$\sum F_{kx} = -F\cos\alpha = -\frac{mg}{R}r\cos\alpha = -\frac{mg}{R}x\,,$$

as it is evident from the diagram that $r\cos\alpha = x$.

We see that the acting force depends on the coordinate x of point M. In order to separate the variables in the differential equation of motion, write it in the form (2.3). Then, eliminating m and introducing the quantity

$$\frac{g}{R} = k^2,$$

we obtain

$$v_x\frac{dv_x}{dx} = -k^2x.$$

Multiplying through by dx, we separate the variables and, integrating, obtain

$$\frac{v_x^2}{2} = k^2\frac{x^2}{2} + C_1.$$

From the initial condition, at $x = a$, $v_x = 0$; hence $C_1 = \frac{1}{2}k^2a^2$. Substituting this expression of C_1, we have

$$v_x = \pm k\sqrt{a^2 - x^2}.$$

As in the investigated position the velocity is directed from M to O, $v_x < 0$, and the sign before the radical should be minus. Then, substituting $\frac{dx}{dt}$ for v_x, we have

$$\frac{dx}{dt} = -k\sqrt{a^2 - x^2}.$$

Separating the variables, we write the equation in the from

$$kdt = -\frac{dx}{\sqrt{a^2-x^2}},$$

and integrating, we obtain

$$kt = \arccos\frac{x}{a} + C_2.$$

Substituting the initial data (at t=0, x=a) in this equation, we find that $C_2 = 0$. The equation of motion for the body in the tunnel is

$$x = a\cos kt.$$

Thus, the body is in harmonic motion with an amplitude a.

Now let us determine the time t_1 when the body will reach the end B of the tunnel. At B the coordinate $x = -a$. Substituting this value in the equation of motion, we obtain $\cos kt_1 = -1$, whence $kt_1 = \pi$ and $t_1 = \frac{\pi}{k}$. But we have assumed $k = \sqrt{\frac{g}{R}}$. Calculating, we find that the time of the motion through the tunnel, given the conditions of the problem, does not depend on the length of the tunnel and is always equal to

$$t_1 = \pi\sqrt{\frac{R}{g}} \approx 42\, min\, 11\, sec.$$

Let us also find the maximum velocity of the body. From the expression for v_x we see that $v = v_{max}$ at x = 0, i.e., at the origin O. The magnitude of the velocity is

$$v_{max} = ka = a\sqrt{\frac{g}{R}}.$$

If, for example, 2a = 0.1R = 637 km, then $v_{max} \approx 395\ m/\,sec = 1{,}422\ km/h.$

Problem 7. A boat of weight P=400 N is pushed and receives an initial velocity $v_0 = 0.5\ m/sec$. Assuming the resistance of the water at low velocities to be proportional to the first power of the velocity and changing according to the equation $R = \mu v$, where the factor μ = 9,3 *N·sec/m*, determine the time in which the velocity will drop by one-half and the distance the boat will travel in that time. Determine also the distance the boat will travel till it stops.

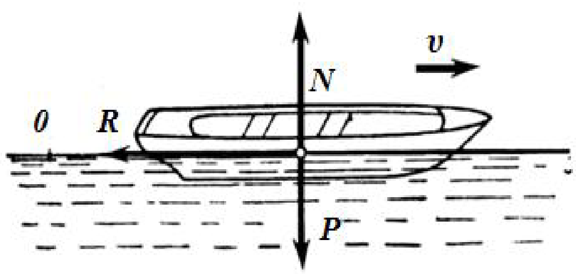

Fig. 10

Solution. Let us choose the origin *O* to coincide with the initial position of the boat, pointing the axis *Ox* in the direction of the motion (Fig.10). In this case the initial conditions will be: at *t*=0, *x*=0 and $v_x = v_0$. Draw the boat in an arbitrary position with the acting forces $\boldsymbol{P}$, $\boldsymbol{N}$, and $\boldsymbol{R}$.

Calculating the projections of the acting forces, we find that

$$\sum F_{kx} = -R = -\mu v.$$

To determine the duration of the motion, we write differential equation. Noting that $v_x = v,$ we have

$$\frac{P}{g}\frac{dv}{dt} = -\mu v.$$

Separating the variables, we obtain

$$\frac{dv}{v} = -\frac{\mu g}{P}dt,$$

whence, integrating, we have

$$\ln v = -\frac{\mu g}{P}t + C_1.$$

Substituting the initial values, we have $C_1 = \ln v_0$, and finally

$$t = \frac{P}{\mu g}\ln\frac{v_0}{v}.$$

The required time t_1, is determined by assuming $v = 0.5v_0$ .We see that in this case the time does not depend on the value of v_0. As $\ln 2 = 0.69$,

$$t_1 = \frac{P}{\mu g}\ln 2 \approx 3\, sec.$$

To determine the distance, it is best to write the differential equation of motion in the form (2.3), as it immediately establishes the relation between x and v. We thus obtain

$$\frac{P}{g}v\frac{dv}{dx} = -\mu v,$$

whence, eliminating v and separating the variables, we find

$$dv = -\frac{\mu g}{P}dx,$$

and consequently

$$v = -\frac{\mu g}{P}x + C_1.$$

Since at $x = 0$ the velocity $v = v_0$, then $C_1 = v_0$, and finally

$$x = \frac{P}{\mu g}(v_0 - v).$$

Assuming $v = 0.5v_0$, we find the required displacement: $x_1 = \frac{Pv_0}{2\mu g} \approx 1.1\, m.$

To find the distance travelled by the boat till it stops, in the last equation we assume $v = 0$. Then $x_2 = \frac{Pv_0}{\mu g} = 2.2\, m.$